Carroll Fermions and Supersymmetry

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Based on work in progress with E. Bergshoeff, A. Campoleoni, A. Fontanella, L. Mele
and work with E. Bergshoeff, J. Figueroa-O'Farrill, K. van Helden, I. Rotko, T. ter Veldhuis
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Introduction

- Carroll symmetries and Carroll invariant field theories might be useful for a variety of applications, e.g.,
  - physics of black hole horizons/null hypersurfaces
  - flat space and celestial holography
  - hydrodynamics (Ciambelli, Marteau, Petkou, Petropoulos, Siampos)
  - tensionless strings (Bagchi)
  - ...
- Carroll field theories are still relatively unexplored. Mostly bosonic so far. (e.g., de Boer, Hartong, Obers, Sybesma, Vandoren; Bagchi, Grumiller, Mehra, Nandi)
- Supersymmetry is very powerful and thus natural to consider to better understand the behavior of Carroll QFTs.
- Need to get complete understanding of fermionic field representations of the (homogeneous) Carroll group, coupling to Carroll geometry, ...
- One approach is to build these up from scratch, e.g., by defining suitable spinor representations from a construction of 'Carrollian' Clifford algebras (Bagchi, Banerjee, Basu, Islam) or by constructing unitary irreducible representations. (Figueroa-O'Farrill, Perez, Prohazka)

Introduction

- While general, this can obscure a possible relativistic field theory origin.
- Other approach: define Carroll fermions via a Carrollian limit.
- 1. Redefine fields, symmetry parameters, ... of relativistic theory with contraction parameter $\tilde{c}$ (inverse speed of light).
  2. Plug in in transformation rules, action, ...
  3. $\tilde{c} \to \infty$ limit of a quantity = leading order term of $\tilde{c}$-expansion.
- End result uses the usual Γ-matrices of the relativistic parent theory.
- Plan:
  1. Show two different types of such limits (electric, magnetic) for spinor fields.
  2. Discuss Carroll geometry, the geometry they couple to from a limit point of view.
  3. Discuss coupling of the two types of Carroll fermions to Carroll geometry and (magnetic Carroll) gravity.
  4. Argue how the two types of Carroll fermion can appear in supermultiplets.

Carroll fermions

- Homogeneous Carroll transformations = {spatial rotations, Carroll boosts}. Action on space-time coordinates $\{t, x^a\}$: (Lévy-Leblond)

$$\delta t = -\lambda^0{}_a\, x^a\,, \qquad \delta x^a = -\lambda^a{}_b\, x^b\,.$$

- Obtained from action of Lorentz transformations on coordinates $X^A = \{X^0, X^a\}$ of Minkowski space-time:

$$\delta X^A = -\Lambda^A{}_B\, X^B\,,$$

by rescaling coordinates and parameters with contraction parameter $\tilde{c}\,(= c^{-1})$:

$$X^0 = \frac{t}{\tilde{c}}\,, \qquad X^a = x^a\,, \qquad \Lambda^{ab} = \lambda^{ab}\,, \qquad \Lambda^{0a} = \frac{1}{\tilde{c}}\lambda^{0a}\,,$$

and taking the $\tilde{c} \to \infty$ limit.

- Extend this limit to the Lorentz transformation rule of a Majorana or Dirac spinor Ψ in flat $4D$ Minkowski space-time:

$$\delta\Psi = \Lambda^A{}_B\, X^B\, \frac{\partial\Psi}{\partial X^A} - \frac{1}{4}\Lambda^{AB}\,\Gamma_{AB}\,\Psi\,,$$

Carroll fermions

- First limit: using the above rescalings, as well as

$$\Psi = \tilde{c}^{-1/2}\,\psi\,,$$

and taking $\tilde{c} \to \infty$ gives the following transformation rule of a Carroll fermion:

$$\delta\psi = \lambda^0{}_a\, x^a\, \frac{\partial\psi}{\partial t} + \lambda^a{}_b\, x^b\, \frac{\partial\psi}{\partial x^a} - \frac{1}{4}\lambda^{ab}\,\Gamma_{ab}\,\psi\,.$$

Applying this limit to the Lagrangian:

$$\mathcal{L} = \bar{\Psi}\Gamma^A\partial_A\Psi - \frac{M}{\tilde{c}}\bar{\Psi}\Psi\,,$$

gives (upon also rescaling $M = \tilde{c}^2\, m$): (Bagchi, Grumiller, Nandi)

$$\mathcal{L}_{\text{electric Carroll}} = \bar{\psi}\Gamma^0\dot{\psi} - m\bar{\psi}\psi\,, \qquad \text{with } \dot{\psi} \equiv \frac{\partial}{\partial t}\psi\,.$$

- This is an ‘electric’ Carroll fermion:
  - Trivial boost transformation, i.e., only via transport term.
  - Lagrangian contains only time derivative.

Carroll fermions

- How to obtain a 'magnetic' Carroll fermion with non-trivial Carroll boost transformation and spatial derivatives in its Lagrangian?
- Start from a Dirac spinor Ψ and introduce the projections:

$$\Psi_\pm \equiv P_\pm \Psi \,,$$

$$\text{with } \; P_\pm \equiv \frac{1}{2}\big(1 \pm \mathrm{i}\Gamma^0\big)\,, \quad \big(\,(P_\pm)^2 = P_\pm\,, \;\; P_\pm P_\mp = 0\,, \;\; P_\pm^\dagger = P_\pm\big)$$

Note that

$$\Gamma_{ab} P_\pm = P_\pm \Gamma_{ab}\,, \qquad \text{and} \qquad \Gamma_{0a} P_\pm = P_\mp \Gamma_{0a}\,,$$

$\Rightarrow$ $\Psi_\pm$ are subrepresentations under $SO(3)$ subgroup but $\Psi_+ \leftrightarrow \Psi_-$ under Lorentz boosts.
- Introduce the (invertible) field redefinition:

$$\Psi_\pm = \tilde{c}^{\pm 1/2+\epsilon}\, \psi_\pm \,.$$

Note that this requires working with a Dirac spinor: for a Majorana spinor

$$\Psi_\pm = \mathrm{i}\gamma^0 C^{-1}\,(\Psi_\mp)^*\;,$$

so that consistency requires rescaling $\Psi_\pm$ in the same way.

Carroll fermions

- The $\tilde{c} \to \infty$ limit of the Lorentz transformation rule of Ψ then gives:

$$\delta\psi_+ = \lambda^0{}_a x^a \frac{\partial\psi_+}{\partial t} + \lambda^a{}_b x^b \frac{\partial\psi_+}{\partial x^a} - \frac{1}{4}\lambda^{ab}\,\Gamma_{ab}\,\psi_+\,,$$
$$\delta\psi_- = \lambda^0{}_a x^a \frac{\partial\psi_-}{\partial t} + \lambda^a{}_b x^b \frac{\partial\psi_-}{\partial x^a} - \frac{1}{4}\lambda^{ab}\,\Gamma_{ab}\,\psi_- - \frac{1}{2}\lambda^{0a}\,\Gamma_{0a}\,\psi_+\,.$$

  indecomposable, reducible representation of the homogeneous Carroll algebra.

- To get Lagrangian for both ψ_+ and ψ_- that includes spatial derivatives, one should start from a 'tachyonic' Dirac Lagrangian:

$$\mathcal{L} = \bar{\Psi}\Gamma^A\Gamma_5\partial_A\Psi - \frac{M}{\tilde{c}}\bar{\Psi}\Psi\,, \qquad \text{with } \Gamma_5 = \mathrm{i}\Gamma^0\Gamma^1\Gamma^2\Gamma^3\,.$$

- Taking $\epsilon = -\frac{1}{2}$ and rescaling $M = \tilde{c}\,m$ gives in the $\tilde{c} \to \infty$ limit:

$$\mathcal{L}_{\text{magnetic Carroll}} = \bar{\psi}_-\Gamma^0\Gamma_5\dot{\psi}_+ + \bar{\psi}_+\Gamma^0\Gamma_5\dot{\psi}_- + \bar{\psi}_+\Gamma^a\Gamma_5\frac{\partial\psi_+}{\partial x^a} - m\bar{\psi}_+\psi_+\,.$$

Carroll geometry

- Coupling of Carroll fermions to curved space and to gravity requires understanding of the geometry they live in in a Cartan formulation (using Vielbeine and spin connections). Stresses local homogeneous Carroll symmetries.
- Can be obtained as limit of Cartan formulation of Lorentzian geometry.
- Lorentzian geometry à la Cartan:
  - Vielbein ${E_\mu}^A$ (inverse ${E_A}^\mu$) transforming under local Lorentz transformations as
  $$\delta {E_\mu}^A = -\Lambda^A{}_B \, {E_\mu}^B \, .$$
  - Spin connection ${\Omega_\mu}^{AB} = -{\Omega_\mu}^{BA}$ transforming as
  $$\delta {\Omega_\mu}^{AB} = \partial_\mu \Lambda^{AB} - 2\Lambda^{[A}{}_C \, {\Omega_\mu}^{|C|B]} \, .$$
  - 1st Cartan structure equations express ${\Omega_\mu}^{AB}$ uniquely in terms of ${E_\mu}^A$ and torsion:
  $$2\partial_{[\mu}{E_{\nu]}}^A + 2{\Omega_{[\mu}}^{AB} E_{\nu]B} = {T_{\mu\nu}}^A \qquad\qquad {T_{\mu\nu}}^A = \text{torsion tensor}$$
  $$\Rightarrow \quad {\Omega_\mu}^{AB} = E^{[A|\nu|}\left(2\,\partial_{[\mu}{E_{\nu]}}^{B]} - {T_{\mu\nu}}^{B]}\right) - \frac{1}{2} E_{\mu C}\, E^{A\nu}\, E^{B\rho}\left(2\,\partial_{[\nu}{E_{\rho]}}^C - {T_{\nu\rho}}^C\right) \, .$$

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  $$\Rightarrow \quad \Omega_\mu{}^{AB} = E^{[A|\nu|}\left(2\,\partial_{[\mu} E_{\nu]}{}^{B]} - T_{\mu\nu}{}^{B]}\right) - \frac{1}{2} E_{\mu C}\, E^{A\nu}\, E^{B\rho}\left(2\,\partial_{[\nu} E_{\rho]}{}^C - T_{\nu\rho}{}^C\right) \, .$$

Carroll geometry

- Carrollian analogue can be obtained by rescaling

$$E_\mu{}^0 = \frac{1}{\tilde{c}}\tau_\mu\,, \qquad E_\mu{}^a = e_\mu{}^a\,, \qquad E_0{}^\mu = \tilde{c}\tau^\mu\,, \qquad E_a{}^\mu = e_a{}^\mu\,,$$
$$\Omega_\mu{}^{0a} = \frac{1}{\tilde{c}}\omega_\mu{}^{0a}\,, \qquad \Omega_\mu{}^{ab} = \omega_\mu{}^{ab}\,, \qquad \Lambda^{ab} = \lambda^{ab}\,, \qquad \Lambda^{0a} = \frac{1}{\tilde{c}}\lambda^{0a}\,,$$

and taking the $\tilde{c} \to \infty$ limit.

- Vielbeine: τ_μ, $e_\mu{}^a$ transforming under local homogeneous Carroll transformations as

$$\delta\tau_\mu = -\lambda^0{}_a\, e_\mu{}^a\,, \qquad \delta e_\mu{}^a = -\lambda^a{}_b\, e_\mu{}^b\,,$$
$$\delta\tau^\mu = 0\,, \qquad \delta e_a{}^\mu = \lambda^0{}_a\,\tau^\mu - \lambda_a{}^b\, e_b{}^\mu\,.$$

τ^μ, $e_a{}^\mu$ are dual to τ_μ, $e_\mu{}^a$:

$$\tau^\mu\tau_\mu = 1\,, \quad \tau^\mu e_\mu{}^a = 0\,, \quad e_a{}^\mu\tau_\mu = 0\,, \quad e_a{}^\mu e_\mu{}^b = \delta_a^b\,, \quad \tau_\mu\tau^\nu + e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu\,.$$

- Can be used to decompose tensors into time-like (index 0) and space-like (index a) components, e.g., for a one-form/vector X_μ/X^μ:

$$X_0 \equiv \tau^\mu X_\mu\,, \quad X_a \equiv e_a{}^\mu X_\mu\,, \qquad X^0 \equiv \tau_\mu X^\mu\,, \quad X^a \equiv e_\mu{}^a X^\mu\,.$$

Carroll geometry

- 2 spin connections:

$$\text{spatial rotation connection } \omega_\mu{}^{ab} \qquad \delta\omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} - 2\lambda^{[a}{}_c\,\omega_\mu{}^{|c|b]}\,,$$

$$\text{Carroll boost connection } \omega_\mu{}^{0a} \qquad \delta\omega_\mu{}^{0a} = \partial_\mu\lambda^{0a} - \lambda^a{}_b\,\omega_\mu{}^{0b} - \lambda^0{}_b\,\omega_\mu{}^{ba}\,.$$

- First Cartan structure equations:

$$2\partial_{[\mu}\tau_{\nu]} + 2\omega_{[\mu}{}^{0a}e_{\nu]a} = T_{\mu\nu}\,, \qquad 2\partial_{[\mu}e_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}e_{\nu]b} = T_{\mu\nu}{}^a\,,$$

with the torsion tensors $T_{\mu\nu}$ and $T_{\mu\nu}{}^a$ transforming as:

$$\delta T_{\mu\nu} = -\lambda^0{}_a\,T_{\mu\nu}{}^a\,, \qquad \delta T_{\mu\nu}{}^a = -\lambda^a{}_b\,T_{\mu\nu}{}^b\,.$$

- Important differences with Lorentzian geometry
  1. Setting torsion components equal to zero does not always just lead to a particular choice of connection, but can also lead to geometric constraints: intrinsic torsion.
  2. $\omega_\mu{}^{ab}$ and $\omega_\mu{}^{0a}$ are no longer uniquely determined by the 1st Cartan structure equations.

Carroll geometry

- In particular, the 1st Cartan structure equations can be split up in two sets:

1. "Intrinsic torsion equations":

$$\tau^{\mu} e_{(a|}{}^{\nu} \left(2\partial_{[\mu} e_{\nu]|b)} + 2\omega_{[\mu|b)}{}^{c} e_{\nu]c}\right) = T_{0(a,b)} \quad \Leftrightarrow \quad 2\tau^{\mu} e_{(a|}{}^{\nu} \partial_{[\mu} e_{\nu]|b)} = T_{0(a,b)} \,.$$

   Setting components of $T_{0(a,b)}$ equal to zero implies geometric constraints. Four consistent possibilities: (Bergshoeff, Figueroa-O'Farrill, van Helden, JR, Rotko, ter Veldhuis)

   1. $T_{0(a,b)} = 0$
   2. only trace part $T_{0a}{}^{a} = 0$
   3. only symmetric traceless part $T_{0\{a,b\}} = 0$
   4. no constraints on $T_{0(a,b)}$

2. Remaining 1st Cartan structure equations are "conventional constraints" that do contain spin connection components. Can be used to express

$$\omega_{\mu}{}^{ab} \,, \qquad \tau^{\mu}\omega_{\mu}{}^{0a} \,, \qquad e^{[a|\mu}\omega_{\mu}{}^{0|b]} \qquad \text{as dependent fields.}$$

   $e^{(a|\mu}\omega_{\mu}{}^{0|b)}$ remain independent: no unique metric compatible connection (for given torsion).

- These (and other related) observations generalize to arbitrary non-Lorentzian (p-brane Galilean and Carrollian) geometries. (Bergshoeff, Figueroa-O'Farrill, van Helden, JR, Rotko, ter Veldhuis)

Carroll geometry

- Gravitational action with local homogeneous Carroll symmetry from $\tilde{c} \to \infty$ limit of first order Einstein-Hilbert action:

$$S_{\text{EH}} \propto \int \mathrm{d}^D x\, E\, E_A{}^\mu E_B{}^\nu \left(2\partial_{[\mu}\Omega_{\nu]}{}^{AB} + 2\Omega_{[\mu}{}^{[A|C|}\Omega_{\nu]C}{}^{B]}\right) .$$

- Gives first order 'magnetic Carroll gravity' action (Bergshoeff, Gomis, Rollier, JR, ter Veldhuis; Campoleoni, Henneaux, Pekar, Pérez, Salgado-Rebolledo)

$$S_{\text{Carr.grav.}} \propto \int \mathrm{d}^D x\, e\, \left[e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(J) + 2\tau^\mu e_a{}^\nu R_{\mu\nu}{}^{0a}(G)\right] ,$$

$$\text{with } R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{[a|c|}\omega_{\nu]c}{}^{b]}$$

$$\text{and } R_{\mu\nu}{}^{0a}(G) = 2\partial_{[\mu}\omega_{\nu]}{}^{0a} + 2\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^{0}{}_{b} .$$

Equations of motion for $\omega_\mu{}^{ab}$ and $\omega_\mu{}^{0a}$ reproduce the Carrollian 1st Cartan structure equations for $T_{\mu\nu} = 0 = T_{\mu\nu}{}^a$. In particular $T_{0(a,b)} = 0$.

- Going to the second order formulation, one finds that the independent spin connection components are Lagrange multipliers for the $T_{0(a,b)} = 0$ intrinsic torsion constraint.

Coupling fermions to Carroll geometry

- We can then consider the limit of the Einstein-Hilbert action, coupled to a Dirac/tachyonic Dirac fermion, in the first order formulation:

$$S = S_{\text{EH}} + S_{\text{ferm}} \,, \qquad \text{with}$$

$$E^{-1}\mathcal{L}_{\text{ferm}} = \bar{\Psi} E_A{}^\mu \Gamma^A \left(\partial_\mu \Psi + \frac{1}{4}\Omega_\mu{}^{BC}\Gamma_{BC}\Psi\right) - \frac{M}{\tilde{c}}\bar{\Psi}\Psi \qquad \text{or}$$

$$E^{-1}\mathcal{L}_{\text{ferm}} = \bar{\Psi} E_A{}^\mu \Gamma^A \Gamma_5 \left(\partial_\mu \Psi + \frac{1}{4}\Omega_\mu{}^{BC}\Gamma_{BC}\Psi\right) - \frac{M}{\tilde{c}}\bar{\Psi}\Psi \,.$$

- Option 1: electric fermion limit:

$$S = S_{\text{Carr.grav.}} + \int \mathrm{d}^4x\, e\, \left[\bar{\psi}\Gamma^0\tau^\mu D_\mu\psi - m\bar{\psi}\psi\right] \,, \qquad \text{with } D_\mu\psi = \partial_\mu\psi + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\psi \,.$$

- Option 2: magnetic fermion limit

$$S = S_{\text{Carr.grav.}} + \int \mathrm{d}^4x\, e\, \Big[\bar{\psi}_+\Gamma^0\Gamma_5\tau^\mu D_\mu\psi_- + \bar{\psi}_-\Gamma^0\Gamma_5\tau^\mu D_\mu\psi_+ + \bar{\psi}_+\Gamma^a\Gamma_5 e_a{}^\mu D_\mu\psi_+ - m\bar{\psi}_+\psi_+\Big] \,,$$

with $D_\mu\psi_+ = \partial_\mu\psi_+ + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\psi_+$ and $D_\mu\psi_- = \partial_\mu\psi_- + \frac{1}{2}\omega_\mu{}^{ab}\Gamma_{ab}\psi_- + \frac{1}{2}\omega_\mu{}^{0a}\Gamma_{0a}\psi_+$.

Coupling fermions to Carroll geometry

- To go to second order formulation, find Carrollian 1st Cartan structure equations, obtained by varying S with respect to $\omega_\mu{}^{ab}$ and $\omega_\mu{}^{0a}$.
- Fermion bilinears give rise to particular non-zero torsion components in Carrollian 1st Cartan structure equations.
- Only conventional constraints acquire non-zero torsion components.
  $\Rightarrow$ expressions for dependent spin connection components now contain fermion bilinears
  $\Rightarrow$ give rise to quartic fermion terms in second order action.
- No fermion bilinear torsion in intrinsic torsion equations: one still has the geometric constraint

$$T_{0(a,b)} = 2\tau^\mu e_{(a|}{}^\nu \partial_{[\mu} e_{\nu]|b)} = 0\,.$$

- Note that the fermionic part of the Lagrangian only contains $\tau^\mu\omega_\mu{}^{ab}$, $\tau^\mu\omega_\mu{}^{0a}$ that become dependent. Does not contain independent spin connection components.

Coupling fermions to Carroll geometry

- It is also possible to take the limit of the Einstein-Hilbert action, coupled to fermions directly in the second order formulation.
- Using $E_\mu{}^0 = \tilde{c}^{-1}\,\tau_\mu$, $E_\mu{}^a = e_\mu{}^a$ in

$$\Omega_C{}^{AB} = E_C{}^\mu \left[2\,E^{[A|\nu|}\partial_{[\mu}E_{\nu]}{}^{B]} - E_{\mu C}\,E^{A\nu}\,E^{B\rho}\,\partial_{[\nu}E_{\rho]}{}^C\right] ,$$

one finds

$$\Omega_0{}^{ab} = \tilde{c}\tau^\mu\omega_\mu{}^{ab}(\tau,e) + \mathcal{O}\left(\frac{1}{\tilde{c}}\right) , \qquad \Omega_c{}^{ab} = e_c{}^\mu\omega_\mu{}^{ab}(\tau,e) ,$$

$$\Omega_0{}^{0a} = \tau^\mu\omega_\mu{}^{0a}(\tau,e) , \qquad \Omega^{a,0b} = \tilde{c}T^{0(a,b)} + \frac{1}{\tilde{c}}e^{[a|\mu}\omega_\mu{}^{0|b]}(\tau,e) ,$$

where $\omega_\mu{}^{ab}(\tau,e)$, $\tau^\mu\omega_\mu{}^{0a}(\tau,e)$, $e^{[a|\mu}\omega_\mu{}^{0|b]}(\tau,e)$ are the dependent spin connection components of torsionless Carroll geometry. Note the divergence $\propto T^{0(a,b)}$ in $\Omega^{a,0b}$.

- $\tilde{c}$-expansion of 2nd order Einstein-Hilbert action

$$S \propto c^2 \int \mathrm{d}^D x\, e\, \left(T_0{}^{(a,b)}T_{0(a,b)} - T_{0a}{}^a T_{0b}{}^b\right) + \mathcal{O}(\tilde{c}^0) .$$

Leading order = $\tilde{c} \to \infty$ limit = 'electric' Carroll gravity. (Henneaux, Pilati, Teitelboim)

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- Using $E_\mu{}^0 = \tilde{c}^{-1}\,\tau_\mu$, $E_\mu{}^a = e_\mu{}^a$ in

$$\Omega_C{}^{AB} = E_C{}^\mu \left[2\,E^{[A|\nu|}\partial_{[\mu}E_{\nu]}{}^{B]} - E_{\mu C}\,E^{A\nu}\,E^{B\rho}\,\partial_{[\nu}E_{\rho]}{}^C\right] ,$$

one finds

$$\Omega_0{}^{ab} = \tilde{c}\tau^\mu\omega_\mu{}^{ab}(\tau,e) + \mathcal{O}\left(\frac{1}{\tilde{c}}\right) , \qquad \Omega_c{}^{ab} = e_c{}^\mu\omega_\mu{}^{ab}(\tau,e) ,$$

$$\Omega_0{}^{0a} = \tau^\mu\omega_\mu{}^{0a}(\tau,e) , \qquad \Omega^{a,0b} = \tilde{c}T^{0(a,b)} + \frac{1}{\tilde{c}}e^{[a|\mu}\omega_\mu{}^{0|b]}(\tau,e) ,$$

where $\omega_\mu{}^{ab}(\tau,e)$, $\tau^\mu\omega_\mu{}^{0a}(\tau,e)$, $e^{[a|\mu}\omega_\mu{}^{0|b]}(\tau,e)$ are the dependent spin connection components of torsionless Carroll geometry. Note the divergence $\propto T^{0(a,b)}$ in $\Omega^{a,0b}$.

- $\tilde{c}$-expansion of 2nd order Einstein-Hilbert action

$$S \propto c^2 \int \mathrm{d}^D x\, e \left(T_0{}^{(a,b)}T_{0(a,b)} - T_{0a}{}^a T_{0b}{}^b\right) + \mathcal{O}(\tilde{c}^0) .$$

Leading order = $\tilde{c} \to \infty$ limit = ‘electric’ Carroll gravity. (Henneaux, Pilati, Teitelboim)

Coupling fermions to Carroll geometry

- Alternatively, first replace the leading order $\tilde{c}^2$ term by the classically equivalent:

$$\int \mathrm{d}^D x\, e \left(\lambda^{(ab)} T_{0(a,b)} - \lambda T_{0a}{}^{a} - \frac{1}{4\tilde{c}^2}\lambda^{(ab)}\lambda_{(ab)} + \frac{1}{4\tilde{c}^2}\lambda^2\right) .$$

  and then take the $\tilde{c} \to \infty$ limit. Gives 2nd order formulation of magnetic Carroll gravity.
- There are no extra $\tilde{c}^2$ divergences when adding fermions.
- For electric fermion:

$$S = S_{\text{magn. gravity}} + \int \mathrm{d}^4 x\, e \left[\bar{\psi}\Gamma^0 \tau^\mu \left(\partial_\mu \psi + \frac{1}{4}\omega_\mu{}^{ab}(\tau, e)\Gamma_{ab}\psi\right) - \frac{1}{2} T_{0a}{}^{a}\bar{\psi}\Gamma_0\psi\right] .$$

- For magnetic fermion:

$$S = S_{\text{magn. gravity}} + \int \mathrm{d}^4 x\, e \Big[\bar{\psi}_+\Gamma^0\Gamma_5\tau^\mu D_\mu\psi_- + \bar{\psi}_-\Gamma^0\Gamma_5\tau^\mu D_\mu\psi_+ + \bar{\psi}_+\Gamma^a\Gamma_5 D_\mu\psi_+ \\ - \frac{1}{2} T_{0a}{}^{a}\bar{\psi}_+\Gamma_0\Gamma_5\psi_- - \frac{1}{2} T_{0a}{}^{a}\bar{\psi}_-\Gamma_0\Gamma_5\psi_+\Big] .$$

  with

$$D_\mu\psi_+ = \partial_\mu\psi_+ + \frac{1}{4}\omega_\mu{}^{ab}(\tau, e)\Gamma_{ab}\psi_+ ,$$

$$\tau^\mu D_\mu\psi_- = \tau^\mu\partial_\mu\psi_- + \frac{1}{4}\omega_0{}^{ab}(\tau, e)\Gamma_{ab}\psi_- + \frac{1}{2}\omega_0{}^{0a}(\tau, e)\Gamma_{0a}\psi_+ .$$

Supersymmetry

- Can one supersymmetrize electric and magnetic Carroll fermions? (see also recent work by Koutrolikos and Najafizadeh)

- Look at simplest cases of multiplets containing only scalars, alongside the fermions: $\mathcal{N}=1$ chiral multiplet and $\mathcal{N}=2$ hypermultiplet.

- $\mathcal{N}=1, D=4$ (off-shell) chiral multiplet:
  - Field content: $\{Z, \Psi_L, F\}$
    with $\Psi_{L/R} = P_{L/R}\Psi = \frac{1}{2}(1 \pm \Gamma_5)\Psi, \quad \Psi$ Majorana
  - Lagrangian:

$$\mathcal{L}_{\text{WZ}} = -\partial_A Z \partial^A Z^* - \bar{\Psi}\Gamma^A \partial_A P_L \Psi + FF^* + \left(\frac{M}{\tilde{c}} FZ - \frac{M}{2\tilde{c}} \bar{\Psi} P_L \Psi + \text{h.c.}\right) .$$

- $\mathcal{N}=2, D=4$ (on-shell) hypermultiplet:
  - Field content: $\{Z^i, \Psi\}$, $i=1,2$,
    with Ψ Dirac and $Z_i = (Z^i)^*$.
  - Lagrangian:

$$\mathcal{L}_{\text{hyp}} = -\frac{1}{2}\partial_A Z^i \partial^A Z_i - \bar{\Psi}\partial\!\!\!/ \Psi - 2\frac{M}{\tilde{c}}\bar{\Psi}\Psi - 2\frac{M^2}{\tilde{c}^2} Z^i Z_i ,$$

Supersymmetry

- The electric limit can be taken straightforwardly. E.g. for the $\mathcal{N} = 1$ chiral multiplet, one uses the rescalings:

$$X^0 = \frac{t}{\tilde{c}}\,, \qquad X^a = x^a\,, \qquad Z = \frac{z}{\tilde{c}}\,, \qquad \Psi = \frac{\psi}{\sqrt{\tilde{c}}}\,, \qquad \epsilon = \frac{\varepsilon}{\sqrt{\tilde{c}}}\,, \qquad F = f\,.$$

  This gives (with $M = m\tilde{c}^2$):

$$\mathcal{L}_{\text{Carroll WZ}} = \dot{z}\dot{z}^* - \bar{\psi}\Gamma^0\dot{\psi}_L + ff^* + \left(mfz - \frac{m}{2}\bar{\psi}\psi_L + \text{h.c.}\right)\,,$$

  which is invariant under the following supersymmetry transformation rules:

$$\delta z = \frac{1}{\sqrt{2}}\bar{\varepsilon}\psi_L\,, \qquad \delta\psi_L = \frac{1}{\sqrt{2}}\left(\Gamma^0\dot{z}\,\varepsilon_R + f\varepsilon_L\right)\,, \qquad \delta f = \frac{1}{\sqrt{2}}\bar{\varepsilon}\Gamma^0\dot{\psi}_L\,.$$

- The superalgebra of this 'electric Carroll chiral multiplet' then closes off-shell:

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] = \frac{1}{2}\left(\bar{\varepsilon}_2\Gamma^0\varepsilon_1\right)\frac{\partial}{\partial t}\,.$$

- The scalar sector, with kinetic term $\dot{z}\dot{z}^*$ gives an 'electric' Carroll scalar. (e.g., de Boer, Hartong, Obers, Sybesma, Vandoren)

Supersymmetry

- The magnetic limit is trickier. Requires different rescalings of $\Psi_{\pm} = P_{\pm}\Psi$ projections, which can not be done sensibly for Ψ Majorana $\quad\Rightarrow\quad$ look at $\mathcal{N} = 2$ hypermultiplet.
- Need for tachyonic version of the fermion part, including Γ_5 in the kinetic term of Ψ.
- To get magnetic scalar part, write the scalar kinetic terms in first order form:

$$-G_{0i}\partial_0 Z^i + G^a{}_i\partial_a Z^i - G_0{}^i\partial_0 Z_i + G_a{}^i\partial^a Z_i - G_{0i}G_0{}^i + G^a{}_i G_a{}^i\,.$$

and rescale

$$G_{0i} = \frac{1}{\tilde{c}}g_{0i}\,,\qquad G_0{}^i = \frac{1}{\tilde{c}}g_0{}^i\,,\qquad G^a{}_i = g^a{}_i\,,\qquad G_a{}^i = g_a{}^i\,.$$

- Limit of Lagrangian = sum of magnetic fermion Lagrangian and

$$-g_{0i}\dot{Z}^i - g_0{}^i\dot{Z}_i - \partial^a Z_i\partial_a Z^i + 2m^2 Z^i Z_i\,.$$

- Only properly supersymmetric when there are no ‘divergences’ in the supersymmetry transformation rules!
- Redefining transformation rules with ‘zilch’ symmetries

$$\delta\phi^\alpha = \Omega^{\alpha\beta}\frac{\delta S}{\delta\phi^\beta}\,,$$

can however remove potential divergences.

Conclusions

- Defined two types of Carrollian limits of a Dirac/tachyonic Dirac fermion: electric and magnetic.
- Cartan formulation of Carrollian geometry, needed for coupling to non-trivial backgrounds and gravity, is available.
- Supersymmetrization of the electric fermion is straightforward. Supersymmetrizing the magnetic one is a bit more tricky.
- To do:
  - connect the fermion limits to classification of unitary irreducible representations (Figueroa-O'Farrill, Perez, Prohazka)
  - study Carrollian superalgebras and their representations in more detail
  - generic Carrollian supermultiplets?
  - Carrollian supergravity?
  - application to (some of) the topics discussed at this workshop?