On the road to the Carrollian conformal boostrap

based on 2305.02884 with Peter West (King's College London) + ongoing work

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Scattering theory: the need for a new framework

Current framework

Despite its innumerable phenomenological successes, the standard approach to scattering amplitudes is unsatisfactory and ill-defined!

- perturbative expansion does not converge
- infrared divergences
- nonrenormalizable UV divergences (e.g. in gravity)
- ambiguous multicolinear limits

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Attempts at fixing these issues exist:

- resurgence
- dressed states
- analytic boostrap
- ...

Conformal Field Theory

When theorists can feel safe

Conformal Field Theory is well-defined.

- No divergences
- Non-perturbative
- Fully calculable (in principle)
- Only known way to *fully* formulate quantum gravity (in AdS)
- Can be completed from the bottom-up through conformal boostrap!

Say no more... Where do I sign?

A Carrollian conformal boostrap?

Expectations

[Bagchi-Banerjee-Basu-Dutta '22, Donnay-Fiorucci-Herfray-Ruzziconi '22]

Scattering amplitudes are correlation functions in Carrollian CFT

- Indeed the Poincaré group is the conformal Carrollian group!

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Open question

Can we construct scattering amplitudes through conformal boostrap?

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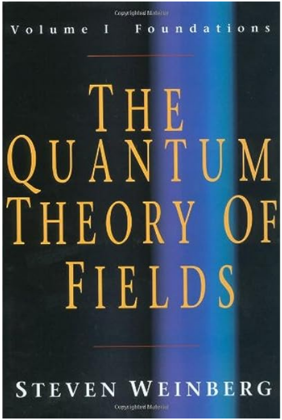
- Non-perturbative
- Fully calculable (in principle)
- Can be completed from the bottom-up!


Volume I Foundations
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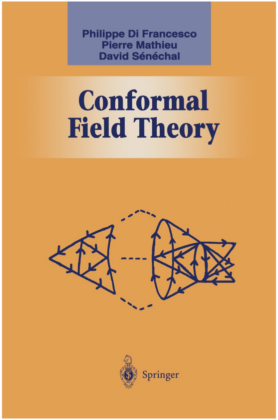
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Conformal Field Theory
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QFT on $\mathbb{M}^{d+1}$

- Hilbert space: ISO(1,d) unitary reps
- Fields: Lorentzian SO(1,d) tensors + ISO(1,d) non-unitary reps
- Interactions: local Lagrangian

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VS

CFT on $\mathbb{R} \times S^d$

- Hilbert space: SO(2,d+1) unitary reps
- Fields: *euclidean* SO(d+1) tensors + SO(2,d+1) unitary reps
- State-operator correspondence
- Interactions: OPE coefficients

New approach to scattering theory

Two spacetime realizations of ISO(1,d)

isometry group of $\mathbb{M}^{d+1}$ and conformal group of $\mathscr{I} \simeq \mathbb{R} \times S^{d-1}$

This suggest two alternative routes:

- Relativistic fields in $\mathbb{M}^{d+1}$ (standard)
- Carrollian conformal fields on $\mathscr{I}$

Goal

Follow the second route and develop a conformal boostrap approach

bonus: BMS symmetries appear naturally as part of the conformal group!
Irreps of the BMS group should be related to dressed states (?)

Poincaré generators at $\mathscr{I}$

By $\mathscr{I}$ we simply mean a null surface $\mathbb{R} \times S^{d-1}$ covered with stereographic coordinates $x^\alpha = (u, x^i)$ and conformal metric

$$ds^2_{\mathscr{I}} = q_{\alpha\beta}\, dx^\alpha dx^\beta = 0\, du^2 + \delta_{ij}\, dx^i dx^j\ .$$

The ISO(1,d) conformal isometries are

$$x'^\alpha = (1 + iaH + ia^i P_i + \frac{i}{2}\omega^{ij} J_{ij} + ib^i B_i + i\lambda D + ikK + ik^i K_i)\, x^\alpha\ ,$$

with

$$\begin{aligned}
P_i &= -i\partial_i\ , & J_{ij} &= i(x_i \partial_j - x_j \partial_i)\ , \\
D &= -i(u\partial_u + x^i \partial_i)\ , & K_i &= -i(x^2 \partial_i - 2x_i(u\partial_u + x^j \partial_j))\ ,\ K = -ix^2 \partial_u\ , \\
H &= -i\partial_u\ , & B_i &= -ix_i \partial_u\ .
\end{aligned}$$

Poincaré generators at $\mathscr{I}$

The dictionary with the standard ISO(1,d) notation is

$$\tilde{J}_{ij} = J_{ij}\,, \quad \tilde{J}_{i0} = -\frac{1}{2}\left(P_i + K_i\right)\,, \quad \tilde{J}_{id} = \frac{1}{2}\left(P_i - K_i\right)\,, \quad \tilde{J}_{0d} = -D\,,$$

$$\tilde{P}_0 = \frac{1}{\sqrt{2}}(H+K)\,, \qquad \tilde{P}_i = -\sqrt{2}\,B_i\,, \qquad \tilde{P}_d = \frac{1}{\sqrt{2}}(K-H)\,,$$

The quadratic Casimir operator is identically zero!

$$\mathcal{C}_2 = \tilde{P}^\mu \tilde{P}_\mu = -(HK + KH) + 2B^i B_i = 2x^2\partial_u^2 - 2x^2\partial_u^2 = 0\,.$$

Group theory knows too much...

Carrollian fields on $\mathscr{I}$ carry massless representations!

Carrollian conformal fields

Just like with relativistic conformal fields [Mack-Salam '69], we can construct Carrollian conformal fields as induced ISO(1,d) representations, starting from a representation of the isotropy group of $\mathscr{I}$ with algebra

$$\mathfrak{h} = \{J_{ij}, B_i, K_i, K, D\}\,.$$

The inducing representation acts on the field $\phi(0) \equiv \phi_{i_1 \dots i_s}(0)$ at the origin,

$$[J_{ij}, \phi(0)] = \Sigma_{ij}\, \phi(0)\,, \qquad [D, \phi(0)] = i\Delta\, \phi(0), \qquad \Delta \in \mathbb{R}\,.$$

Since we want finite-components field, we must impose

$$[B_i, \phi(0)] = [K_i, \phi(0)] = 0\,,$$

$$[B_i, K_j] = i\delta_{ij} K \quad \Rightarrow \quad [K, \phi(0)] = 0\,.$$

Note that $K_\alpha = (K, K_i)$ and $P_\alpha = (H, P_i)$ act as lowering and raising operator for the conformal dimension.

Carrollian conformal fields

The induced representation is constructed through

$$\phi(x) \equiv U(x)\,\phi(0)\,U(x)^{-1}\,, \qquad U(x) \equiv e^{-ix^\alpha P_\alpha} = e^{-i(uH + x^i P_i)}\,.$$

Some simple algebra gives

$$\begin{aligned}
[H, \phi(x)] &= -i\partial_u \phi(x)\,,\\
[P_i, \phi(x)] &= -i\partial_i \phi(x)\,,\\
[J_{ij}, \phi(x)] &= -i\left(i\Sigma_{ij} - x_i\partial_j + x_j\partial_i\right)\phi(x)\,,\\
[D, \phi(x)] &= -i\left(-\Delta + u\partial_u + x^i\partial_i\right)\phi(x)\,,\\
[K, \phi(x)] &= -ix^2\partial_u \phi(x)\,,\\
[K_i, \phi(x)] &= -i\left(2x_i\Delta + 2ix^j\Sigma_{ij} - 2ux_i\partial_u - 2x_ix^j\partial_j + x^2\partial_i\right)\phi(x)\,,\\
[B_i, \phi(x)] &= -ix_i\partial_u \phi(x)\,.
\end{aligned}$$

Carrollian conformal fields

Let's quickly check what kind of states these fields carry! For this we simply fix a momentum frame and assess the action of the little group $ISO(d-1)$.

The point $x^\alpha = 0$ corresponds to momentum frame

$$\tilde{P}_\mu = \frac{1}{\sqrt{2}}(H, 0, ..., 0, -H)\,,$$

with little group $\{J_{ij}\,, K_i\} = \mathfrak{iso}(d-1)$. The action of these generators corresponds to that of a spin-s massless inducing representation!

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What is the value of Δ?

The dilation operator D is outside the little group ISO(d-1), hence in a *unitary* representation its action is induced and Δ is not an independent parameter!

Carrollian conformal fields

The conformal dimension is fixed to [Peter's talk]

$$\Delta = -\frac{d-1}{2}\,.$$

We can also summarise the Poincaré transformations by

$$\delta\phi(x) = \left(\zeta^\alpha \partial_\alpha - \frac{i}{2}\partial_{[i}\zeta_{j]}\,\Sigma^{ij} - \Delta\Omega\right)\phi = \left(L_\zeta - \Delta\Omega\right)\phi(x)\,,$$

with scaling factor $\Omega = \partial_\alpha \zeta^\alpha = \lambda - 2k_i x^i$ and

$$\zeta^u = a + b_i x^i + k x^2 + (\lambda - 2k_i x^i)u\,,$$
$$\zeta^i = a^i + \omega^i{}_j\, x^j + \lambda x^i + k^i x^2 - 2k_j x^j x^i\,.$$

Holographic correspondence

Carrollian fields are obtained by pull-back of relativistic bulk fields to $\mathscr{I}$!

$^{(*)}$ The conformal dimension is shifted by the spin s, $\Delta_{bulk} = s - \frac{d-1}{2}$

State-operator correspondence [Peter's talk]

Massless particle states $\psi_\sigma(p) \equiv |p, \sigma\rangle$ with arbitrary momentum p can be constructed by boosting a reference state with $p_-^{(0)} = 1$,

$$\psi_\sigma(p) \equiv e^{ix^i \tilde{J}_{-i}}\, e^{i \ln \omega \tilde{J}_{+-}}\, \psi_\sigma(p^{(0)})\,.$$

and we recover the standard celestial parametrisation

$$p^+ = \omega\,, \qquad p^- = -\omega\, x^i x_i\,, \qquad p^i = \omega\, x^i\,.$$

The following Fourier transform allows us to reach the Carrollian field basis

$$\psi_\sigma(u, x^i) \equiv \int_0^\infty d\omega\, \omega^{\frac{d-3}{2}} e^{-i\omega u} e^{ix^i P_i} e^{-i \ln \omega D}\, \psi_\sigma(p^{(0)})\,.$$

State-operator correspondence

$\psi_\sigma(u, x^i)$ transforms exactly like the Carrollian field $\phi_{i_1 \ldots i_s}(x^\alpha)$!

Perpectives

- Massive representations at timelike infinity $i^{\pm}$
  [WIP with Figueroa-O'Farrill, Have, Prohazka, Salzer]
- Unifying framework for massless + massive?
- General 2- and 3-point functions in arbitrary d [in preparation]
- OPE expansion
- Conformal blocks
- Crossing equations?
- Boostrap?
- ...