Einstein's equations in the covariant Newman-Unti gauge: flat from AdS

Simon Pekar

CPHT, Ecole Polytechnique

3rd Carroll Workshop, Thessaloniki — 2 October 2023

2309.15182 with A. Campoleoni, A. Delfante, M. Petropoulos, D. Rivera-Betancour and M. Vilatte

Introduction

Solution space of Einstein's equations

- AlAdS_4: boundary metric & energy-momentum tensor
- AlMink_4: shearless boundary metric & Bondi shear & *Chthonian* (deep)

as well as flux-balance laws

Dynamical equations obeyed by Chthonian dof [Godazgar^2, Pope '18] [Freidel, Pranzetti '21]

Λ-BMS gauge offers deep insights in the $\Lambda \to 0$ limit [Compère, Fiorucci, Ruzziconi '19]

Introduction

Solution space of Einstein's equations

- AlAdS_4: boundary metric & energy-momentum tensor
- AlMink_4: shearless boundary metric & Bondi shear & *Chthonian* (deep)

as well as flux-balance laws

Dynamical equations obeyed by Chthonian dof [Godazgar^2, Pope '18] [Freidel, Pranzetti '21]

Λ-BMS gauge offers deep insights in the $\Lambda \to 0$ limit [Compère, Fiorucci, Ruzziconi '19]

Plan:

1. covariant Newman-Unti gauge for $\Lambda \leq 0$
2. transverse decomposition of energy-momentum and Cotton tensors
3. solve Einstein's equations (up to order $1/r^2$)
4. *Laurent-expansion* in powers of $\Lambda = -3k^2$: Carrollian replicas
5. flux-balance laws for Chthonian degrees of freedom

Gauges for General Relativity ($D = 4$)

Fefferman-Graham

Holographic boundary ($\rho \to \infty$) $G^{(-2)}_{\mu\nu}$ and $T_{\mu\nu} := \frac{3k}{16\pi G} G^{(1)}_{\mu\nu}$ [Fefferman, Graham '85]

$$\mathrm{d}s^2_{\text{bulk}} = \frac{\mathrm{d}\rho^2}{k^2\rho^2} + \sum_{s\geq -2} \frac{1}{k^s\rho^s} G^{(s)}_{\mu\nu}(x)\,\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1}$$

invariant under boundary diffeos, but no smooth zero-k limit

Gauges for General Relativity ($D = 4$)

Fefferman-Graham

Holographic boundary ($\rho \to \infty$) $G^{(-2)}_{\mu\nu}$ and $T_{\mu\nu} := \frac{3k}{16\pi G} G^{(1)}_{\mu\nu}$ [Fefferman, Graham '85]

$$\mathrm{d}s^2_{\text{bulk}} = \frac{\mathrm{d}\rho^2}{k^2\rho^2} + \sum_{s\geq -2} \frac{1}{k^s\rho^s} G^{(s)}_{\mu\nu}(x)\, \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1}$$

invariant under boundary diffeos, but no smooth zero-k limit

Partial Bondi gauge

Given G_{ij}, U^i, V and β functions of all coordinates [Geiller, Zwikel '22]

$$\mathrm{d}s^2_{\text{bulk}} = e^{2\beta}\frac{V}{r}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\,\mathrm{d}r + G_{ij}\left(\mathrm{d}x^i - U^i\mathrm{d}u\right)\left(\mathrm{d}x^j - U^j\mathrm{d}u\right) \tag{2}$$

valid regardless of k, not invariant under boundary diffeos

Covariant Newman-Unti gauge

Newman-Unti gauge

Integral transformation of $r \longrightarrow$ affine coordinate [Newman, Unti '62]

$$ds^2_{\text{bulk}} = \frac{V}{r} du^2 - 2\, du\, dr + G_{ij}\left(dx^i - U^i du\right)\left(dx^j - U^j du\right) \tag{3}$$

with $G_{ij} = r^2 g_{ij} + \mathcal{O}(r)$, $U^i = v^i + \mathcal{O}(1/r)$ and $V = -k^2 r^3 + \mathcal{O}(r^2)$

Covariant Newman-Unti gauge

Newman-Unti gauge

Integral transformation of $r \longrightarrow$ affine coordinate [Newman, Unti '62]

$$\mathrm{d}s^2_{\text{bulk}} = \frac{V}{r}\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + G_{ij}\left(\mathrm{d}x^i - U^i\mathrm{d}u\right)\left(\mathrm{d}x^j - U^j\mathrm{d}u\right) \tag{3}$$

with $G_{ij} = r^2 g_{ij} + \mathcal{O}(r)$, $U^i = v^i + \mathcal{O}(1/r)$ and $V = -k^2 r^3 + \mathcal{O}(r^2)$

Covariantisation (boundary coordinates x^μ)

Covariantise under boundary diffs [Ciambelli, Marteau, Petkou, Petropoulos, Siampos '18]

$$\mathrm{d}u \longrightarrow -\frac{1}{k^2}\mathrm{u} = -\frac{1}{k^2}u_\mu \mathrm{d}x^\mu$$

$$-k^2\mathrm{d}u^2 + g_{ij}\left(\mathrm{d}x^i - v^i\mathrm{d}u\right)\left(\mathrm{d}x^j - v^j\mathrm{d}u\right) \longrightarrow \mathrm{d}s^2_{\text{bdy}} = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

Go to Cartan's frame $\mathrm{d}s^2_{\text{bdy}} = \eta_{AB}\,\theta^A\,\theta^B$ and $\mathrm{u} = u_A\,\theta^A$ with $u_A u^A = -k^2$

Weyl invariance

The Weyl connection

Redefinition $r \to \mathscr{B}(x) r$ gives rise to boundary Weyl invariance

$$\mathrm{d}s^2_{\text{bdy}} \longrightarrow \mathscr{B}^{-2}\,\mathrm{d}s^2_{\text{bdy}}\,, \qquad \mathrm{u} \longrightarrow \mathscr{B}^{-1}\,\mathrm{u} \tag{4}$$

once a Weyl connection A is introduced [Ciambelli, Leigh '19]

$$\mathrm{A} \longrightarrow \mathrm{A} - \mathrm{d}\ln \mathscr{B} \tag{5}$$

Non-zero asymptotic charge for FG [Ciambelli, Delfante, Ruzziconi, Zwikel '23]

Weyl-covariant Newman-Unti gauge up to order r^2

$$\mathrm{d}s^2_{\text{bulk}} = \frac{2}{k^2}\mathrm{u}\,(\mathrm{d}r + r\mathrm{A}) + r^2\,\mathrm{d}s^2_{\text{bdy}} + \mathcal{O}(r) \tag{6}$$

From A and u, build Weyl-covariant derivative s.t. $\mathscr{D}_A u^A = 0,\ u^A \mathscr{D}_A u_B = 0$

Radial expansion

Assume a radial expansion (bulk metric has Weyl weight 0)

$$
\begin{aligned}
\mathrm{d}s^2_{\text{bulk}} = \frac{2}{k^2}\mathrm{u}\,(\mathrm{d}r + r\mathrm{A}) + r^2\,\mathrm{d}s^2_{\text{bdy}} + r\,\mathscr{C}_{AB}\,\theta^A\,\theta^B + \frac{1}{k^4}\mathscr{F}_{AB}\,\theta^A\,\theta^B \\
+ \sum_{s\geq 1}\frac{1}{r^s}\left(\frac{1}{k^4}f^{(s)}\,\mathrm{u}^2 + \frac{2}{k^2}f^{(s)}_A\,\mathrm{u}\,\theta^A + f^{(s)}_{AB}\,\theta^A\,\theta^B\right)
\end{aligned}
\tag{7}
$$

for the moment, everything is arbitrary:

- the $(w = -2)$ boundary metric $\mathrm{d}s^2_{\text{bdy}}$
- the $(w = 1)$ tensor $\mathscr{C}_{AB}$ is the shear along ∂_r
- the $(w = 2)$ tensor $\mathscr{F}_{AB}$
- the $(w = s + 2)$ tensors $f^{(s)}$, $f^{(s)}_A$ and $f^{(s)}_{AB}$ with $u^A f^{(s)}_A = 0$ and $u^A f^{(s)}_{AB} = 0$

Transversality, vorticity and shear

Dualise (traceless) transverse objects using

$$\hat{\eta}_{AB} := -\epsilon_{ABC}\frac{u^C}{k} \qquad \longrightarrow \qquad *X^A := \hat{\eta}^A{}_B X^B \tag{8}$$

Introduce the vorticity two-form and scalar

$$\omega := \frac{1}{2}\left(\mathrm{du} + \frac{1}{k^2}\mathrm{u} \wedge \mathrm{A}\right) \qquad \text{and} \qquad \gamma := \frac{1}{2k^2}\hat{\eta}^{AB}\omega_{AB} \tag{9}$$

Fundamental object: the shear of the congruence u

$$\boxed{\sigma_{AB} := \nabla_{(A}u_{B)} + \frac{1}{k^2}u_{(A}a_{B)} - \frac{\Theta}{2}h_{AB}} \tag{10}$$

with $a_A = u^B\nabla_B u_A$ and $\Theta = \nabla_A u^A$

The energy-momentum and Cotton tensors

The energy-momentum tensor (*electric*)

Transverse decomposition of the energy-momentum tensor (weight 3)

$$T_{AB} := (\varepsilon + p)\frac{u_A u_B}{k^2} + p\,\eta_{AB} + \tau_{AB} + \frac{1}{k^2}u_A q_B + \frac{1}{k^2}u_B q_A \qquad (11)$$

with $u^A \tau_{AB} = 0$, $u^A q_A = 0$, $\varepsilon = 2p$ and $\tau_A{}^A = 0$

The energy-momentum and Cotton tensors

The energy-momentum tensor (*electric*)

Transverse decomposition of the energy-momentum tensor (weight 3)

$$T_{AB} := \frac{3\varepsilon}{2}\frac{u_A u_B}{k^2} + \frac{\varepsilon}{2}\eta_{AB} + \tau_{AB} + \frac{1}{k^2}u_A q_B + \frac{1}{k^2}u_B q_A \tag{11}$$

with $u^A\tau_{AB} = 0$, $u^A q_A = 0$, $\varepsilon = 2p$ and $\tau_A{}^A = 0$

The energy-momentum and Cotton tensors

The energy-momentum tensor (*electric*)

Transverse decomposition of the energy-momentum tensor (weight 3)

$$T_{AB} := \frac{3\varepsilon}{2}\frac{u_A u_B}{k^2} + \frac{\varepsilon}{2}\eta_{AB} + \tau_{AB} + \frac{1}{k^2}u_A q_B + \frac{1}{k^2}u_B q_A \qquad (11)$$

with $u^A \tau_{AB} = 0$, $u^A q_A = 0$, $\varepsilon = 2p$ and $\tau_A{}^A = 0$

The Cotton tensor (*magnetic*)

Transverse decomposition of the Cotton tensor (weight 3)

$$C_{AB} := \epsilon_A{}^{CD}\, \nabla_C \left(R_{BD} - \frac{R}{4}\eta_{BD} \right) \qquad (12)$$

symmetric, traceless and conserved (equation vs identity: *electric* vs *magnetic*)

$$\frac{1}{k}C_{AB} := \frac{3c}{2}\frac{u_A u_B}{k^2} + \frac{c}{2}\eta_{AB} - \frac{c_{AB}}{k^2} + \frac{1}{k^2}u_A c_B + \frac{1}{k^2}u_B c_A \qquad (13)$$

Solving Einstein's equations: up to order 1

Einstein tensor $\mathscr{E}_{MN} := R^{\text{bulk}}_{MN} - \frac{1}{2}R^{\text{bulk}}g^{\text{bulk}}_{MN} - 3k^2 g^{\text{bulk}}_{MN}$ $(M, N = r, 0, 1, 2)$

Einstein's equations $\mathscr{E}_{MN} = 0$ split into dynamical and constraint

Constraint: up to order r

Equality of geometric and Bondi shear (now traceless and transverse)

$$k^2 \mathscr{C}_{AB} = -2\,\sigma_{AB} \tag{14}$$

Define the news tensor (Weyl-invariant, traceless, transverse) $\mathscr{N}_{AB} = u^C \mathscr{D}_C \mathscr{C}_{AB}$

Solving Einstein's equations: up to order 1

Einstein tensor $\mathscr{E}_{MN} := R^{\text{bulk}}_{MN} - \frac{1}{2}R^{\text{bulk}}g^{\text{bulk}}_{MN} - 3k^2 g^{\text{bulk}}_{MN}$ $(M, N = r, 0, 1, 2)$

Einstein's equations $\mathscr{E}_{MN} = 0$ split into dynamical and constraint

Constraint: up to order r

Equality of geometric and Bondi shear (now traceless and transverse)

$$k^2 \mathscr{C}_{AB} = -2\,\sigma_{AB} \tag{14}$$

Define the news tensor (Weyl-invariant, traceless, transverse) $\mathscr{N}_{AB} = u^C \mathscr{D}_C \mathscr{C}_{AB}$

Constraint: up to order 1

$$\begin{aligned}\mathscr{F}_{AB} &= 2u_{(A}\mathscr{D}^C\left(\sigma_{B)C} + \omega_{B)C}\right) - \frac{\mathscr{R}}{2}u_A u_B \\ &+ \left(\frac{1}{2}\sigma_{CD}\sigma^{CD} + k^2\gamma^2\right)h_{AB} + 2\,\omega_{(A}{}^C\sigma_{B)C}\end{aligned} \tag{15}$$

with $\mathscr{R} = R + 4\,\nabla_B A^B - 2\,A_B A^B$ and $h_{AB} = \eta_{AB} + \frac{u_A u_B}{k^2}$ (u: observer at rest)

Solving Einstein's equations: up to order $1/r$

The metric up to order $1/r$ is determined

$$\frac{1}{k^4} f^{(1)} \mathrm{u}^2 + \frac{2}{k^2} f_A^{(1)} \mathrm{u}\, \theta^A + f_{AB}^{(1)} \theta^A \theta^B = \frac{8\pi G}{k^4} \left(\varepsilon\, \mathrm{u}^2 + \frac{4}{3} \Delta q_A\, \mathrm{u}\, \theta^A + \frac{2k^2}{3} \Delta\tau_{AB}\, \theta^A \theta^B \right) \tag{16}$$

where

$$\Delta q_A = q_A - \frac{1}{8\pi G} {*c_A} \quad \text{and} \quad \Delta\tau_{AB} = \tau_{AB} + \frac{1}{8\pi G k^2} {*c_{AB}} \tag{17}$$

- $f_{AB}^{(1)}$ is constrained to be traceless by Einstein's equations
- energy-momentum (*electric*) data appear decomposed
- receive contributions from geometry under the dualised Cotton (*magnetic*)

Solving Einstein's equations: conservation equations

Dynamical part (choosing $u = -k\theta^0$, $A = 0, 1, 2$ and $a = 1, 2$)

$$\frac{k}{8\pi G}\mathcal{E}_{00} = \frac{1}{r^2}\mathscr{D}_B T^B{}_0 + \mathcal{O}(\tfrac{1}{r^3}) = -\frac{1}{kr^2}\mathscr{L} + \mathcal{O}(\tfrac{1}{r^3}) \quad (18)$$

$$\frac{k}{8\pi G}\mathcal{E}_{0a} = \frac{1}{r^2}\mathscr{D}_B T^B{}_a + \mathcal{O}(\tfrac{1}{r^3}) = \frac{1}{r^2}\mathscr{T}_a + \mathcal{O}(\tfrac{1}{r^3}) \quad (19)$$

Einstein's equations require

$$\mathscr{L} = u^A \mathscr{D}_A \varepsilon + \mathscr{D}_A q^A + \sigma_{AB}\tau^{AB} = 0 \quad (20)$$

$$\mathscr{T}^a = \frac{1}{2}\mathscr{D}^a \varepsilon + \mathscr{D}_B \tau^{aB} + \frac{2}{k^2} q_B \omega^{Ba} + \frac{1}{k^2}\left(u^B \mathscr{D}_B q^a + \sigma^{aB} q_B\right) = 0 \quad (21)$$

Analogue of Bondi evolution equations (mass-loss, etc.)

Solving Einstein's equations: up to order $1/r^2$

Constraint equations $\mathscr{E}_{rM} = \mathscr{E}_{ab} = 0$ fix the metric up to order $1/r^2$

$$f^{(2)} = \frac{8\pi G}{3k^2}\left(\sigma_{CD}\Delta\tau^{CD} + \mathscr{D}_C \Delta q^C\right) + \gamma\, c \tag{22}$$

$$f^{(2)}_A = -\frac{8\pi G}{3k^4}\sigma_{AC}\Delta q^C + \frac{4\pi G}{k^2}\left(h_{AC}\mathscr{D}_D\Delta\tau^{CD} + \frac{8}{3}\gamma * \Delta q_A\right) \tag{23}$$

$$\begin{aligned} f^{(2)}_{AB} = -\frac{4\pi G}{k^4}&\left(\frac{4}{3}u^C\mathscr{D}_C\Delta\tau_{AB} + \frac{2}{3}h_{AC}h_{BD}\mathscr{D}^{(C}\Delta q^{D)} \right. \\ &\left. -\frac{1}{3}h_{AB}h^{CD}\mathscr{D}_C\Delta q_D + 2\sigma_{(A}{}^{C}\Delta\tau_{B)C}\right) \\ &-\frac{1}{2k^4}\left(8\pi G\varepsilon\sigma_{AB} - c * \sigma_{AB}\right) + \frac{32\pi G}{3k^2}\gamma * \Delta\tau_{AB} \end{aligned} \tag{24}$$

Interplay between *electric* (τ_{AB}, q_A, ε) and *magnetic* (c_{AB}, c_A, c) quantities

[Illustration: *Tea Party with the Mad Hatter*, John Tenniel 1865]

Boundary geometry

Carrollian limit of the triad and its inverse (choosing $\mathrm{u} = -k\theta^0$)

$$\mu = \lim_{k\to 0} \frac{\mathrm{u}}{k^2} \qquad \text{and} \qquad \hat{\theta}^a = \lim_{k\to 0} \theta^a \tag{25}$$

where the *field of observers* $\boxed{\upsilon^A = \lim_{k\to 0} u^A}$ verifies $\mu(\upsilon) = -1$

The boundary metric $\mathrm{d}\ell^2 = \delta_{ab}\hat{\theta}^a\hat{\theta}^b$ is invariant under local rotations

Coefficients of the Weyl connection, shear and vorticity

$$\theta = \lim_{k\to 0} \Theta \qquad \varphi_a = \lim_{k\to 0} \frac{a_a}{k^2} \qquad \xi_{ab} = \lim_{k\to 0} \sigma_{ab} \qquad \varpi_{ab} = \lim_{k\to 0} \frac{\omega_{ab}}{k^2} \tag{26}$$

Transverse dualisation on $a, b \rightsquigarrow *\varpi = -\frac{1}{k}\epsilon^{ab}\varpi_{ab}$

Flat limit: up to order 1

Constraint: up to order r

Geometric and dynamic shear decouple ($\mathscr{C}_{ab}$ remains traceless transverse)

$$k^2\,\mathscr{C}_{ab} = -2\,\xi_{ab} \quad \underset{k\to 0}{\longrightarrow} \quad \mathscr{C}_{ab} \text{ arbitrary }, \; \xi_{ab} = 0 \tag{27}$$

News tensor $\hat{\mathscr{N}}_{ab} = \hat{\mathscr{D}}_{\upsilon}\mathscr{C}_{ab}$ differs from Bondi's by $\frac{\theta}{2}\mathscr{C}_{ab}$ [Barnich, Troessaert '10]

Constraint: up to order 1

Upon substituting ξ_{ab} for $-\frac{k^2}{2}\mathscr{C}_{ab}$ in the expression of $\mathscr{F}$

$$\lim_{k\to 0}\frac{\mathscr{F}}{k^4} = \left(\frac{1}{8}\mathscr{C}_{ab}\mathscr{C}^{ab} + {*\varpi}^2\right)\mathrm{d}\ell^2 - \left(\frac{1}{2}\hat{R} + \hat{\nabla}_a\varphi^a\right)\mu^2 - \hat{\mathscr{D}}^b\mathscr{C}_{ab}\,\mu\,\theta^a - 2*\hat{\mathscr{D}}_a{*\varpi}\,\mu\,\theta^a + {*\varpi}\,{*\mathscr{C}_{ab}}\,\theta^a\,\theta^b \tag{28}$$

is regular in the $k\to 0$ limit (some terms drop) and contains $\mathscr{C}_{ab}$

Flat limit: the energy-momentum tensor

Crucial step: analytic expansion of the energy-momentum in Laurent series

$$\varepsilon := \sum_{n\in\mathbb{Z}} k^{2n}\varepsilon^{(n)} \tag{29}$$

$$q_a := \sum_{n\geq 2} \frac{\zeta_a^{(n)}}{k^{2n}} + \frac{\zeta_a}{k^2} + Q_a + k^2\pi_a + \sum_{n\geq 2} k^{2n}\pi_a^{(n)} \tag{30}$$

$$\tau_{ab} := -\sum_{n\geq 3} \frac{\zeta_{ab}^{(n)}}{k^{2n}} - \frac{\zeta_{ab}}{k^4} - \frac{\Sigma_{ab}}{k^2} - \Xi_{ab} - k^2 E_{ab} - \sum_{n\geq 2} k^{2n} E_{ab}^{(n)} \tag{31}$$

Method

- substitute in $\mathscr{L} = 0$, $\mathscr{T}^a = 0$ and line-element $f^{(s\geq 1)}$, $f_a^{(s\geq 1)}$, $f_{ab}^{(s\geq 1)}$
- replace systematically ξ_{ab} by $-\frac{k^2}{2}\mathscr{C}_{ab}$
- require the line-element to be finite when $k \to 0$

Flat limit: up to order $1/r$

Algebraic relations

Finiteness at order $1/r$ requires

$$\varepsilon^{(n<0)} = 0 \qquad \zeta_a^{(n\geq 2)} = 0 \qquad \zeta_{ab}^{(n\geq 3)} = 0 \tag{32}$$

Some components of q^a and τ^{ab} are locked in terms of Cotton coeffs

$$\zeta_a = \frac{*c_a^{(-1)}}{8\pi G} \quad Q_a = \frac{*c_a^{(0)}}{8\pi G} \quad \zeta_{ab} = \frac{*c_{ab}^{(-1)}}{8\pi G} \quad \Sigma_{ab} = \frac{*c_{ab}^{(0)}}{8\pi G} \quad \Xi_{ab} = \frac{*c_{ab}^{(1)}}{8\pi G}$$

Line element at order $1/r$

$$\begin{aligned}\lim_{k\to 0}\left(\frac{1}{k^4} f^{(1)}\,\mathrm{u}^2 + \frac{2}{k^2} f_A^{(1)}\,\mathrm{u}\,\theta^A + f_{AB}^{(1)}\,\theta^A\,\theta^B\right) \\ = 8\pi G\varepsilon^{(0)}\,\mu^2 - \frac{4}{3}N_a\,\mu\,\hat{\theta}^a - \frac{16\pi G}{3}E_{ab}\,\hat{\theta}^a\hat{\theta}^b\end{aligned} \tag{33}$$

where $N_a = -8\pi G\pi_a + *c_a^{(1)}$ and E_{ab} is the covariant stress of [Freidel, Pranzetti '21]

Flat limit: conservation equations

Flux-balance laws

Example: $\lim_{k\to 0} \mathscr{L} =$

$$\hat{\mathscr{D}}_\upsilon \varepsilon^{(0)} + \hat{\mathscr{D}}_a Q^a - \frac{1}{16\pi G}\left(\hat{\mathscr{D}}_a \hat{\mathscr{D}}_b \hat{\mathscr{N}}^{ab} + \mathscr{C}^{ab} \hat{\mathscr{D}}_a \hat{\mathscr{R}}_b + \frac{1}{2}\mathscr{C}_{ab} \hat{\mathscr{D}}_\upsilon \hat{\mathscr{N}}^{ab}\right) \quad (34)$$

Bondi mass aspect $M = 4\pi G \varepsilon^{(0)} - \frac{1}{8}\mathscr{C}^{ab}\hat{\mathscr{N}}_{ab}$ [Compère, Fiorucci, Ruzziconi '19]

Finite part of the line element at order $1/r^2$

$$\hat{f}^{(2)} = \lim_{k\to 0} f^{(2)} = -\frac{1}{3}\hat{\mathscr{D}}_a N^a + 2 *\varpi\, \nu \quad (35)$$

$$\hat{f}_a^{(2)} = \lim_{k\to 0} f_a^{(2)} = -\frac{1}{6} N^b \mathscr{C}_{ba} - \frac{4}{3} *\varpi *N_a - 4\pi G \hat{\mathscr{D}}_b E^b{}_a \quad (36)$$

where $\nu = \frac{1}{2}\lim_{k\to 0} c = \frac{1}{2} c^{(0)} - \frac{1}{4}\hat{\mathscr{D}}_a \hat{\mathscr{D}}_b *\mathscr{C}^{ab} - \frac{1}{8}\mathscr{C}_{ab} *\hat{\mathscr{N}}^{ab}$

Flat limit: up to order $1/r^2$

Tensorial contribution at order $1/r^2$

$$f_{ab}^{(2)} = \frac{1}{k^2}\underbrace{\left(\frac{16\pi G}{3}\hat{\mathscr{D}}_\upsilon E_{ab} + \frac{1}{3}\hat{\mathscr{D}}_{\langle a}N_{b\rangle} + 2\pi G\varepsilon^{(0)}\mathscr{C}_{ab} - \frac{\nu}{2}*\mathscr{C}_{ab}\right)}_{\text{flux-balance law for } E_{ab} \text{ agrees with [Freidel, Pranzetti '21]}}$$
$$+\frac{16\pi G}{3}\underbrace{\left(\hat{\mathscr{D}}_\upsilon E_{ab}^{(2)} - \frac{1}{2}\hat{\mathscr{D}}_{\langle a}\pi_{b\rangle}^{(2)} + \frac{3}{8}\varepsilon^{(1)}\mathscr{C}_{ab} - \frac{3}{8\pi G}*\varpi^3 *\mathscr{C}_{ab}\right)}_{\text{new tensor } F_{ab}} \qquad (37)$$
$$-4\pi G\mathscr{C}^c{}_{\langle a}E_{b\rangle c} + \mathcal{O}(k^2)$$

Only F_{ab} is relevant, not $E_{ab}^{(2)}$, $\pi_a^{(2)}$ or $\varepsilon^{(1)}$ (flux-balance law obtained at $1/r^3$)

Pattern:

- $f^{(s\geq 2)}$ and $f_a^{(s\geq 2)}$ have a smooth $k\to 0$ limit
- $f_{ab}^{(s\geq 2)}$ has singular contributions: flux-balance laws
- regular contribution repackages $E_{ab}^{(s)}$, $\pi_a^{(s)}$ and $\varepsilon^{(s-1)}$ into a single $F_{ab}^{(s)}$

Recap

Ingredients

- covariant Newman-Unti gauge + Weyl connection + $1/r$-expansion
- choice of boundary metric and energy-momentum tensor
- solve Einstein's equations in AlAdS$_4$ (metric + flux-balance laws)

Method for taking the flat limit

- transverse decomposition of metric, energy-momentum tensor, ...
- *Laurent expansion* of the energy-momentum tensor
- take the $k \to 0$ limit, informed with $k^2\, \mathscr{C}_{ab} = -2\, \sigma_{ab}$
- finiteness of the line element: additional flux-balance equations

Results and future direction

Results

- Carrollian boundary-covariant metric reconstruction for AlMink_4
- correct solution space and flux-balance laws (Bondi mass-loss etc.)
- Carrollian origin of the Chthonian degrees of freedom
- class of resummable metrics for $\mathscr{C}_{ab} = N^a = F_{(s)ab} = 0$

Future directions

- asymptotic symmetries and (subleading) charges
- link Newman-Penrose formalism [Mittal, Petropoulos, Rivera-Betancour, Vilatte '22]
- generalisations: higher dimensions? logarithms?
- consequences for flat holography?

Weyl-covariant derivative

In a $1/r$-expansion, objects have Weyl weight w, e.g.

$$\begin{aligned} \mathscr{D}_A\Phi &:= \nabla_A\Phi + wA_A\Phi \\ \mathscr{D}_A v_B &:= \nabla_A v_B + wA_A v_B + A_B v_A - \eta_{AB}A^C v_C \end{aligned} \tag{38}$$

Weyl-covariant derivative is metric-compatible

$$\mathscr{D}_A\eta_{BC} = 0 \tag{39}$$

Has effective torsion ($\mathrm{d}A = \frac{1}{2}F_{AB}\theta^A \wedge \theta^B$) and curvature

$$\begin{aligned} [\mathscr{D}_A, \mathscr{D}_B]\,\Phi &= w\Phi F_{AB} \\ [\mathscr{D}_A, \mathscr{D}_B]\,v^C &= \mathscr{R}^C{}_{DAB}v^D + (w+1)F_{AB}v^C \end{aligned} \tag{40}$$

Expression of the Weyl connection

If we require $\mathscr{D}$ to preserve the congruence u, i.e.

$$\mathscr{D}_A u^A = 0 \qquad \text{and} \qquad u^A \mathscr{D}_A u_B = 0 \tag{41}$$

then its components are fixed

$$\mathsf{A} = \frac{1}{k^2}\left(\mathsf{a} - \frac{\Theta}{2}\mathsf{u}\right) \quad \text{where} \quad a_A = u^B \nabla_B u_A \quad \text{and} \quad \Theta = \nabla_A u^A \tag{42}$$

Weyl-covariant derivative preserves transversality

$$u^A \mathscr{D}_A h_{BC} = 0 \qquad \text{where} \qquad h_{AB} := \eta_{AB} + \frac{u_A u_B}{k^2} \tag{43}$$

Carrollian Cotton tensor

Decomposition of the Cotton coefficients in powers of k^2

$$c = c^{(1)}\, k^2 + c^{(0)} + \frac{c^{(-1)}}{k^2} + \frac{c^{(-2)}}{k^4} \tag{44}$$

$$c_a = c_a^{(1)}\, k^2 + c_a^{(0)} + \frac{c_a^{(-1)}}{k^2} \tag{45}$$

$$c_{ab} = c_{ab}^{(1)}\, k^2 + c_{ab}^{(0)} + \frac{c_{ab}^{(-1)}}{k^2} \tag{46}$$

Conservation of the Cotton $\longrightarrow$ 8 Carrollian identities for $n \in \{-2, \ldots, 1\}$

$$-\hat{\mathscr{D}}_\upsilon c^{(n)} - \hat{\mathscr{D}}^a * c_a^{(n)} + c_{ab}^{(n+1)} \xi^{ab} = 0 \tag{47}$$

$$\frac{1}{2}\hat{\mathscr{D}}_a c^{(n)} + 2 * \varpi\, * c_a^{(n)} - \hat{\mathscr{D}}^b c_{ab}^{(n+1)} + \hat{\mathscr{D}}_\upsilon c_a^{(n+1)} + c_b^{(n+1)}\, \xi_a{}^b = 0 \tag{48}$$

Solving Einstein's equations

Constraint equations, up to now

$$\mathcal{E}_{rr} = -3\eta^{AB} f^{(1)}_{AB} \frac{1}{r^5} - 6\left(\eta^{AB} f^{(2)}_{AB} + \frac{3}{2k^2}\sigma^{AB} f^{(1)}_{AB}\right)\frac{1}{r^6} + \mathcal{O}(\tfrac{1}{r^7})$$

$$k\mathcal{E}_{r0} = \left(-f^{(2)} - 2k^2\eta^{AB} f^{(2)}_{AB} + \frac{1}{2}h^{AB}\mathscr{D}_A f^{(1)}_B - \frac{5}{2}\sigma^{AB} f^{(1)}_{AB} + c\,\gamma\right)\frac{1}{r^4} + \mathcal{O}(\tfrac{1}{r^5})$$

$$\mathcal{E}_{ra} = \left(2f^{(2)}_a - \frac{3}{2}h^B_a \mathscr{D}^C f^{(1)}_{BC} + \frac{1}{k^2}\left(\sigma_a{}^B + 4\,\omega_a{}^B\right) f^{(1)}_B\right)\frac{1}{r^4} + \mathcal{O}(\tfrac{1}{r^5})$$

$$\mathcal{E}_{ab} = \Big(-f^{(2)}h_{ab} + c\,\gamma h_{ab} + 4\,\omega^C{}_{(a} f^{(1)}_{b)C} + 2k^2\hat{\eta}^C{}_a\hat{\eta}^D{}_b f^{(2)}_{CD} - 2u^C\mathscr{D}_C f^{(1)}_{ab}$$
$$+ \hat{\eta}^C{}_a\hat{\eta}^D{}_b\mathscr{D}_{(C} f^{(1)}_{D)} + \frac{1}{k^2}\left(\hat{\eta}^C{}_{(a}\sigma_{b)C}\,c - f^{(1)}\sigma_{ab}\right) + 4\,\sigma^C{}_{(a} f^{(1)}_{b)C}\Big)\frac{1}{r^2}$$
$$+ \mathcal{O}(\tfrac{1}{r^3})$$

Dictionary between Bondi and covariant Newman-Unti

Comparison with [Compère, Fiorucci, Ruzziconi '19] [Freidel, Pranzetti '21]

- $\upmu = -\mathrm{d}u \quad (\mathrm{d}\upmu = \varphi_a \hat{\uptheta}^a \wedge \upmu + \varpi_{ab} \hat{\uptheta}^a \wedge \hat{\uptheta}^b \implies *\varpi = 0\,,\ \varphi^a = 0)$
- $\mathscr{C}^{ab} = C^{ab}_{\text{Bondi}}$
- $\hat{\mathscr{N}}^{ab} = \hat{\mathcal{N}}^{ab}_{\text{Bondi}} - \frac{\theta}{2}\mathscr{C}^{ab}$
- $\theta = \ell_{\text{Bondi}}$
- $4\pi G \varepsilon^{(0)} = \mathcal{M}_{\text{Bondi}}$
- $\nu = \frac{1}{2}c^{(0)} - \tilde{\mathcal{M}}_{\text{Bondi}}$
- $N^a = \mathcal{P}^a = N^a_{\text{Bondi}} + \frac{1}{4}\left(\mathscr{C}^{ab}\hat{\nabla}^c\mathscr{C}_{bc} + \frac{3}{8}\hat{\nabla}^a(\mathscr{C}^{bc}\mathscr{C}_{bc})\right)$
- $E^{ab} = -\frac{1}{16\pi G}\mathcal{T}^{ab} = -\frac{3}{16\pi G}\left(\mathcal{E}^{ab}_{\text{Bondi}} - \frac{1}{16}\mathscr{C}^{ab}\mathscr{C}^{cd}\mathscr{C}_{cd}\right)$

Flat limit: recursive pattern

- $f^{(s\geq 2)}$ have a smooth zero-k limit involving

$$\hat{\mathscr{D}}^a \hat{f}_a^{(s-1)} \quad \text{and} \quad *\varpi^{2k}\varepsilon^{(0)} \quad \text{or} \quad *\varpi^{2k+1}\nu \tag{49}$$

- $f_a^{(s\geq 2)}$ have a smooth zero-k limit involving

$$\hat{\mathscr{D}}^b \hat{f}_{ab}^{(s-1)} \quad \text{and} \quad *\varpi * \hat{f}_a^{(s-1)} \quad \text{and} \quad \mathscr{C}_a{}^b \hat{f}_b^{(s-1)} \tag{50}$$

- $f_{ab}^{(s\geq 2)}$ have singular pieces imposing flux-balance laws

$$\begin{aligned} f_{ab}^{(s)} &= \frac{\#}{k^{2s-2}} \hat{\mathscr{D}}_\upsilon^{s-2} \mathcal{FB}_{ab}^{(1)} + \cdots + \frac{\#}{k^2} \mathcal{FB}_{ab}^{(s-1)} \\ &\quad + \underbrace{\left(F_{ab}^{(s)} \left[E^{(s)}, \pi^{(s)}, \varepsilon^{(s-1)} \right] + \ldots \right)}_{\hat{f}_{ab}^{(s)}} + \mathcal{O}(k^2) \end{aligned} \tag{51}$$

$$F_{ab}^{(s)} = \left\{ \hat{\mathscr{D}}_\upsilon^{s-1} E_{ab}^{(s)} \,,\; \hat{\mathscr{D}}_\upsilon^{s-2} \hat{\mathscr{D}}_{\langle a} \pi_{b\rangle}^{(s)} \,,\; \varepsilon^{(s-1)} *\varpi^{s-2} \mathscr{C}_{ab} \,,\; \ldots \right\} \tag{52}$$

$$\mathcal{FB}_{ab}^{(s)} = \hat{\mathscr{D}}_\upsilon F_{ab}^{(s)} + \ldots \tag{53}$$

The resummable case

AlAdS case

Take $\sigma_{AB} = \Delta q^A = \Delta\tau_{AB} = 0$ (therefore $f_A^{(s)} = f_{AB}^{(s)} = 0$) and

$$f^{(2s+1)} = (-)^s 8\pi G\,\varepsilon\,\gamma^{2s} \quad \text{and} \quad f^{(2s+2)} = (-)^s c\,\gamma^{2s+1} \tag{54}$$

then the $1/r$ expansion resums to (where $\rho^2 = r^2 + \gamma^2$)

$$ds^2_{\text{res. Einstein}} = \frac{2}{k^2}\mathrm{u}\,(dr + rA) + r^2 ds^2_{\text{bdy}} + \frac{\mathscr{F}}{k^4} + \frac{\mathrm{u}^2}{k^4\rho^2}\,(8\pi G\varepsilon r + c\gamma) \tag{55}$$

Flat case

Take $\mathscr{C}_{ab} = N^a = F_{(s)ab} = 0$ and

$$f^{(2s+1)} = (-)^s 8\pi G\,\varepsilon^{(0)} * \varpi^{2s} \quad \text{and} \quad f^{(2s+2)} = (-)^s c^{(0)} * \varpi^{2s+1} \tag{56}$$

then the $1/r$ expansion resums to ($\rho^2 = r^2 + *\varpi^2$)

$$ds^2_{\text{res. Ricci-flat}} = 2\,\mu\,(dr + rA) + \rho^2 d\ell^2 + \frac{\mu^2}{\rho^2}\left(8\pi G\varepsilon^{(0)} r + c^{(0)} * \varpi\right) \tag{57}$$

Link with Newman-Penrose

In the resummable case (algebraically special)

$$d\ell^2 = \frac{2}{P(u,\zeta,\bar{\zeta})^2} d\zeta d\bar{\zeta} \tag{58}$$

The Weyl scalars are

$$\Psi_0 = \Psi_1 = 0 \tag{59}$$

$$\Psi_2 = \frac{i\hat{\tau}}{2(r - i * \varpi)^3} \tag{60}$$

$$\Psi_3 = \frac{iP\chi_\zeta}{(r - i * \varpi)^2} + \mathcal{O}\left(1/(r - i * \varpi)^3\right) \tag{61}$$

$$\Psi_4 = \frac{iX_\zeta{}^{\bar{\zeta}}}{r - i * \varpi} + \mathcal{O}\left(1/(r - i * \varpi)^2\right) \tag{62}$$

with $\hat{\tau} = -c^{(0)} + i\, 8\pi G \varepsilon^{(0)}$