Definition of spin charge in asymptotically-flat spacetimes

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Based on work (to appear soon) in collaboration with

L. Freidel and F. Moosavian

Marie Skłodowska-Curie Actions

By the pioneering work of Bargmann and Wigner, elementary particles are defined to be the irreducible unitary representations of the isometry group of a spacetime. [Bargmann, Wigner 1948]

- Casimir elements of a Lie algebra provide a convenient way to distinguish irreducible representations of a Lie algebra $\mathfrak{g}$.

The corresponding physical quantities are given by the value of Casimir elements in a representation $\mathcal{R}$ of $\mathfrak{g}$

The Lie algebras we are interested in are of the form of a semi-direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$$

$\mathfrak{k} =$ Quotient $\qquad$ $\mathfrak{n} =$ Abelian normal ideal

$\oplus$ means that $\mathfrak{k}$ acts on $\mathfrak{n}$ by the Lie bracket

For $p \in \mathfrak{n}^*$, the isotropy algebra is defined as $\mathbf{iso}(p) := \left\{u \in \mathfrak{k} \,\middle|\, \mathrm{ad}^*_u\, p = 0\right\}$

This subalgebra has an important role in the study of representation theory of $\mathfrak{g}$

Ex. : Poincaré, BMS (and its extension), conformal Carrollian symmetries

- Excluding gravity

We can study physical theories in a fixed spacetime M:

We can define elementary particles with respect to $\mathrm{g}[M]$, the isometry group of the spacetime.

Casimir elements of $\mathrm{g}[M]$ can be used to define the conserved quantities of the theory.

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- Including gravity

In quantum gravity, there is no fixed spacetime and the study its isometry group à la Bargmann–Wigner to define elementary particles as its irreducible unitary representations do not make much sense.

▸ However, one can fix the asymptotic structure $\mathcal{S}_\infty$ and define the asymptotic symmetry group as those elements of the bulk diffeomorphisms that preserve $\mathcal{S}_\infty$:
Isometry group replaced by the asymptotic symmetry group that act on the gravitational phase space.

The view of defining fundamental excitations in gravity as irreducible unitary representations of an asymptotic symmetry group has been advocated by [McCarthy 1973]

▸ Conserved quantities in the presence of gravity can be defined with respect to relevant asymptotic symmetries, via their Casimir elements (using the notion of moment-map):
Generalization of the familiar example of the Poincaré algebra.

Outline

1. Poincaré algebra
    - Pauli–Lubanski pseudo-vector
2. Issues with BMS algebra
    - Absence of spin generator
3. GBMS algebra
    - Coadjoint orbits and Casimir functionals
4. Spin generator
    - Isotropy algebra and its finite dimensional subalgebra
5. GBMS moment map
    - Gravitational Casimirs

1. Poincaré algebra

The isometry group of the 4D Minkowski spacetime is the Poincaré algebra

$$\mathsf{iso}(3,1) = \mathsf{so}(3,1) \bigoplus \mathbb{R}^4$$

$$[\boldsymbol{P}_\mu, \boldsymbol{P}_\nu] = 0, \qquad [\boldsymbol{J}_{\mu\nu}, \boldsymbol{P}_\rho] = \mathfrak{i}(\eta_{\mu\rho}\boldsymbol{P}_\nu - \eta_{\nu\rho}\boldsymbol{P}_\mu),$$
$$[\boldsymbol{J}_{\mu\nu}, \boldsymbol{J}_{\rho\sigma}] = \mathfrak{i}(\eta_{\mu\rho}\boldsymbol{J}_{\nu\sigma} + \eta_{\nu\sigma}\boldsymbol{J}_{\mu\rho} - \eta_{\mu\sigma}\boldsymbol{J}_{\nu\rho} - \eta_{\nu\rho}\boldsymbol{J}_{\mu\sigma}),$$

$\boldsymbol{P}_\mu =$ Translations generators

$\boldsymbol{J}_{\mu\nu} =$ Angular momentum generating Lorentz transformations

Casimir elements:

$$\widehat{\mathcal{C}}_2(\mathsf{iso}(3,1)) = -\boldsymbol{P}_\mu \boldsymbol{P}^\mu\,, \qquad \widehat{\mathcal{C}}_4(\mathsf{iso}(3,1)) = \boldsymbol{W}_\mu \boldsymbol{W}^\mu$$

- Pauli—Lubanski pseudo-vector $\boldsymbol{W}_\mu := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\boldsymbol{P}^\nu \boldsymbol{J}^{\rho\sigma}$

1. Invariant under translation and it transforms as a vector under Lorentz transformations

$$[\boldsymbol{P}_\mu, \boldsymbol{W}_\nu] = 0, \qquad [\boldsymbol{J}_{\mu\nu}, \boldsymbol{W}_\rho] = \mathrm{i}(\eta_{\nu\rho}\boldsymbol{W}_\mu - \eta_{\mu\rho}\boldsymbol{W}_\nu)$$

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2. Is orthogonal to the 4-momentum operator $(\boldsymbol{P}_\mu \boldsymbol{W}^\mu = 0)$. Moreover, if we diagonalize $\boldsymbol{P}_\mu$ and evaluate $\boldsymbol{W}_\mu$ when $\boldsymbol{P}_\mu = k_\mu$ we find that it satisfies the isotropy subalgebra so(3) of Lorentz transformation preserving the vector k_μ :

$\boldsymbol{W}^2$ as one of the Casimir elements of the Poincaré algebra provides an unambiguous definition of the spin of a particle in any special-relativistic system

2. Issues with BMS algebra

Abelian ideal of supertranslations

$$\text{BMS} = \mathrm{SO}(3,1)^{\uparrow} \ltimes \mathbb{R}^{S}_{-1}$$

proper orthochronous Lorentz subgroup

- Supertranslation ambiguity [Sachs, 1962]; [Geroch, 1977]:

Given a cut *C* located at $u = T_C(\sigma)$ and a CKV $Y^A \partial_A \in TS$

canonical Lorentz generators are represented by

$$\xi^C_Y := Y^A\partial_A + Y^A\partial_A T_C \partial_u + \frac{1}{2} D_A Y^A (u - T_C)\partial_u$$ tangent to *C*

2 different cuts lead to two different notion of Lorentz generator related to each other by a supertranslation with parameter $\Delta T = T_C - T_{C'}$.

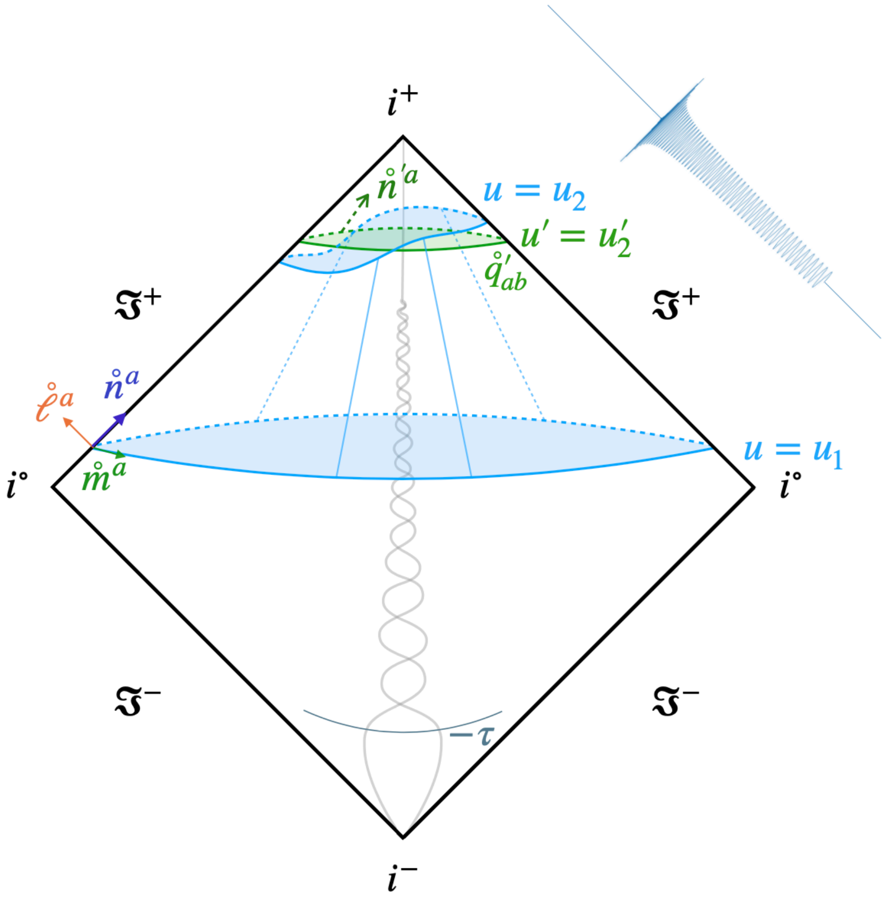

Figure credit:
[Ashtekar, De Lorenzo, Khera, gr-qc/1910.02907]

bms does not have a canonical Lorentz subalgebra in the presence of radiation

It appears that enlarging Poincaré to BMS creates a problem…

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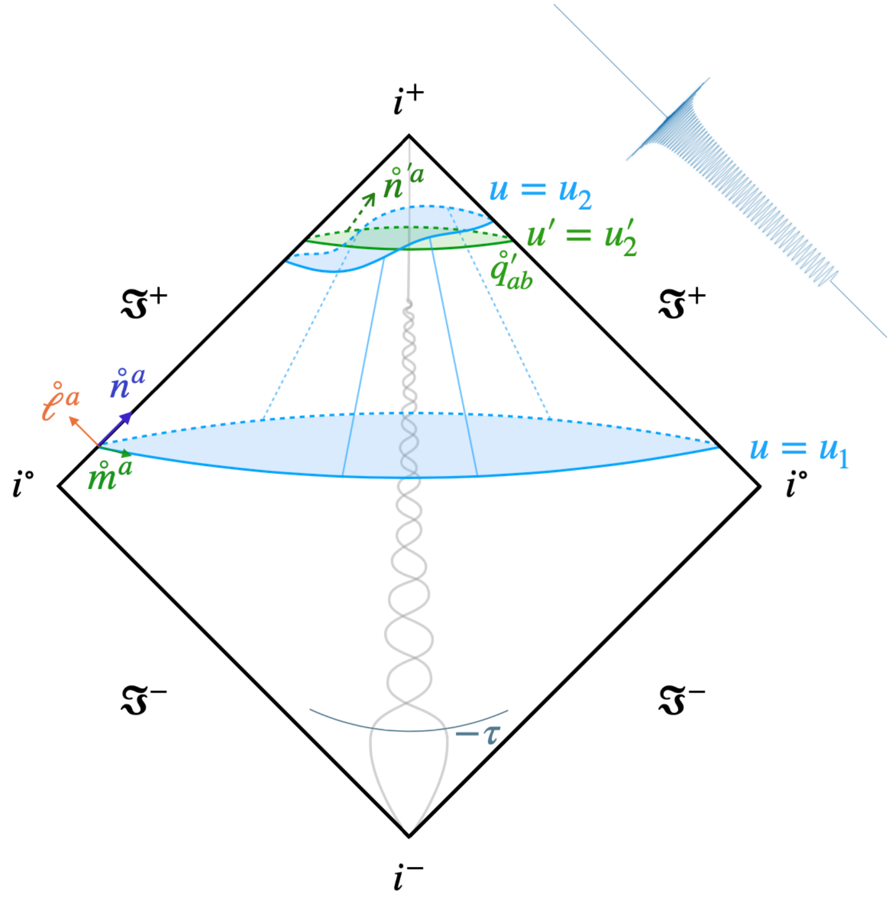

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It appears that enlarging Poincaré to BMS creates a problem…

- Undefined angular momentum aspect: The knowledge of the BMS charges $J_{Y_{\mathrm{KV}}}$, with Y_{KV} CKV of $(S, \mathring{q}_{AB})$,

only determines an equivalence class $[j_A]$, $j'_A = j_A + D^B\tau_{BA}$ with τ_{BA} symmetric traceless tensor

- Absence of spin generator

Isotropy group	Fixed Points
$\mathbb{Z}_2$	$m(z,\bar{z})$
Γ	$m(\lvert z\rvert)$
$\mathrm{SU}(2)$	m_0

Possible little groups of the BMS group ($\Gamma =$ double covering group of so(2)) [McCarthy 1973]

⚠ A generic supertranslation would have the group $\mathbb{Z}_2$ as its little group and hence we cannot associate any Casimir element to it

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➥ No natural way to define the Pauli–Lubanski and spin generators from $\mathfrak{bms}$

It turns out that BMS is actually too small!

If the mass aspect is in the orbit of the constant mass aspect, there is an SU(2) isotropy group: recently used in [Compère, Gralla, Wei 2023] to define the spin in scattering problems

3. GBMS algebra (to the rescue)

GBMS has a desired feature that makes it suitable for the definition of PL and spin generators, i.e. it has a suitable isotropy algebra with an infinite set of Casimir functionals

Method of coadjoint orbits = Classical analog of the representation theory [Kirillov 1976]

1. There exists a moment-map $\mu_{\mathrm{gbms}} : \mathbf{\Gamma} \to \mathfrak{gbms}^*$ for $\mathfrak{gbms}$-action on gravitational phase space.

2. This allows us to purely work at the algebraic level i.e. with (the dual of) $\mathfrak{gbms}$, instead of using phase space generators, to construct (classical) conserved quantities $\mathcal{O} \in C^\infty(\mathfrak{gbms}^*)$ for $\mathfrak{gbms}$-action

3. Pull these Casimir functionals back to the phase space through the moment-map

Given the spacetime vector field on $\mathcal{I}^+$

$$\xi_{(Y,T)} = T(\sigma)\partial_u + Y^A(\sigma)\partial_A + W_Y(\sigma)(u\partial_u - r\partial_r)\,, \qquad W_Y := \frac{1}{2}D_A Y^A$$

gbms Lie bracket $[\xi_{(Y_1,T_1)}, \xi_{(Y_2,T_2)}]_{\text{gbms}} = \xi_{(Y_{12},T_{12})}$

[Campiglia, Laddha 2014];
[Compere, Fiorucci, Ruzziconi 2018]

with

$$Y_{12} = [Y_1, Y_2]_S,$$
$$T_{12} = (Y_1[T_2] - T_2 W_{Y_1}) - (Y_2[T_1] - T_1 W_{Y_2})$$

$$\rightarrow \quad \text{gbms} = \text{diff}(S) \oplus \mathbb{R}^S_{-1}$$

$\mathbb{R}^S_\Delta$ = space of functions on the celestial sphere of conformal weight Δ

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$\mathbb{R}^S_\Delta =$ space of functions on the celestial sphere of conformal weight Δ

$V_{(\Delta,s)} =$ Space of spin-s tensor fields of weight Δ (symmetric and traceless contravariant tensor $\tau_{A_1\cdots A_s}$)

$\oplus$: $\quad \delta_Y \mathcal{O}_{(\Delta,s)} = \mathcal{L}_Y[\mathcal{O}_{(\Delta,s)}] + (\Delta - s)W_Y \mathcal{O}_{(\Delta,s)}, \qquad \forall\ \mathcal{O}_{(\Delta,s)} \in V_{(\Delta,s)} \quad \rightarrow \quad T \in V_{(-1,0)}\,,\ Y \in V_{(-1,-1)}$

[Freidel, DP, Raclariu 2021]

Choice of an area form $\boldsymbol{\epsilon} := \sqrt{q}\mathrm{d}^2\sigma\,,$ with $\int_S \boldsymbol{\epsilon} = 1$ preserved by gbms : $\delta_{(Y,T)}\sqrt{q} = 0 = \delta_{(Y,T)}\boldsymbol{\epsilon}$

- gbms coadjoint action [★]

$m =$ Mass aspect (scalar dual to T)

$(m, j) \in$ gbms*

$j = j_A \mathrm{d}\sigma^A =$ Angular momentum aspect (one form dual to $Y = Y^A \partial_A$)

Canonical charges: $M_T := \int_S T m \, \boldsymbol{\epsilon}, \qquad J_Y = \int_S Y^A j_A \, \boldsymbol{\epsilon}$

Canonical pairing $\langle \cdot | \cdot \rangle$: gbms × gbms* $\to \mathbb{R}$ $\qquad \langle j, m | Y, T \rangle = M_T + J_Y$

[★] See [Barnich, Ruzziconi 2021] for coadjoint orbit study of EBMS group and [Ciambelli, Leigh 2022] for the corner symmetry group

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infinitesimal coadjoint action of $(Y, T) \in$ gbms on $(j, m) \in$ gbms* :

$$\langle \delta_{(Y_1, T_1)}(j, m) | Y_2, T_2 \rangle = -\langle j, m | Y_{12}, T_{12} \rangle$$

➥
$$\delta_{(Y,T)} m = Y^A \partial_A m + 3 W_Y m,$$
$$\delta_{(Y,T)} j_A = \mathcal{L}_Y j_A + 2 W_Y j_A + \frac{3}{2} m \partial_A T + \frac{T}{2} \partial_A m$$

$\to$

$$m \in V_{(3,0)}\,, \qquad j_A \in V_{(3,1)}$$
$$(Tm + Y^A j_A) \in V_{(2,0)}$$

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- Casimir functionals

Symmetry algebra of 2-dimensional barotropic fluids [Marsden, Ratiu and Weinstein, 1984], [Arnold, Khesin, 1999]

$\mathsf{h} := \mathsf{diff}(S) \bigoplus \mathbb{R}_0^S$ variables parametrizing h^* : Density $\rho \in V_{(2,0)}$ Momentum $p_A \in V_{(1,1)}$

Casimirs for this algebra constructed in [Donnelly, Freidel, Moosavian, Speranza, 2021]:

Enstrophies = moments of the vorticity $\boldsymbol{w}_{\mathrm{Fluid}} := \mathrm{d}p$

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Enstrophies = moments of the vorticity $\boldsymbol{w}_{\text{Fluid}} := \mathrm{d}p$

$$\boxed{\rho := m^{\frac{2}{3}} \in V_{(2,0)}, \qquad p_A := \rho^{-1} j_A \in V_{(1,1)}}$$

$$\delta_T \rho = 0\,, \qquad \delta_Y \rho = D_A(\rho Y^A) \qquad \text{and} \qquad \delta_T p_A = \frac{3}{2}\partial_A(\sqrt{\rho}T), \qquad \delta_Y p_A = \mathcal{L}_Y p_A$$

Vorticity $w := \rho^{-1}\epsilon^{AB}\partial_A p_B\,,$ where $\boldsymbol{\epsilon} = \frac{1}{2}\epsilon_{AB}\mathrm{d}\sigma^A \wedge \mathrm{d}\sigma^B$ and $\delta_T w = 0\,, \quad \delta_Y w = Y[w]$

Casimir functionals $\boxed{\mathcal{C}_n(\mathsf{gbms}) := \int_S w^n \rho \boldsymbol{\epsilon}}$ s.t. $\delta_{(Y,T)}\mathcal{C}_n(\mathsf{gbms}) = \int_S D_A(w^n \rho Y^A)\boldsymbol{\epsilon} = 0$

4. Spin generator

- Isotropy subalgebra

To determine the isotropy algebra of $\mathfrak{gbms}$, we choose $m \in (\mathbb{R}_{-1})^*$

and study those $\mathfrak{gbms}$ transformations that preserves it:

$$\delta_{(Y,T)} m = Y^A \partial_A m + 3 W_Y m = \frac{3m^{\frac{1}{3}}}{2} \operatorname{div}_{\boldsymbol{\rho}}(Y) = 0\,,$$

rescaled measure $\boldsymbol{\rho} := \rho \boldsymbol{\epsilon} = m^{\frac{2}{3}} \boldsymbol{\epsilon}$ and $\operatorname{div}_{\boldsymbol{\rho}}(Y) := \frac{1}{\boldsymbol{\rho}} \partial_A \left(\boldsymbol{\rho} Y^A\right)$

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$\rightarrow$ The isotropy subalgebra of gbms is the one that preserves $\boldsymbol{\rho}$:

$$\mathbf{iso}(\mathsf{gbms}) = \mathsf{sdiff}_{\boldsymbol{\rho}}(S) \bigoplus \mathbb{R}^S_{-1}$$

generated by vector fields $Y_\phi := -\rho^{-1} \epsilon^{AB} \partial_A \phi \partial_B\,,$ $\phi \in C^\infty(S)$

with Lie bracket $[Y_\phi, Y_\psi]_S = Y_{\{\phi,\psi\}_\rho}\,,$ $\{\phi, \psi\}_\rho := \rho^{-1} \epsilon^{AB} \partial_A \phi \partial_B \psi$

↗ Poisson bracket on S equipped with a rescaled symplectic structure

Consider a smeared version of the vorticity

$$\boxed{w[\phi] := \int_S \phi\rho w \boldsymbol{\epsilon} = \int_S \phi\, \epsilon^{AB} \partial_A p_B\, \boldsymbol{\epsilon}, \qquad \phi \in C^\infty(S)}$$

transformation properties under gbms : $\quad \delta_T w[\phi] = 0, \qquad \delta_Y w[\phi] = -w[Y[\phi]]$

$$\rightarrow \quad \big\{w[\phi], w[\psi]\big\}_{\mathfrak{g}^*} = -w[\{\phi, \psi\}_\rho]$$

$w[\phi]$ implements the action of $\mathsf{sdiff}_{\boldsymbol{\rho}}(S)$ on gbms^*

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$w[\phi]$ implements the action of sdiff$_{\boldsymbol{\rho}}(S)$ on gbms*

1. $w[\phi]$ is invariant under supertranslations
2. $w[\phi]$ generates the isotropy algebra of gbms

➥ $w[\phi] =$ Spin generator

- Finite dimensional subalgebra

Given an orientation preserving diffeomorphism of the sphere $F : S \to S$ we can construct a density

s.t. $F^* \boldsymbol{\epsilon} = \rho_F \boldsymbol{\epsilon}$ and $\rho_F = \frac{1}{2}\epsilon^{AB}\partial_A F^C \partial_B F^D (\epsilon_{CD} \circ F)$

and the mass aspect, the curvature tensor and the angular momentum aspect transform as

$$m \to m^F := \rho_F^{\frac{3}{2}} m \circ F,$$

$$R \to R^F := \rho_F R \circ F + \rho_F \Delta \ln \rho_F$$

$$j_A \to j_A^F := \rho_F \, (j_B \circ F) \partial_A F^B$$

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Notions of preferred frames:

Bondi frame $\partial_A R^{\mathrm{B}} = 0, \qquad \partial_A m^{\mathrm{B}} \neq 0$

Center-of-mass frame [Flanagan, Nichols 2015] $\partial_A m^{\mathrm{CM}} = 0\,, \qquad \partial_A R^{\mathrm{CM}} \neq 0\,, \qquad j_A^{\mathrm{CM}} = j_0 \epsilon_A{}^B \partial_B n_3$

Therefore, two frames are related by a diffeomorphism $F : S^{\mathrm{CM}} \to S^{\mathrm{B}}$ under which

$$m^{\mathrm{B}} = M^{\mathrm{CM}} \rho_{F^{-1}}^{\frac{3}{2}}\,, \qquad R^{\mathrm{CM}} = \rho_F (R^{\mathrm{B}} + \Delta \ln \rho_F)$$

Let us consider a boost operator that maps the rest frame vector $t_\mu = (1,0,0,0)$ onto a unit vector in the hyperboloid of velocity $v \in \mathbb{R}^3 : P_{v\,\mu} = \gamma_v(1, v_i)$:

The boost yields a transformation given by

$$F^i_v(n) = n'^i = \alpha_v^{-1}\left(n^i + \left[\frac{\gamma_v v^j n_j}{\gamma_v + 1} - 1\right]\gamma_v v^i\right)$$

with $n_i = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and $\rho_{F_v} = \dfrac{1}{\gamma_v^2(1 - v^i n_i)^2}$

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with $n_i = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and $\rho_{F_v} = \dfrac{1}{\gamma_v^2(1 - v^i n_i)^2}$

In particular, this means that the mass aspect of a particle with velocity v is given by

$$m_v(n) = \frac{m^{\mathsf{CM}}}{[\gamma_v(1 - v^i n_i)]^3}$$

expression for the Bondi mass aspect in any general boosted frame [Bondi, van der Burg, Metzner, Sachs 1962],

define the total mass (Casimir of $\mathfrak{gbms}$) $M := \left(\int_S m^{\frac{2}{3}}\right)^{\frac{3}{2}}$ → mass aspect in the center-of-mass frame $\rho^{\mathsf{CM}} = M^{\frac{2}{3}}$

Poincaré embedding:

[Barnicha, Troessaert 2011];
[Flanagan, Nichols 2015];
[Compère, Oliveri, Seraj 2019]

Four-momentum $P_\mu := (E, P_i)$ with $P_0 := \int_S m\boldsymbol{\epsilon}\,, \quad P_i := \int_S n_i\, m\boldsymbol{\epsilon}$

rotation and boost generators $J_i := \int_S Y^A_{J_i}\, j_A\,, \quad K_i := \int_S Y^A_{K_i}\, j_A\,,$

with $Y^A_{J_i} := \epsilon^{AB}\partial_B n_i\,, \quad Y^A_{K_i} := q^{AB}\partial_B n_i$ satisfying $D_A Y_B + D_B Y_A = \frac{1}{2} D_C Y^C q_{AB}$

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$$\{P_\mu, P_\nu\}_{\mathfrak{g}^*} = 0\,,$$
$$\{P_\mu, J_{\nu\rho}\}_{\mathfrak{g}^*} = \eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\,,$$

$$\{J_i, J_j\}_{\mathfrak{g}^*} = -\varepsilon_{ij}{}^k J_k,$$
$$\{J_i, K_j\}_{\mathfrak{g}^*} = -\varepsilon_{ij}{}^k K_k,$$
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Pauli–Lubanski generator $\boxed{W_{v\mu} := \rho^{\mathsf{CM}\frac{1}{2}} w[\rho_v^{\frac{1}{2}} n_\mu]}$ with $\rho_v = \dfrac{\rho^{\mathsf{CM}}}{\gamma_v^2(1 - v^i n_i)^2}$

- Supertranslation invariant $\{P_\mu, W_{v\nu}\}_{\mathfrak{g}^*} = 0$
- Covariant under Lorentz transformations $\{J_{\mu\nu}, W_{v\rho}\}_{\mathfrak{g}^*} = (\eta_{\nu\rho}W_{v\mu} - \eta_{\mu\rho}W_{v\nu})$
- Satisfying the algebra $\{W_{v\mu}, W_{v\nu}\}_{\mathfrak{g}^*} = \varepsilon_{\mu\nu\rho\sigma}P^\rho_v W^\sigma_v$

5. GBMS moment map

To connect these GBMS algebraic results with the gravitational physics, one has to identify

(1) the appropriate phase space on which $\mathfrak{gbms}$ acts by a Hamiltonian action

(2) the momentum map for this Hamiltonian action

Asymptotic expansions of Well scalars around future null infinity:

$$\Psi_0 = \sum_{s=0}^{\infty} \frac{\Psi_0^{(s)}}{r^{5+s}}\,, \quad \Psi_1 = \frac{\Psi_1^0}{r^4} + \mathcal{O}(r^{-5})\,, \quad \Psi_2 = \frac{\Psi_2^0}{r^3} + \mathcal{O}(r^{-4})\,, \quad \Psi_3 = \frac{\Psi_3^0}{r^2} + \mathcal{O}(r^{-3})\,, \quad \Psi_4 = \frac{\Psi_4^0}{r} + \mathcal{O}(r^{-2})$$

↑ Incoming radiation

↑ Outgoing radiation at $\mathcal{I}^+$

$$\Psi_4^0 = \mathcal{N}\,, \quad \Psi_3^0 = \mathcal{I}\,, \quad \Psi_2^0 = \mathcal{M}_{\mathbb{C}}\,, \quad \Psi_1^0 = \mathcal{J}\,, \quad \Psi_0^{(0)} = \mathcal{T}$$ where

$$\mathcal{N}^{AB} := \ddot{C}_{AB}\,, \quad \mathcal{I}^A := \frac{1}{2} D_B N^{AB} + \frac{1}{4}\partial^A R\,, \quad \mathcal{M}_{\mathbb{C}} = \mathcal{M} + i\tilde{\mathcal{M}}\,, \quad \mathcal{M} := M + \frac{1}{8} N^{AB} C_{AB}$$

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$$\Psi_4^0 = \mathcal{N}\,, \quad \Psi_3^0 = \mathcal{I}\,, \quad \Psi_2^0 = \mathcal{M}_{\mathbb{C}}\,, \quad \Psi_1^0 = \mathcal{J}\,, \quad \Psi_0^{(0)} = \mathcal{T} \qquad \text{where}$$

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Electric and strongly non-radiative phase space $\mathbf{\Gamma}_{\mathrm{ESNR}}$:

$$\mathcal{N}^{AB} \underset{\downarrow}{=} 0\,, \qquad \mathcal{I}^A = 0\,, \qquad \tilde{\mathcal{M}} = 0$$

$$N_{AB} = n_{AB}\,, \qquad C_{AB} = u n_{AB} + c_{AB}$$

Conserved charges under the time evolution in $\mathbf{\Gamma}_{\mathrm{ESNR}}$:

$$Q_{(\tau,\upsilon,\zeta)} := \int_S \left(\tau\mathcal{M} + \frac{1}{2}\upsilon^A\mathcal{J}_A + \frac{1}{3}\zeta^{AB}\mathcal{T}_{AB}\right)\boldsymbol{\epsilon}$$

demanding $\partial_u Q_{(\tau,\upsilon,\zeta)} = 0$

$$\rightarrow \quad \begin{aligned} \zeta^{AB}(u) &= Z^{AB}\,, \\ \upsilon^A(u) &= Y^A + \frac{2u}{3}D_B Z^{AB}\,, \\ \tau(u) &= T + \frac{u}{2}(D_A Y^A - Z^{AB}c_{AB}) + \frac{u^2}{6}D_A D_B Z^{AB} - \frac{1}{4}u^2 Z^{AB}n_{AB} \end{aligned}$$

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$$Q_{(\tau,\upsilon,\zeta)} := \int_S \left(\tau\mathcal{M} + \frac{1}{2}\upsilon^A\mathcal{J}_A + \frac{1}{3}\zeta^{AB}\mathcal{T}_{AB}\right)\boldsymbol{\epsilon}$$ demanding $$\partial_u Q_{(\tau,\upsilon,\zeta)} = 0$$

$$\rightarrow \qquad \begin{aligned} \zeta^{AB}(u) &= Z^{AB}\,, \\ \upsilon^A(u) &= Y^A + \frac{2u}{3}D_B Z^{AB}\,, \\ \tau(u) &= T + \frac{u}{2}(D_AY^A - Z^{AB}c_{AB}) + \frac{u^2}{6}D_AD_BZ^{AB} - \frac{1}{4}u^2Z^{AB}n_{AB} \end{aligned}$$

$$Q_{(Y,T,Z)} = \int_S \left(T\mathsf{m} + Y^A\mathsf{j}_A + Z^{AB}\mathsf{t}_{AB}\right)\boldsymbol{\epsilon}$$ with

$$\begin{aligned} \mathsf{m} &= \mathcal{M}\,, \\ \mathsf{j}_A &= \frac{1}{2}\left[\mathcal{J}_A - uD_A\mathcal{M}\right]\,, \\ \mathsf{t}_{AB} &= \frac{1}{3}\left[\mathcal{T}_{AB} - uD_{\langle A}\mathcal{J}_{B\rangle} + \frac{u^2}{2}D_{\langle A}D_{B\rangle}\mathcal{M} - \frac{3}{2}(\partial_u^{-1}C_{AB})\mathcal{M}\right] \end{aligned}$$

[Compere, Fiorucci, Ruzziconi 2018]; [Freidel, DP 2021]; [Donnay, Ruzziconi 2021]

gbms-action on the conserved charge aspects parametrizing $\mathbf{\Gamma}_{\mathrm{ESNR}}$

$$\delta_{(Y,T)}\mathsf{m} = [\mathcal{L}_Y + 3W_Y]\,\mathsf{m}\,,$$
$$\delta_{(Y,T)}\mathsf{j}_A = [\mathcal{L}_Y + 2W_Y]\,\mathsf{j}_A + \frac{3}{2}\mathsf{m}\partial_A T + \frac{T}{2}\partial_A \mathsf{m}\,,$$
$$\delta_{(Y,T)}\mathsf{t}_{AB} = [\mathcal{L}_Y + W_Y]\,\mathsf{t}_{AB} + \frac{8}{3}\mathsf{j}_{\langle A}\partial_{B\rangle}T + \frac{2}{3}TD_{\langle A}\mathsf{j}_{B\rangle}$$

Moment map $\boldsymbol{\mu}_{\mathsf{gbms}} : \mathbf{\Gamma}_{\mathrm{ESNR}} \to \mathsf{gbms}^*$

$$\boldsymbol{\mu}_{\mathsf{gbms}}(\mathsf{m}, \mathsf{j}_A, \mathsf{t}_{AB}) = (m, j_A)$$

$\boldsymbol{\mu} : \mathbf{\Gamma} \to \mathsf{g}^*$ s.t.

$$\{F, G\}_{\mathsf{g}^*} \circ \boldsymbol{\mu} = \{F \circ \boldsymbol{\mu}, G \circ \boldsymbol{\mu}\}_{\mathbf{\Gamma}}$$

gbms-action on the conserved charge aspects parametrizing $\boldsymbol{\Gamma}_{\rm ESNR}$

$$\begin{aligned}
\delta_{(Y,T)}\mathsf{m} &= \left[\mathcal{L}_Y + 3W_Y\right]\mathsf{m}\,,\\
\delta_{(Y,T)}\mathsf{j}_A &= \left[\mathcal{L}_Y + 2W_Y\right]\mathsf{j}_A + \frac{3}{2}\mathsf{m}\partial_A T + \frac{T}{2}\partial_A\mathsf{m}\,,\\
\delta_{(Y,T)}\mathsf{t}_{AB} &= \left[\mathcal{L}_Y + W_Y\right]\mathsf{t}_{AB} + \frac{8}{3}\mathsf{j}_{\langle A}\partial_{B\rangle}T + \frac{2}{3}TD_{\langle A}\mathsf{j}_{B\rangle}
\end{aligned}$$

Moment map $\boldsymbol{\mu}_{\mathsf{gbms}} : \boldsymbol{\Gamma}_{\rm ESNR} \to \mathsf{gbms}^*$

$$\boldsymbol{\mu}_{\mathsf{gbms}}(\mathsf{m},\mathsf{j}_A,\mathsf{t}_{AB}) = (m, j_A)$$

$$\boldsymbol{\mu} : \boldsymbol{\Gamma} \to \mathsf{g}^* \quad \text{s.t.}$$

$$\{F,G\}_{\mathsf{g}^*}\circ\boldsymbol{\mu} = \{F\circ\boldsymbol{\mu}, G\circ\boldsymbol{\mu}\}_{\boldsymbol{\Gamma}}$$

- Gravitational Casimirs

Gravitational vorticity $\mathsf{w} := \mu^*_{\mathsf{gbms}}w = \frac{1}{2}\mathcal{M}^{-\frac{2}{3}}\epsilon^{AB}\partial_A\left(\mathcal{M}^{-\frac{2}{3}}\mathcal{J}_B\right)$

$$\mathsf{C}_n(\boldsymbol{\Gamma}_{\rm ESNR}) := \mu^*_{\mathsf{gbms}}\mathcal{C}_n(\mathsf{gbms}) = \int_S \mathcal{M}^{\frac{2}{3}}\mathsf{w}^n\,\boldsymbol{\epsilon}$$

Gravitational spin generator $\mathcal{S}[\phi] := \mu^*_{\mathsf{gbms}}w[\phi] = \frac{1}{2}\int_S \phi\,\epsilon^{AB}\partial_A\left(\mathcal{M}^{-\frac{2}{3}}\mathcal{J}_B\right)\boldsymbol{\epsilon}$

	Poincaré	GBMS
Lie Group	$\mathrm{SO}(3,1)^{\uparrow} \ltimes \mathbb{R}^4$	$\mathrm{Diff}(S) \ltimes \mathbb{R}^S_{-1}$
Lie Algebra	$\mathsf{so}(3,1) \bigoplus \mathbb{R}^4$	$\mathsf{diff}(S) \bigoplus \mathbb{R}^S_{-1}$
Lie Coalgebra Elements	$(j_{\mu\nu}, p_\mu)$	(j_A, m)
Type of Orbits	Massive	Massive
Label of Orbits	$-p^2 > 0$	$m > 0$
Isotropy Subalgebra	$\mathsf{so}(3) \bigoplus \mathbb{R}^4$	$\mathsf{sdiff}_{\boldsymbol{\rho}}(S) \bigoplus \mathbb{R}^S_{-1}$
Spin Generators	w_μ	$w[\phi]$
Casimirs	$(-p^2, w^2)$	$\mathcal{C}_n = \int_S \rho w^n \boldsymbol{\epsilon} \qquad n = 0, 1, \ldots$

Table 1: The comparison between the Poincaré algebra in four dimensions $\mathsf{iso}(3,1)$ and gbms. In this table, $\boldsymbol{\rho} = \rho\boldsymbol{\epsilon} = m^{\frac{2}{3}}\boldsymbol{\epsilon}$ is a rescaling of the round-sphere area form $\boldsymbol{\epsilon}$, w_μ denotes the components of the Pauli–Lubański pseudo-vector, and $w[\phi]$ is the smeared vorticity with $\phi \in C^\infty(S)$.

Pauli–Lubanski pseudo-vector in the component form $\boldsymbol{W}_0 = \boldsymbol{J}^i \boldsymbol{P}_i, \qquad \boldsymbol{W}_i = E \boldsymbol{J}_i - (\boldsymbol{P} \times \boldsymbol{K})_i$

with $\boldsymbol{P}_\mu = (E, \boldsymbol{P}_i)\,, \quad \boldsymbol{J}_i := \varepsilon_i{}^{jk} \boldsymbol{J}_{jk}$ rotation generators and $\boldsymbol{K}_i := \boldsymbol{J}_{0i}$ boost generators

- Rest frame: $\boldsymbol{P}^{\text{rest}}_\mu = (m, \boldsymbol{0})$ pure boost transformation $\boldsymbol{\Lambda}_{\boldsymbol{P}}$ s.t. $(\boldsymbol{P}^{\text{rest}} \cdot \boldsymbol{\Lambda}_{\boldsymbol{P}})_\mu = \boldsymbol{P}_\mu$

- Intrinsic spin vector = Spatial components of the Pauli–Lubanski pseudo-vector in the rest frame

$$m \boldsymbol{S}_i := (\boldsymbol{W} \cdot \boldsymbol{\Lambda}_{\boldsymbol{P}}^{-1})_i \qquad \to \qquad \boldsymbol{S}_i = \frac{1}{m} \left(\boldsymbol{W}_i - \frac{\boldsymbol{J}_j \boldsymbol{P}^j}{m + E} \boldsymbol{P}_i \right)$$

It satisfies $[\boldsymbol{P}_\mu, \boldsymbol{S}_i] = 0\,, \qquad [\boldsymbol{S}_i, \boldsymbol{S}_j] = i \epsilon_{ij}{}^k \boldsymbol{S}_k\,, \qquad [\boldsymbol{J}_i, \boldsymbol{S}_j] = i \epsilon_{ij}{}^k \boldsymbol{S}_k$ but not preserved under boost

$\boldsymbol{S}_i$ is uniquely determined by these conditions (see e.g [Bogolubov et al., "General principles of QFT" 1990])

The intrinsic spin implements the action of the isotropy subalgebra so(3) of Poincaré and $\boldsymbol{W}^2 = m^2 \boldsymbol{S}_i \boldsymbol{S}^i$

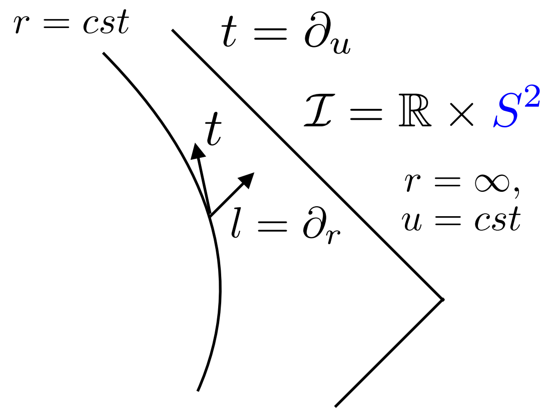
Weyl BMS group

Bondi-Sachs coordinates $x^\mu = (u, r, \sigma^A)$:

$$\mathrm{d}s^2 = -2e^{2\beta}\mathrm{d}u(\mathrm{d}r + \Phi \mathrm{d}u) + r^2\gamma_{AB}\left(\mathrm{d}\sigma^A - \frac{\Upsilon^A}{r^2}\mathrm{d}u\right)\left(\mathrm{d}\sigma^B - \frac{\Upsilon^B}{r^2}\mathrm{d}u\right)$$

The Bondi gauge conditions:

$$g_{rr} = 0\,, \quad g_{rA} = 0\,, \quad \partial_r\sqrt{\gamma} = 0 \qquad \text{(i)}$$



BMSW boundary conditions:

$$g_{ur} = -1 + \mathcal{O}(r^{-2}), \qquad g_{uA} = \mathcal{O}(1), \qquad g_{uu} = \mathcal{O}(1), \qquad \gamma_{AB} = \mathcal{O}(1) \quad \text{(ii)}$$

Original BMS boundary conditions: $g_{uu} = -1 + \mathcal{O}(r^{-1}), \quad \gamma_{AB} = \mathring{q}_{AB} + \mathcal{O}(r^{-1})$

[Bondi, van der Burg, Metzner, Sachs 1962]

Metric coefficients:

$$\Phi = \frac{R(q)}{4} - \frac{M}{r} + o\left(r^{-1}\right)\,, \quad \beta = -\frac{1}{32}\frac{C_{AB}C^{AB}}{r^2} + o\left(r^{-2}\right)\,,$$

$$\Upsilon^A = -\frac{1}{2}D_B C^{BA} - \frac{1}{r}\left(\frac{2}{3}\mathcal{P}^A - \frac{1}{2}C^{AB}D^C C_{CB} - \frac{1}{16}\partial^A\left(C_{BC}C^{BC}\right)\right) + o\left(r^{-1}\right)\,,$$

$$\gamma_{AB} = q_{AB} + \frac{1}{r}C_{AB} + \frac{1}{4r^2}q_{AB}\left(C_{CD}C^{CD}\right) + \frac{1}{r^3}\left(\frac{1}{3}\mathcal{T}_{AB} + \frac{1}{16}C_{AB}(C_{CD}C^{CD})\right) + o\left(r^{-3}\right)\,,$$

Minkowski spacetime: $M, \mathcal{P}_A, C_{AB}, \mathcal{T}_{AB} \to 0\,, R \to 2$