Weyl-Fefferman-Graham and covariant Bondi gauges

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Based on [A. Campoleoni, L. Ciambelli, AD, C. Marteau, P. M. Petropoulos, R. Ruzziconi (2208.07575)] and [LC, AD, RR, C. Zwikel (2308.15480)]

Outline

I. Plan and Motivations

- Study of the classical phase space of 3D asymptotically AdS gravity

  Select the allowed metric fluctuations at infinity [Brown-Henneaux '86]

- No requirement to fix any particular gauge but it is often convenient

  For example: Fefferman-Graham, Bondi gauge [Starobinsky '83, Fefferman-Graham '85] [Bondi-van der Burg-Metzner '62, Sachs '62]

- In this talk: Weyl-Fefferman-Graham gauge [Ciambelli-Leigh '19, Jia-Karydas '21]

  Interesting implications for holography

  Link with the covariant Bondi gauge, allow for a smooth flat-space limit

Outline

II. Fefferman-Graham gauge in 3D

- Useful gauge fixing for holography: [Fefferman-Graham '85]

Any asymptotically AdS_3 space can be written near the boundary as ($\ell = 1$)

$$ds^2_{\text{AdS}} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\mathrm{d}\rho^2}{\rho^2} + h_{ij}(\rho, x)\mathrm{d}x^i \mathrm{d}x^j$$

ρ is a spacelike coordinate s.t. $\rho = 0$ locates the bdy
x^i are bdy coords: $x^1 = t$ is timelike and $x^2 = \theta$ is spacelike

- Boundary geometry $\to$ leading order in the asymptotic expansion

$$h_{ij}(\rho, x) = \frac{1}{\rho^2} h^{(0)}_{ij}(x) + \mathcal{O}(1)$$

- FG expansion does not transform Weyl-covariantly under [Henningson-Skenderis '98]

$$\rho \to \rho' = \frac{\rho}{\mathfrak{B}(x)}\,, \qquad x^i \to x'^i = x^i + \xi^i(\rho, x)$$

II. Weyl-Fefferman-Graham gauge in 3D

- Solution: relax the FG ansatz to the WFG gauge [Ciambelli-Leigh '19]

$$ds^2_{\text{AdS}} = \left(\frac{d\rho}{\rho} - k_i(\rho,x)dx^i\right)^2 + h_{ij}(\rho,x)dx^i dx^j\,, \quad k_i(\rho,x) = k_i^{(0)}(x) + \mathcal{O}(\rho^2)$$

$\hookrightarrow$ $h_{ij}^{(0)}(x)$ is the bdy metric and $k_i^{(0)}(x)$ is a bdy Weyl connection

- Question: Is the Weyl structure associated with an asymptotic symmetry?

- Einstein gravitation $\to$ Chern-Simons gauge theory ($L_{EH} = L_{CS}[\mathrm{A}] - L_{CS}[\widetilde{\mathrm{A}}]$)
  [Achucarro-Townsend '86, Witten '88]

$$\mathrm{L}_{CS}[\mathrm{A}] = \frac{1}{16\pi G}\mathrm{Tr}\left(\mathrm{A}\wedge d\mathrm{A} + \frac{2}{3}\mathrm{A}\wedge\mathrm{A}\wedge\mathrm{A}\right)$$

Isometry algebra of AdS_3 $= \mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

$$[L_n, L_m] = (n-m)L_{n+m}$$

II. WFG gauge in 3D: asymptotic symmetries

- Boundary conformally-flat parametrization: $(x^{\pm} = t \pm \theta)$

$$h_{ij}^{(0)} \mathrm{d}x^i \mathrm{d}x^j = \mathrm{e}^{2\phi(x^+,x^-)} \mathrm{d}x^+ \mathrm{d}x^-$$

- Solution space – Chern-Simons connections:

$$\begin{aligned} A_\rho &= -\tfrac{1}{\rho} L_0 + \mathcal{O}(\rho^2)\,, \\ A_+ &= \tfrac{1}{\rho} \mathrm{e}^{\phi} L_1 + (2\, k_+^{(0)} - \partial_+\phi) L_0 - \rho\, \mathrm{e}^{-\phi} h_{++}^{(2)} L_{-1} + \mathcal{O}(\rho^2)\,, \\ A_- &= \partial_-\phi\, L_0 + \rho\, \mathrm{e}^{-\phi} \partial_-(k_+^{(0)} - \partial_+\phi) L_{-1} + \mathcal{O}(\rho^3)\,, \end{aligned}$$

where $h_{++}^{(2)} = \ell_+(x^+) - (k_+^{(0)} - \partial_+\phi)^2 - \partial_+(k_+^{(0)} - \partial_+\phi)$

- Surface charges: $(\delta_\lambda \phi = \sigma\,,\ \delta_\lambda k_i^{(0)} = h_i^{(0)})$

$$Q = -\frac{1}{8\pi G} \int_0^{2\pi} \mathrm{d}\theta \Big[\phi(\partial_t \sigma - h_t^{(0)}) - \sigma(\partial_t \phi - k_t^{(0)})\Big]$$

The Weyl connection is associated with an asymptotic symmetry!

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where $h^{(2)}_{++} = \ell_+(x^+) - (k^{(0)}_+ - \partial_+\phi)^2 - \partial_+(k^{(0)}_+ - \partial_+\phi)$

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- Solution space – Chern-Simons connections:

$$\begin{aligned} \mathrm{A} &= b^{-1}\,(\mathrm{d}+\mathrm{a})\,b\,, \qquad b(\rho) = \exp\left(-\log\rho\, L_0\right), \\ a_+ &= \mathrm{e}^{\phi} L_1 + (2\,k_+^{(0)} - \partial_+\phi) L_0 - \mathrm{e}^{-\phi} h_{++}^{(2)} L_{-1}\,, \\ a_- &= \partial_-\phi\, L_0 + \mathrm{e}^{-\phi}\partial_-(k_+^{(0)} - \partial_+\phi) L_{-1}\,, \end{aligned}$$

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The Weyl connection is associated with an asymptotic symmetry!

II. WFG gauge in 3D: passage to Covariant Bondi

Solution space – Chern-Simons connections: $(A = b^{-1}(d + a)b)$

- WFG $(b(\rho) = \exp(-\log \rho \, L_0))$

$$
\begin{aligned}
a_+ &= e^{\phi} L_1 + (2\, k_+^{(0)} - \partial_+ \phi) L_0 - e^{-\phi} h_{++}^{(2)} L_{-1} \,, \\
a_- &= \partial_- \phi \, L_0 + e^{-\phi} \partial_- (k_+^{(0)} - \partial_+ \phi) L_{-1} \,, \\
h_{++}^{(2)} &= \ell_+(x^+) - (k_+^{(0)} - \partial_+ \phi)^2 - \partial_+ (k_+^{(0)} - \partial_+ \phi)
\end{aligned}
$$

- Covariant Bondi $(b(r) = \exp(r\, L_{-1}))$

$$
\begin{aligned}
a_+ &= e^{\varphi + \zeta} L_1 + \partial_+ (\varphi - \zeta) L_0 - e^{-(\varphi + \zeta)} h_{++}^{(2)} L_{-1} \,, \\
a_- &= \partial_- (\varphi + \zeta) L_0 - e^{-(\varphi + \zeta)} \partial_+ \partial_- \zeta \, L_{-1} \,, \\
h_{++}^{(2)} &= \ell_+(x^+) - (\partial_+ \zeta)^2 + \partial_+^2 \zeta
\end{aligned}
$$

- Passage from WFG to CB

$$
k_i^{(0)}(x) \to \partial_i \varphi(x) \,, \qquad \phi(x) \to \varphi(x) + \zeta(x)
$$

Outline

III. Covariant Bondi gauge in AdS

- Key idea 1: relax the AdS Bondi gauge → dependence on the boundary dyad

$$\mathrm{d}s^2_{\mathrm{AdS}} = \frac{2}{k^2}\mathrm{u}\,(\mathrm{d}r + r\,\mathrm{A}) + r^2 \mathrm{d}s^2_{\mathrm{bdy}} + \frac{8\pi G}{k^4}\mathrm{u}\,(\varepsilon\,\mathrm{u} + \chi\,{*\mathrm{u}})$$

- Boundary metric and Cartan frame: $(\mathrm{u} = -\frac{k}{2}\mathrm{e}^{\varphi}(\mathrm{e}^{\zeta}\mathrm{d}x^+ - \mathrm{e}^{-\zeta}\mathrm{d}x^-))$

$$\mathrm{d}s^2_{\mathrm{bdy}} = \frac{1}{k^2}\left(-u_i u_j + {*u_i}{*u_j}\right)\mathrm{d}x^i\mathrm{d}x^j = \mathrm{e}^{2\varphi}\mathrm{d}x^+\mathrm{d}x^-$$

Weyl connection: [Loganayagam '08]

$$\mathrm{A} = \frac{1}{k^2}\left(\Theta^*{*\mathrm{u}} - \Theta\,\mathrm{u}\right), \qquad \Theta = \nabla_i u^i, \qquad \Theta^* = \nabla_i {*u^i}$$

- Energy-momentum tensor: [Brown-York '93]

$$\mathrm{T} = \mathrm{T}(\varepsilon, \chi)\ : \qquad \nabla_i T^{ij} = 0\,, \qquad T^i{}_i = \frac{R}{16\pi G k}$$

III. CB gauge in AdS: asymptotic symmetries

- Residual symmetries: [Ciambelli-Marteau-Petropoulos-Ruzziconi '20]

$$v = \left(\xi^i - \frac{1}{k^2 r}\,\eta * u^i\right)\partial_i + \left(r\,\sigma + \frac{1}{k^2}\left(*u^j\,\partial_j\eta + \Theta^*\eta\right) + \frac{4\pi G}{k^2 r}\,\chi\,\eta\right)\partial_r$$

where $(\mathrm{d}s^2_{\mathrm{bdy}} = g_{ij}\mathrm{d}x^i\mathrm{d}x^j)$

$$\delta_{(\xi,\sigma,\eta)}g_{ij} = \mathcal{L}_\xi g_{ij} + 2\,\sigma\,g_{ij}$$

and

$$\begin{pmatrix} \mathrm{u}' \\ *\mathrm{u}' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} \mathrm{u} \\ *\mathrm{u} \end{pmatrix}$$

- Charges associated with the Weyl–Lorentz symmetries: $(\delta_v\varphi = \hat{\sigma},\ \delta_v\zeta = \hat{\eta})$

$$Q = \frac{1}{8\pi G k^2}\int_0^{2\pi}\mathrm{d}\theta\left[\zeta\,\partial_u\hat{\eta} - \hat{\eta}\,\partial_u\zeta\right]$$

Anomaly in the Lorentz symmetry in the dual theory $(F_{ij} = \partial_i A_j - \partial_j A_i)$

$$\delta_{(\xi,\sigma,\eta)}S_{\mathrm{L}} = \int\left(\eta\,\frac{F}{8\pi G}\right)\mathrm{Vol}_{\partial\mathcal{M}}$$

III. Flat limit and boundary Carroll frames

- Key idea 2: null gauges in AdS admit a proper flat limit

  $\hookrightarrow$ in the flat limit: timelike AdS bdy $\to$ null manifold

  $\hookrightarrow$ bdy metric $\to$ degenerate $\Rightarrow$ Carrollian geometry

- Bulk metric: [Campoleoni-Ciambelli-Marteau-Petropoulos-Siampos '19]

$$\mathrm{d}s^2_{\text{Flat}} = \lim_{k\to 0} \mathrm{d}s^2_{\text{AdS}} = 2\,\mu\,(\mathrm{d}r + r\mathcal{A}) + r^2\,(\mu^*)^2 + 8\pi G\,\mu\,(\epsilon\,\mu + \alpha\,\mu^*)$$

$\hookrightarrow$ small-k behavior for the line element quantities:

$$\mu = \lim_{k\to 0} \frac{\mathrm{u}}{k^2}\,, \quad \mu^* = \lim_{k\to 0} \frac{*\mathrm{u}}{k}\,, \quad \upsilon = \lim_{k\to 0} u\,, \quad \upsilon_* = \lim_{k\to 0} \frac{*u}{k}\,,$$

$$\alpha = \lim_{k\to 0} \frac{\chi}{k}\,, \quad \epsilon = \lim_{k\to 0} \varepsilon\,, \quad \mathcal{A} = \lim_{k\to 0} \mathrm{A} = \mu^*\,\theta^* - \mu\,\theta$$

III. Flat limit: asymptotic symmetries

- Residual symmetries: [Ciambelli-Marteau-Petropoulos-Ruzziconi '20]

$$v = \left(\xi^i - \tfrac{1}{r}\,\lambda\,\upsilon^i_*\right)\partial_i + \left(r\,\sigma + \upsilon^j_*\partial_j\lambda + \theta^*\lambda + \tfrac{4\pi G}{r}\,\alpha\,\lambda\right)\partial_r$$

$\hookrightarrow$ bdy diffeomorphisms $\xi^i(x)$, Weyl rescalings $\sigma(x)$ and Carroll boosts $\lambda(x)$

$$\lambda(x) = \lim_{k\to 0}\frac{\eta(x)}{k}$$

- Conformal gauge: parametrization of the Carrollian dyad ($\beta = \lim_{k\to 0}\frac{\zeta}{k}$)

$$\mu = -e^{\varphi}(du + \beta\,d\theta)\,, \quad \mu^* = e^{\varphi}d\theta$$

Charges associated with the Weyl–boost symmetries: ($\delta_v\varphi = \hat{\sigma}$, $\delta_v\beta = \hat{\lambda}$)

$$Q = \frac{1}{8\pi G}\int_0^{2\pi} d\theta\left[\beta\,\partial_u\hat{\lambda} - \hat{\lambda}\,\partial_u\beta\right]$$

Anomalies: ($\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i$)

$$\delta_{(\xi,\sigma,\lambda)}S_{\mathrm{C}} = \int\left(\lambda\,\frac{\mathcal{F}}{8\pi G}\right)\mathrm{vol}_{\partial\mathcal{M}}$$

Outline

IV. Summary

Main goal:

- Explore the charges of 3D gravity in Weyl–Fefferman–Graham and covariant Bondi gauges

Results:

- Residual symmetries, surface charges and anomalies
- FG fixing can constrain the physical solution space
- New holographic Carrollian prediction

Future possibilities:

- Relate to asymptotic corner group [Donnelly-Freidel '16, Freidel-Geiller-Pranzetti '20, Ciambelli-Leigh-Pai '21]
- Explicit diffeomorphisms, synthesis of other gauge relaxation proposals [Grumiller-Riegler '16, Ciambelli-Marteau-Petropoulos-Ruzziconi '20, Geiller-Goeller-Zwikel '21]
- Extension to higher dimensions [Ciambelli-Leigh '19, Petkou-Petropoulos-Betancour-Siampos '22, Campoleoni-AD-Pekar-Petropoulos-Betancour-Vilatte '23]

IV. Summary

Lewis Carroll, Hermann Weyl, Charles Fefferman and Robin Graham. Created with the assistance of DALL·E 2.

Thank you for listening!

Three-dimensional aspects

- Isometry algebra of AdS$_3$ $= \mathfrak{so}(2,2)$

$$[M_a, M_b] = \epsilon_{abc} M^c, \quad [M_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \left(\frac{\mathcal{G}}{\ell}\right)^2 \epsilon_{abc} M^c$$

- Dreibein or first order formalism ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$)

$$g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b, \qquad de^a + \epsilon^{abc}\,\omega_b \wedge e_c = 0$$

- Einstein gravitation → Chern-Simons gauge theory ($\mathscr{A} = \frac{1}{\mathcal{G}} e^a P_a + \omega^a M_a$)
  [Achucarro-Townsend '86, Witten '88]

$$S_{EH} = \frac{1}{16\pi} \int_M \mathrm{Tr}\left(\mathscr{A} \wedge d\mathscr{A} + \frac{2}{3}\, \mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A}\right)$$

- No local degrees of freedom → no gravitational radiation

  Global properties → asymptotic surface charges [M. Banados (1996)]

Covariant phase space formalism: Chern-Simons

Chern-Simons theory, gauge theory with a simple Lie group G

- CS action

$$S = \int_M L\,, \qquad L = \mathrm{tr}\left(A \wedge \mathrm{d}A + \tfrac{2}{3}\, A \wedge A \wedge A\right)$$

$\mathfrak{g}$-valued differential one-form

$$A = A_\mu{}^a \mathrm{d}x^\mu J_a$$

- Arbitrary field variation $(A \to A + \delta A)$: $\delta L = (\mathrm{eom})\delta A + \mathrm{d}\Theta$

$$\mathrm{eom} = \mathrm{d}A + A \wedge A \approx 0\,, \qquad \Theta = -\mathrm{tr}\,(A \wedge \delta A)$$

Gauge symmetry

$$\delta_\lambda A = I_{V_\lambda} \delta A = \mathrm{d}\lambda + [A, \lambda]$$

- Conserved codimension-2 charge

$$\Omega = \int_{\partial M} \delta\Theta\,, \qquad I_{V_\lambda}\Omega = -\delta H_\lambda\,, \qquad H_\lambda \approx -2\int_{\partial^2 M} \mathrm{tr}\,(\lambda\, \delta A)$$

Geometry of the Weyl-Fefferman-Graham gauge 1/3

WFG gauge is preserved under radial diffeos inducing bdy Weyl transfos

$$\rho \to \rho' = \frac{\rho}{\mathfrak{B}(x)}\,, \qquad x^i \to x'^i = x^i\,,$$

so that the radial expansion transforms Weyl-covariantly

$$\begin{aligned} k_i(\rho, x) &\to {k'}_i(\rho', x') = k_i(\mathfrak{B}(x)\rho', x) - \partial_i \ln \mathfrak{B}(x)\,, \\ h_{ij}(\rho, x) &\to {h'}_{ij}(\rho', x) = h_{ij}(\mathfrak{B}(x)\rho', x)\,, \end{aligned}$$

i.e.

$$k_i^{(2n)}(x) \to k_i^{(2n)}(x)\,\mathfrak{B}(x)^{2n} - \delta_{n,0}\,\partial_i \ln \mathfrak{B}(x)\,, \qquad h_{ij}^{(2n)}(x) \to h_{ij}^{(2n)}(x)\,\mathfrak{B}(x)^{2n-2}$$

The leading term in k_i transforms inhomogeneously, i.e., as a Weyl connection

$$k_i^{(0)} \to k_i^{(0)} - \partial_i \ln \mathfrak{B}\,.$$

Geometry of the Weyl-Fefferman-Graham gauge 2/3

Choice of dual form and vector basis for the WFG metric:

$$E^\rho = \frac{\mathrm{d}\rho}{\rho} - k_i(\rho, x)\mathrm{d}x^i\,, \qquad E^i = \mathrm{d}x^i\,,$$
$$E_\rho = \rho\partial_\rho \equiv D_\rho\,, \qquad E_i = \partial_i + \rho k_i(\rho, x)\partial_\rho \equiv D_i\,.$$

The Lie brackets of the vectors are given by

$$[D_\rho, D_i] = D_\rho k_i\, D_\rho\,, \qquad [D_i, D_j] = f_{ij}\, D_\rho\,,$$

where $f_{ij} \equiv D_i k_j - D_j k_i$ is the curvature associated to k_i.

Coefficients of the bulk Levi-Civita connection ∇ in the frame $\{D_\rho, D_i\}$:

$$\nabla_{D_i} D_j = \Gamma^k_{ij} D_k + \Gamma^\rho_{ij} D_\rho\,.$$

Geometry of the Weyl-Fefferman-Graham gauge 3/3

Taking into account the WFG radial expansions, one obtains at leading order

$$(\Gamma^{(0)})^k_{ij} = \frac{1}{2} h^{kl}_{(0)} \left((\partial_i - 2k^{(0)}_i) h^{(0)}_{jl} + (\partial_j - 2k^{(0)}_j) h^{(0)}_{il} - (\partial_l - 2k^{(0)}_l) h^{(0)}_{ij} \right),$$

$\hookrightarrow$ coefficients of a torsion-free connection with Weyl metricity [G. B. Folland '70]

The induced connection $\nabla^{(0)}$ acts as

$$\nabla^{(0)}_i h^{(0)}_{jk} = 2k^{(0)}_i h^{(0)}_{jk} \,.$$

For a generic Weyl-weight ω_T tensor T of arbitrary type, one can construct the Weyl covariant connection as

$$\hat{\nabla}^{(0)}_i T \equiv \nabla^{(0)}_i T + \omega_T k^{(0)}_i T \,.$$

So the connection $\hat{\nabla}^{(0)}$ is metric and $\hat{\nabla}^{(0)}_i T$ is Weyl covariant.

All geometric quantities built with this connection are Weyl covariant.

WFG Holographic renormalization

Starting from the WFG solution space, the renormalized action is

$$S_{ren} = \frac{1}{16\pi\mathcal{G}}\int \mathrm{d}^3x\,(R+2) + \frac{1}{8\pi\mathcal{G}}\int \mathrm{d}^2x\sqrt{-\gamma}\,(K-1)$$
$$+ \frac{1}{16\pi\mathcal{G}}\int \mathrm{d}^2x\sqrt{-\gamma}\,k_i\,\gamma^{ij}\,k_j + \frac{\rho^2\log\rho}{16\pi\mathcal{G}}\int \mathrm{d}^2x\sqrt{-\gamma}\,\hat{R}^{(0)}\,,$$

where $n_\mu = -\sqrt{-\gamma}\delta^\rho_\mu$, $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and $K = g^{\mu\nu}\nabla_\mu n_\nu$.
The renormalized presymplectic potential is

$$\Theta_{ren} = -\sqrt{-h^{(0)}}\left(\frac{1}{2}T^{ij}\delta h^{(0)}_{ij} - J^i\delta k^{(0)}_i\right)$$

where

$$T^{ij} = -\frac{2}{\sqrt{-h^{(0)}}}\frac{\delta S_{ren}}{\delta h^{(0)}_{ij}} \approx \frac{1}{8\pi\mathcal{G}}\left(h^{ij}_{(2)} + \frac{1}{2}h^{ij}_{(0)}R^{(0)} + \hat{\nabla}^{(i}_{(0)}k^{j)}_{(0)}\right),$$

$$J^i = \frac{1}{\sqrt{-h^{(0)}}}\frac{\delta S_{ren}}{\delta k^{(0)}_i} \approx \frac{1}{8\pi\mathcal{G}}\,k^i_{(0)}\,.$$

WFG Holographic Ward identities

Evaluating the variation of the on-shell action under the action of boundary diffeomorphisms, one can obtain

$$\nabla_i^{(0)} T^i{}_j = J^i f_{ij}^{(0)} + \nabla_i^{(0)} J^i k_j^{(0)},$$

where $f_{ij}^{(0)} = \nabla_i^{(0)} k_j^{(0)} - \nabla_j^{(0)} k_i^{(0)}$.

The variation of the action under boundary Weyl transformations yields

$$T^i{}_i + \hat{\nabla}_i^{(0)} J^i = \frac{c}{24\pi} \hat{R}^{(0)},$$

which unveils the presence of a holographic Weyl anomaly [Henningson-Skenderis '98].

WFG boundary term

One can always add a finite boundary counterterm to the action as

$$\bar{S}_{ren} = S_{ren} + S_\circ , \qquad S_\circ = \int \mathrm{d}^2 x \, L_\circ[h_{ij}^{(0)}, k_i^{(0)}]$$

where $L_\circ[h_{ij}^{(0)}, k_i^{(0)}]$ is a boundary Lagrangian involving the boundary geometry

$$L_\circ = \lim_{\rho \to 0} \left[\frac{1}{16\pi\mathcal{G}} k_i \gamma^{ij} \partial_j \sqrt{-\gamma} \right] = \frac{1}{16\pi\mathcal{G}} k_i^{(0)} h_{(0)}^{ij} \partial_j \sqrt{-h^{(0)}} \, .$$

The renormalized presymplectic potential reads

$$\bar{\Theta}_{ren} = -\sqrt{-h^{(0)}} \left(\frac{1}{2} \bar{T}^{ij} \delta h_{ij}^{(0)} - \bar{J}^i \delta k_i^{(0)} \right) ,$$

where

$$\bar{T}^{ij} = T^{ij} + J^{(i} \partial^{j)} \sqrt{-h^{(0)}} + \frac{1}{2} h^{ij} \nabla_k J^k \, , \qquad \bar{J}^i = J^i + \frac{1}{16\pi\mathcal{G}} \partial^i \log \sqrt{-h^{(0)}} \, .$$

WFG corner term

One can add a finite corner term to the renormalized action:

$$\tilde{S}_{ren} = S_{ren} + S_C\,, \qquad S_C = \int \mathrm{d}^2 x\, \partial_i L^i_C[h^{(0)}_{ij}, k^{(0)}_i]$$

where $L^i_C[h^{(0)}_{ij}, k^{(0)}_i]$ is a corner Lagrangian involving the boundary geometry

$$L^i_C = \lim_{\rho\to 0}\left[-\frac{1}{16\pi\mathcal{G}}\sqrt{-\gamma}\gamma^{ij}k_j\right] = -\frac{1}{16\pi\mathcal{G}}\sqrt{-h^{(0)}}h^{ij}_{(0)}k^{(0)}_j\,.$$

Then the renormalized symplectic term is

$$\tilde{\Theta}_{ren} = \sqrt{-h^{(0)}}\left(-\frac{1}{2}\tilde{T}^{ij}\delta h^{(0)}_{ij} + J^i\delta K^{(0)}_i\right),$$

where

$$\tilde{T}^{ij} = T^{ij} + \frac{1}{2}h^{ij}_{(0)}\hat{\nabla}^{(0)}_k J^k\,, \qquad K^{(0)}_i = k^{(0)}_i - \frac{1}{2}\partial_i \ln\sqrt{-h^{(0)}}\,.$$

Link with the Bondi gauge

In the Bondi gauge the metric is given by

$$\mathrm{d}s^2 = \frac{V}{r} e^{2\beta} \mathrm{d}u^2 - 2 e^{2\beta} \mathrm{d}u \, \mathrm{d}r + r^2 e^{2\varphi} (\mathrm{d}\phi - U \mathrm{d}u)^2$$

In the fluid/gravity derivative expansion (DE)

$$\mathrm{d}s^2 = 2\ell^2 u_\mu \mathrm{d}x^\mu (\mathrm{d}r + rA_\nu \mathrm{d}x^\nu) + r^2 g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + 8\pi\mathcal{G}\ell^4 u_\mu \mathrm{d}x^\mu (\epsilon u_\nu \mathrm{d}x^\nu + \chi * u_\nu \mathrm{d}x^\nu)$$

Bondi gauge as a sub-gauge of the DE: $u_\phi = 0$

$\rightarrow$ constraint for definite gauge fixing

$\hookrightarrow$ identification between the DE and Bondi solution spaces [Ciambelli-Marteau-Petropoulos-Ruzziconi '20]

Covariant Bondi gauge in AdS: residual symmetries

- Asymptotic Killing vectors: [Ciambelli-Marteau-Petropoulos-Ruzziconi '20]

$$v = \left(\xi^\mu - \frac{1}{k^2 r}\,\eta *u^\mu\right)\partial_\mu + \left(r\,\sigma + \frac{1}{k^2}\left(*u^\nu\,\partial_\nu\eta + \Theta^*\eta\right) + \frac{4\pi\mathcal{G}}{k^2 r}\,\chi\,\eta\right)\partial_r$$

$\hookrightarrow$ bdy diffeomorphisms $\xi^\mu(x)$, Weyl rescalings $\sigma(x)$ and Lorentz boosts $\eta(x)$

$$\delta_{(\xi,\sigma,\eta)}\mathrm{u} = \mathcal{L}_\xi \mathrm{u} + \sigma\,\mathrm{u} + \eta *\mathrm{u}\,, \quad \delta_{(\xi,\sigma,\eta)}*\mathrm{u} = \mathcal{L}_\xi *\mathrm{u} + \sigma *\mathrm{u} + \eta\,\mathrm{u}$$

where

$$\delta_{(\xi,\sigma,\eta)} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} + 2\,\sigma\, g_{\mu\nu}$$

and

$$\begin{pmatrix}\mathrm{u}'\\ *\mathrm{u}'\end{pmatrix} = \begin{pmatrix}\cosh\eta & \sinh\eta\\ \sinh\eta & \cosh\eta\end{pmatrix}\begin{pmatrix}\mathrm{u}\\ *\mathrm{u}\end{pmatrix}$$

- Question: What are the asymptotic symmetries?

Covariant Bondi gauge in AdS: symplectic structure

- Einstein–Hilbert presymplectic potential: [Iyer-Wald '94]

$$\Theta_{\text{EH}}[G;\delta G] = \frac{\sqrt{-G}}{32\pi\mathcal{G}} \left[\nabla^N \delta G_{PN}\, G^{PM} - \nabla^M \delta G_{PN}\, G^{PN}\right] \epsilon_{MQS}\, \mathrm{d}x^Q \wedge \mathrm{d}x^S$$

Radial divergences: need for renormalization

$$\Theta^{(r)}_{\text{EH}}[G;\delta G] = r^2\,\Theta_{(2)} + r\,\Theta_{(1)} + \Theta_{(0)} + \mathcal{O}\left(r^{-1}\right)$$

Ambiguous definition:

$$\Theta_{\text{EH}}[G;\delta G] \to \Theta_{\text{EH}}[G;\delta G] + \delta Z[G] - \mathrm{d}\,Y[G;\delta G]$$

- Choices of prescription:
  i. same results as obtained in FFG [de Haro-Solodukhin-Skenderis (2000)]
  ii. presymplectic potential that remains finite in the flat-space limit

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  i. ~~same results as obtained in FFG~~ [de Haro-Solodukhin-Skenderis (2000)]

  ii. presymplectic potential that remains finite in the flat-space limit

Covariant Bondi gauge in AdS: surface charges

- Conformal gauge: conformally flat bdy metric $(x^{\pm} = \phi \pm k\, u)$

$$\mathrm{d}s^2 = \mathrm{e}^{2\varphi}\, \mathrm{d}x^+ \mathrm{d}x^-$$

Parametrization of the Cartan frame: $(\varphi = \varphi(x^+, x^-),\, \zeta = \zeta(x^+, x^-))$

$$\mathrm{u} = -\frac{k}{2}\, \mathrm{e}^{\varphi} \left(\mathrm{e}^{\zeta}\, \mathrm{d}x^+ - \mathrm{e}^{-\zeta}\, \mathrm{d}x^- \right), \qquad *\mathrm{u} = \frac{k}{2}\, \mathrm{e}^{\varphi} \left(\mathrm{e}^{\zeta}\, \mathrm{d}x^+ + \mathrm{e}^{-\zeta}\, \mathrm{d}x^- \right)$$

- Charges associated with the Weyl–Lorentz symmetries: $(\delta_v \varphi = \varpi,\, \delta_v \zeta = h)$

$$Q_{(\varpi, h)} = \frac{1}{4\pi \mathcal{G} k} \int_0^{2\pi} \mathrm{d}\phi \left(h\, (\partial_- - \partial_+)\, \zeta \right)$$

$\hookrightarrow$ integrable and non-conserved: Lorentz is anomalous, Weyl is pure gauge

Covariant Bondi gauge in AdS: anomalies

- Anomaly in the Lorentz symmetry in the dual theory ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

$$\delta_{(\xi,\sigma,\eta)} S_{\rm L} = \int \left(\eta \, \frac{F}{8\pi \mathcal{G}} \right) \mathsf{Vol}_{\partial \mathcal{M}}$$

$\hookrightarrow$ flat limit: yes

- If we choose the first prescription $\rightarrow$ anomaly in the Weyl symmetry in the dual theory [Alessio-Barnich-Ciambelli-Mao-Ruzziconi '20]

$$\delta_{(\xi,\sigma,\eta)} S_{\rm W} = \int \left(\sigma \, \frac{R}{8\pi \mathcal{G}} \right) \mathsf{Vol}_{\partial \mathcal{M}}$$

$\hookrightarrow$ flat limit: no

- Displacement of the anomaly: two different representatives in the same cohomology class $\rightarrow$ BRST formulation [Ciambelli-Leigh-Jia (to appear)]

Recall of relevant aspects of the Fefferman–Graham gauge

Variation of the on-shell action: [de Haro-Solodukhin-Skenderis (2000)]

$$\delta S = \frac{1}{2} \int_{\partial} {}^{D}x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}$$

The bulk metric induces a conformal class of metrics on the boundary: the boundary metric should be understood as a representative. Moving from one representative to another is not innocuous: [Alessio-Barnich-Ciambelli-Mao-Ruzziconi (2020)].

Evaluation of the on-shell action on a bulk diffeomorphism:

$$\delta_{(\xi,\sigma)} S \sim c \int_{\partial} {}^{2}x \sqrt{-g}\, \sigma R\,.$$

A Cartan frame could be introduced in the FFG gauge:

$$g_{\mu\nu} = \ell^2 \left(-u_\mu u_\nu + {*u_\mu} {*u_\nu}\right).$$

However, the variation of the action would remain sensitive to the energy–momentum tensor and to the metric variation, but not separately to δu and $\delta {*u}$.

Boundary energy–momentum tensor

Brown–York energy–momentum tensor: [Campoleoni-Ciambelli-Marteau-Petropoulos-Siampos ’19]

$$T_{\mu\nu} = \frac{1}{2k}\left(\widetilde{T}_{\mu\nu} + \widehat{T}_{\mu\nu}\right),$$

where

$$\widetilde{\mathrm{T}} = \frac{\varepsilon}{k^2}\left(\mathrm{u}^2 + *\mathrm{u}^2\right) + \frac{\chi}{k^2}\left(\mathrm{u} *\mathrm{u} + *\mathrm{u}\,\mathrm{u}\right) + \frac{R}{8\pi\mathcal{G}k^2} *\mathrm{u}^2,$$

$$\widehat{\mathrm{T}} = \frac{1}{8\pi G k^4}\left(u^\mu\partial_\mu\Theta + *u^\mu\partial_\mu\Theta^* - \frac{k^2}{2}R\right)\left(\mathrm{u}^2 + *\mathrm{u}^2\right) - \frac{1}{4\pi G k^4} *u^\mu\partial_\mu\Theta\left(\mathrm{u} *\mathrm{u} + *\mathrm{u}\,\mathrm{u}\right)$$

such that

$$\nabla_\mu T^{\mu\nu} = 0, \qquad T^\mu{}_\mu = \frac{R}{16\pi\mathcal{G}k}.$$

These equations can be spelled in terms of ε, χ, and the Cartan frame as

$$u^\mu\left(\partial_\mu + 2A_\mu\right)\varepsilon = - *u^\mu\left(\partial_\mu + 2A_\mu\right)\left(\chi - \frac{F}{4\pi\mathcal{G}}\right),$$

$$u^\mu\left(\partial_\mu + 2A_\mu\right)\chi = - *u^\mu\left(\partial_\mu + 2A_\mu\right)\varepsilon.$$

Anomalous holographic Ward identities – AdS

Variation of the on-shell action:

$$\delta S = \int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{-g}\,(J^\mu\,\delta u_\mu + J_*^\mu\,\delta{*u_\mu})\,,$$

where the couple of currents

$$J^\mu = -\frac{1}{k^2}\,T^{\mu\nu}u_\nu + \frac{1}{16\pi\mathcal{G}k^5}\,u^\mu(\Theta^2 - \Theta^{*2}) - \frac{1}{8\pi\mathcal{G}k^3}\,\mathcal{E}^{\mu\nu}\partial_\nu\Theta^*\,,$$

$$J_*^\mu = \frac{1}{k^2}\,T^{\mu\nu}{*u_\nu} - \frac{1}{16\pi\mathcal{G}k^5}\,{*u^\mu}(\Theta^2 - \Theta^{*2}) + \frac{1}{8\pi\mathcal{G}k^3}\,\mathcal{E}^{\mu\nu}\partial_\nu\Theta\,.$$

↪ Role analogous to the energy–momentum tensor in the FFG gauge

$$\mathscr{T}^\mu{}_\nu = J^\mu u_\nu + J_*^\mu\,{*u_\nu}\,.$$

↪ Holographic Ward identities: [Bertlmann '96]

$$\nabla_\mu\mathscr{T}^{\mu\nu} = -\frac{1}{8\pi\mathcal{G}k}\,F^{\mu\nu}A_\mu\,, \qquad \mathscr{T}^\mu{}_\mu = 0\,, \qquad \mathscr{T}_{[\mu\nu]} = \frac{1}{16\pi\mathcal{G}k}\,F_{\mu\nu}\,.$$

Holographic renormalization 1/4

The conformal boundary being at $r \to \infty$, we focus on the r-component of the Einstein–Hilbert presymplectic potential in this limit:

$$\Theta^{(r)}_{\text{EH}}[G;\delta G] = r^2\,\Theta_{(2)} + r\,\Theta_{(1)} + \Theta_{(0)} + \mathcal{O}\left(r^{-1}\right)\ ,$$

where, in terms of boundary data, we have

$$\Theta_{(2)} = -\frac{k}{8\pi\mathcal{G}}\left(\delta\ln\sqrt{-g}\right)\text{Vol}_{\partial\mathcal{M}}\ ,$$

$$\Theta_{(1)} = \frac{1}{16\pi\mathcal{G}k}\left[-2\,\frac{\delta(\Theta\sqrt{-g})}{\sqrt{-g}} - \nabla_\mu\delta u^\mu\right]\text{Vol}_{\partial\mathcal{M}}\ ,$$

$$\begin{aligned}\Theta_{(0)} = \Bigg(&\frac{1}{2}\,T^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2k\sqrt{-g}}\,\delta(\sqrt{-g}\,\varepsilon) + \frac{1}{16\pi\mathcal{G}k^3\sqrt{-g}}\,\delta\left[\sqrt{-g}\left(\Theta^2 - \Theta^{*2}\right)\right]\\ &- \frac{1}{16\pi\mathcal{G}k\sqrt{-g}}\,\delta(\sqrt{-g}R) + \frac{1}{8\pi\mathcal{G}k^3}\,\nabla_\mu(\delta\Theta\,u^\mu)\\ &- \frac{1}{16\pi\mathcal{G}k^3}\,\nabla_\mu\left[\delta(\Theta^* * u^\mu)\right]\Bigg)\text{Vol}_{\partial\mathcal{M}}.\end{aligned}$$

Holographic renormalization 2/4

This potential diverges, but both divergent terms are pure-ambiguity:

$$\Theta_{\text{ren}}[G;\delta G] = \Theta^{(r)}_{\text{EH}}[G;\delta G] + \delta Z[G] - Y[G;\delta G]\,,$$

where Z and Y have their own large-r expansions

$$Z[G] = r^2\, Z_{(2)} + r\, Z_{(1)} + Z_{(0)}\,, \qquad Y[G;\delta G] = r\, Y_{(1)} + Y_{(0)}\,,$$

whose divergent pieces are

$$Z_{(2)} = \frac{k}{8\pi G}\,\text{Vol}_{\partial\mathcal{M}}\,, \qquad Z_{(1)} = \frac{1}{8\pi G k}\,\Theta\,\text{Vol}_{\partial\mathcal{M}}\,, \qquad Y_{(1)} = \frac{1}{16\pi G k}\,{}_{\mu\alpha}\delta u^{\alpha} x^{\mu}\,.$$

At this stage, a choice is expected for the zeroth order of the ambiguities.

The two choices that we propose are

$$Y_{(0)} = -\frac{\mathcal{E}_{\mu\alpha}}{8\pi\mathcal{G}k^3}\left(u^{\alpha}\delta\Theta - \frac{\delta(*u^{\alpha}\Theta^*)}{2}\right)\text{d}x^{\mu}, Z_{(0)} = \left(-\frac{\varepsilon}{2k} - \frac{\Theta^2 - \Theta^{*2} + k^2 R}{16\pi\mathcal{G}k^3}\right)\text{Vol}_{\partial$$

$$Y_{(0)} = \frac{\mathcal{E}_{\mu\alpha}}{16\pi\mathcal{G}k^3}\left(\delta * u^{\alpha}\,\Theta^* - * u^{\alpha}\,\delta\Theta^*\right)\text{d}x^{\mu}, \quad Z_{(0)} = -\frac{\varepsilon}{2k}\,\text{Vol}_{\partial\mathcal{M}}\,.$$

Weyl. This renormalized presymplectic potential matches that obtained in the FFG gauge [de Haro-Solodukhin-Skenderis (2000)]

$$\Theta^{\mathrm{W}}_{\text{ren}}[G;\delta G]\big|_{\partial\mathcal{M}} = \frac{1}{2}\,T^{\mu\nu}\delta g_{\mu\nu}\text{Vol}_{\partial\mathcal{M}} = \frac{1}{k^2}\,T^{\mu\nu}\big(-u_\mu\,\delta u_\nu + *u_\mu\,\delta *u_\nu\big)\text{Vol}_{\partial\mathcal{M}}\,.$$

It vanishes also for milder boundary conditions, since it is proportional to the variation of the boundary metric, meaning that one has to fix the Cartan frame only up to Lorentz transformations.

Lorentz. The prescription leads to the renormalized presymplectic potential

$$\Theta^{\mathrm{L}}_{\text{ren}}[G;\delta G]\big|_{\partial\mathcal{M}} = \big(J^\mu\,\delta u_\mu + J^\mu_*\,\delta *u_\mu\big)\text{Vol}_{\partial\mathcal{M}}\,,$$

where we have introduced the currents

$$J^\mu = -\frac{1}{k^2}\,T^{\mu\nu}u_\nu + \frac{1}{16\pi\mathcal{G}k^5}\,u^\mu(\Theta^2 - \Theta^{*2}) - \frac{1}{8\pi\mathcal{G}k^3}\,\mathcal{E}^{\mu\nu}\partial_\nu\Theta^*\,,$$
$$J^\mu_* = \frac{1}{k^2}\,T^{\mu\nu}*u_\nu - \frac{1}{16\pi\mathcal{G}k^5}\,*u^\mu(\Theta^2 - \Theta^{*2}) + \frac{1}{8\pi\mathcal{G}k^3}\,\mathcal{E}^{\mu\nu}\partial_\nu\Theta\,.$$

A milder condition can be imposed to get stationarity of the action: this time it is the conformal class of the Cartan frame that needs to be fixed.

AdS. Renormalized presymplectic 2-form of the second prescription:

$$\omega^{\mathrm{L}}_{\mathrm{ren}}\big|_{\partial\mathcal{M}} = \frac{1}{\sqrt{-g}}\Big(\delta(\sqrt{-g}\,J^\mu)\wedge\delta u_\mu + \delta(\sqrt{-g}\,J^\mu_*)\wedge\delta{*u_\mu}\Big)\mathsf{Vol}_{\partial\mathcal{M}}$$
$$= \omega^{\mathrm{W}}_{\mathrm{ren}}\big|_{\partial} + \frac{1}{8\pi\mathcal{G}k^3}\nabla_\mu\left[\frac{\delta(\sqrt{-g}\,u^\mu)}{\sqrt{-g}}\wedge\delta\Theta - \frac{\delta(\sqrt{-g}\,{*u^\mu})}{\sqrt{-g}}\wedge\delta\Theta^*\right]\mathsf{Vol}_{\partial\mathcal{M}}\,,$$

where

$$\omega^{\mathrm{W}}_{\mathrm{ren}}\big|_{\partial\mathcal{M}} = \frac{1}{2\sqrt{-g}}\,\delta\big(\sqrt{-g}\,T^{\mu\nu}\big)\wedge\delta g_{\mu\nu}\mathsf{Vol}_{\partial\mathcal{M}}\,.$$

It is this corner term that renders the presymplectic form finite in the $k\to0$ limit.

Flat. Flat limit of the renormalized presymplectic current:

$$\omega^{\mathrm{C}}_{\mathrm{ren}}|_{\partial\mathcal{M}} = \lim_{k\to0}\omega^{\mathsf{L}}_{\mathrm{ren}}|_{\partial\mathcal{M}} = \mathscr{D}^{-1}\Big(\delta(\mathscr{D}\,j^\mu)\wedge\delta\mu^*_\mu + \delta(\mathscr{D}\,j^\mu_*)\wedge\delta\mu_\mu\Big)\mathsf{vol}_{\partial\mathcal{M}}\,,$$

where the density is

$$\mathscr{D} = |\varepsilon^{\mu\nu}\mu_\mu\mu^\star_\nu| = \lim_{k\to0}\frac{\sqrt{-g}}{k}\,.$$

Anomalous holographic Ward identities – flat

Flat limit of the renormalized presymplectic potential:

$$\Theta^{\mathrm{C}}_{\mathrm{ren}}[G;\delta G]|_{\partial\mathcal{M}} = \lim_{k\to 0}\Theta^{\mathrm{L}}_{\mathrm{ren}}[G;\delta G]|_{\partial\mathcal{M}} = \left(j^{\mu}\,\delta\mu_{\mu} + j_{*}^{\mu}\,\delta\mu^{*}_{\mu}\right)\mathrm{vol}_{\partial\mathcal{M}}\,,$$

where the couple of currents

$$j = \lim_{k\to 0} k^3 J = \frac{1}{2}\,\epsilon\,\upsilon + \frac{1}{8\pi\mathcal{G}}\,\upsilon_{*}\,\mathcal{F}\,,$$
$$j_{*} = \lim_{k\to 0} k^2 J_{*} = \frac{1}{2}\,\epsilon\,\upsilon_{*} + \frac{1}{2}\,\alpha\,\upsilon\,.$$

↪ Carrollian energy–momentum tensor:

$$t^{\mu}{}_{\nu} = \lim_{k\to 0} k\,\mathscr{T}^{\mu}{}_{\nu} = j^{\mu}\,\mu_{\nu} + j_{*}^{\mu}\,\mu^{*}_{\nu}\,.$$

↪ Holographic Ward identities: $(D_{\mu}\,t^{\mu}{}_{\nu} = \lim_{k\to 0}\nabla_{\mu}\,\mathscr{T}^{\mu}{}_{\nu})$

$$D_{\mu}t^{\mu}{}_{\nu} = -\frac{1}{8\pi\mathcal{G}}\,\mathcal{F}_{\mu\nu}\,\mathcal{A}^{\mu}\,, \qquad t^{\mu}{}_{\mu} = 0\,, \qquad t^{\mu}{}_{\nu}\mu^{*}_{\mu}\,\upsilon^{\nu} = -\frac{\mathcal{F}}{8\pi\mathcal{G}}\,.$$

Chern–Simons formulation

The isometry algebra of AdS_3, i.e. the algebra $\mathfrak{so}(2,2)$, reads:

$$[M_B, M_C] = \epsilon_{BCD} M^D , \quad [M_B, P_C] = \epsilon_{BCD} P^D , \quad [P_B, P_C] = (k\, G)^2 \, \epsilon_{BCD} M^D ,$$

We introduce a differential one-form, valued in this algebra:

$$\mathscr{A} = \frac{1}{\mathcal{G}} \left(E_N{}^B P_B + \omega_N{}^B M_B \right) \mathrm{d}x^N .$$

Up to bdy terms, one can rewrite the three-dimensional Einstein–Hilbert action:

$$S_{\text{EH}} = \frac{1}{16\pi} \int \mathrm{Tr} \left(\mathscr{A} \wedge \mathrm{d}\mathscr{A} + \frac{2}{3} \mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A} \right) ,$$

$$\mathrm{Tr}\,(M_B M_C) = \mathrm{Tr}\,(P_B P_C) = 0 \,, \qquad \mathrm{Tr}\,(M_B P_C) = \eta_{BC} \,.$$

For $k \neq 0$ one can take advantage of $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$:

$$S_{\text{EH}} = S_{\text{CS}}\left[\, A \,\right] - S_{\text{CS}}\left[\, \widetilde{A} \,\right] ,$$

$$S_{\text{CS}}[A] = \frac{1}{16\pi \mathcal{G} k} \int \mathrm{Tr} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) .$$

Chern–Simons formulation: AdS charges 1/2

We choose the $\mathfrak{sl}(2,\mathbb{R})$ basis

$$[J_m, J_n] = (m-n)\, J_{m+n}\,, \qquad m,n \in \{-1,0,1\}\,,$$

and implement the radial dependence of the connection A as a gauge transformation:

$$A(x^+,x^-,r) = b^{-1}(r)\, a_\mu(x^+,x^-)\, b(r)\, \mathrm{d}x^\mu + b^{-1}(r)\partial_r\, b(r)\, \mathrm{d}r\,,$$

with $b(r)$ suitable $SL(2,\mathbb{R})$ group element

$$b(r) = \exp\left(r\, k\, J_{-1}\right)\,.$$

Then we have for the first gauge copy

$$a_+ = \mathrm{e}^{\varphi+\zeta}\, J_1 - \mathrm{e}^{-(\varphi+\zeta)}\left(\ell_+ - (\partial_+\zeta)^2 + \partial_+^2\zeta\right) J_{-1} + \partial_+(\varphi-\zeta)\, J_0\,,$$

$$a_- = -\mathrm{e}^{-(\varphi+\zeta)}\,\partial_+\partial_-\zeta\, J_{-1} + \partial_-(\varphi+\zeta)\, J_0\,.$$

The parameters of the gauge transformations, $\mathrm{d}A = \mathrm{d}\Lambda + [A,\Lambda]$, are

$$\Lambda(x^+,x^-,r) = b^{-1}(r)\left(\sum_{m=-1}^{+1} \epsilon^m(x^+,x^-)\, J_m\right) b(r)\,.$$

The associated surface charges are obtained by integrating, if possible, the following variations calculated at fixed value of the coordinate u:

$$\delta Q[\Lambda] = -\frac{1}{8\pi \mathcal{G} k} \int_0^{2\pi} \mathrm{d}\phi \, \mathrm{Tr}\left[\Lambda \, \delta A_\phi\right] = -\frac{1}{8\pi \mathcal{G} k} \int_0^{2\pi} \mathrm{d}\phi \, \mathrm{Tr}\left[b \, \Lambda \, b^{-1} \left(\delta a_+ + \delta a_-\right)\right],$$

$\hookrightarrow$ the radial coordinate is gauge out $\Rightarrow$ no radial divergences

The total surface charges are then

$$Q_{\text{tot}}[\Lambda, \widetilde{\Lambda}] = Q[\Lambda] - \widetilde{Q}[\widetilde{\Lambda}] = -\frac{1}{8\pi \mathcal{G} k} \int_0^{2\pi} \mathrm{d}\phi \left[\ell_+ \, Y^+ - \ell_- \, Y^- + 2 \, \zeta \left(\partial_+ - \partial_-\right) h\right],$$

which are identical to those obtained in the metric formulation.

Chern–Simons formulation: flat charges

We choose the $\mathfrak{iso}(1,2)$ basis ($m, n \in \{-1, 0, 1\}$)

$$[M_m, M_n] = (m-n)\, M_{m+n}\,, \quad [M_m, P_n] = (m-n)\, P_{m+n}\,, \quad [P_m, P_n] = 0\,,$$

and we obtain the following CS connection:

$$\begin{aligned} a_\phi &= \frac{e^{-\varphi}}{\sqrt{2}} \Big(4\pi G\, \varepsilon_0 - \frac{1}{2}(\partial_u\beta)^2 + \partial_u\partial_\phi\beta\Big) M_1 - \Big(\partial_\phi\varphi - \partial_u\beta\Big) M_0 - \frac{e^{\varphi}}{\sqrt{2}} M_{-1} \\ &+ \frac{e^{\varphi}\,\beta}{\sqrt{2}} P_{-1} + \frac{e^{-\varphi}}{\sqrt{2}} \Big(4\pi G\,(\alpha_0 - u\,\partial_\phi\varepsilon_0) - \partial_\phi^2\beta + \partial_u\beta\,\partial_\phi\beta \\ &+ \frac{\beta}{2}\big(8\pi G\,\varepsilon_0 - (\partial_u\beta)^2 + 2\,\partial_u\partial_\phi\beta\big)\Big) P_1\,, \\ a_u &= \frac{e^{-\varphi}}{\sqrt{2}} \Big[\partial_u^2\beta\, M_1 - \Big(4\pi G\,\varepsilon_0 - \frac{1}{2}(\partial_u\beta)^2 + \partial_u\partial_\phi\beta - \beta\,\partial_u^2\beta\Big) P_1\Big] \\ &- \partial_u\varphi\, M_0 + \frac{e^{\varphi}}{\sqrt{2}} P_{-1}\,. \end{aligned}$$

The total surface charges are then

$$\delta Q_{\text{tot}}[\Lambda] = \int_0^{2\pi} d\phi \left[\frac{1}{2}\left(H\,\delta\varepsilon_0 - Y\,\delta\alpha_0\right) + \frac{1}{4\pi\mathcal{G}}\,\partial_u\tilde{h}\,\delta\beta\right].$$

Chern–Simons formulation: boundary term

AdS. One can choose a different boundary term such that

$$S_{\text{tot}}\left[A, \widetilde{A}\right] = S_{\text{EH}} + \frac{1}{16\pi\mathcal{G}k}\int \mathrm{d}^2x\,\mathrm{Tr}\left(A_u\,A_\phi - \widetilde{A}_u\,\widetilde{A}_\phi\right),$$

and its on-shell variation gives

$$\delta S_{\text{tot}}\left[A, \widetilde{A}\right] = \frac{1}{2\pi\mathcal{G}k}\int \delta\zeta\,\mathrm{e}^{-2\varphi}\,\partial_+\partial_-\zeta\,\mathrm{Vol}_{\partial\mathcal{M}}\,.$$

It corresponds to the “Lorentz” presymplectic potential in conformal gauge.

Flat. In the flat limit, the bdy term to be added is

$$S_{\text{bdy}}[\mathscr{A}] = \frac{1}{8\pi\mathcal{G}}\int \mathrm{d}^2x\,\mathrm{Tr}\big(\mathscr{A}_\phi\,\mathscr{A}_u\big)$$

and the on-shell variation of the total action is

$$\delta S_{\text{tot}}[\mathscr{A}] = \delta S_{\text{EH}}[\mathscr{A}] + \delta S_{\text{bdy}}[\mathscr{A}] = -\frac{1}{8\pi\mathcal{G}}\int \delta\beta\,\mathrm{e}^{-2\varphi}\,\partial_u^2\beta\,\mathrm{vol}_{\partial\mathcal{M}}\,.$$

It corresponds to the “Carroll-boost” presymplectic potential in conformal gauge.

Asymptotic symmetry algebra

The charges generate global symmetries when acting on a generic functional F of the phase space as $\delta_{(\Lambda,\tilde{\Lambda})} F = \{Q_{\text{tot}}[\Lambda, \widetilde{\Lambda}], F\}$.

AdS. The non-vanishing brackets are ($c = \frac{3}{2k\mathcal{G}}$)

$$\begin{aligned} \mathrm{i}\,\{L^{\pm}_p, L^{\pm}_q\} &= (p-q)\,L^{\pm}_{p+q} + \frac{c}{12}\,p^3\,\delta_{p+q,0}\,,\\ \mathrm{i}\,\{Z_{pq}, Z_{rs}\} &= -\frac{c}{3}\,(r-q)\,\mathrm{e}^{2\mathrm{i}k(q+s)u}\,\delta_{p+r,q+s}\,. \end{aligned}$$

Flat. The non-vanishing brackets are ($c_M = \frac{3}{\mathcal{G}}$)

$$\begin{aligned} \mathrm{i}\,\{Y_p, Y_q\} &= (p-q)\,Y_{p+q}\,,\\ \mathrm{i}\,\{Y_p, T_q\} &= (p-q)\,T_{p+q} + \frac{c_M}{12}\,p^3\,\delta_{p+q,0}\,,\\ \mathrm{i}\,\{B_{pq}, B_{rs}\} &= -\frac{c_M}{6}\,(r-q)\,\mathrm{e}^{2\mathrm{i}(q+s)u}\,\delta_{p+r,q+s}\,. \end{aligned}$$