Singular Supertranslations and Boundary Chern-Simons Theory

Sangmin Choi

3rd Carroll Workshop, Thessaloniki (6 Oct 2023)
Based on 2212.00170, 2205.07923 with R Akhoury and M Perry.

European Research Council
Established by the European Commission

ERC STARTING GRANT HOLOHAIR 852386

Roadmap

1. Review

- BMS supertranslation
- Supertranslation with a black hole
- Dual supertranslation

2. Main story

- Singular supertranslation
- Boundary Chern-Simons theory

BMS Supertranslation

- 4D asymptotically flat spacetimes are parametrized by Bondi coordinates near $\mathcal{I}^+$:

$$
\begin{aligned}
ds^2 = &-du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\
&+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^z C_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} + \cdots
\end{aligned}
$$

- Diffeomorphisms that preserve Bondi gauge:

BMS group = Lorentz & **super**translation

- Supertranslation generalizes spacetime translation:
  - A supertranslation is parametrized by a function $f(\theta, \phi)$ on the sphere.
  - For smooth functions there is one generator for each spherical harmonics $Y_m^\ell(\theta, \phi)$.
  - The 4 lowest harmonics $\ell = 0, 1$ correspond to spacetime translations.

- One can also generalize Lorentz transformations:

Extended/Generalized BMS group = superrotations & supertranslation

[Bondi, van der Burg, Metzner, Sachs]

BMS Supertranslation

- Supertranslation is an *asymptotic* symmetry of the theory, as opposed to an actual symmetry or a metric isometry.

- It is a "symmetry" of the S-matrix:

$$\langle \text{out}|[Q,S]|\text{in}\rangle = 0 \qquad\qquad [Q,S] \equiv Q^+ S - S Q^-$$

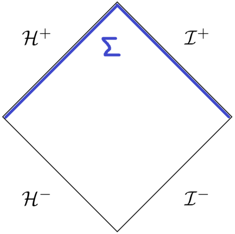
  In the momentum Fock basis, supertranslation charge conservation law corresponds to Weinberg's soft graviton theorem.

- Supertranslation is *physical*, unlike a local gauge symmetry which is a redundancy in the description of the theory.

[Strominger, He, Lysov, Mitra, Akhoury, Choi, Kol, . . .]

Supertranslation with a Black Hole

- In the presence of a Schwarzschild black hole, there are horizon supertranslations and Cauchy data on $\mathcal{H}^+$ as well.



- Generators break into two parts

$$Q^{\Sigma}[f] = Q^{\mathcal{H}^+}[f] + Q^{\mathcal{I}^+}[f]$$

$Q^{\mathcal{H}^+}[f]$ generates supertranslations on $\mathcal{H}^+$.

[Hawking, Perry, Strominger]

Supertranslation with a Black Hole

- Linear metric perturbation h_{AB} (analogous to C_{AB} on $\mathcal{I}^+$)

$$\{\partial_v h_{AB}(v,z,\bar{z}), h_{CD}(v',z',\bar{z}')\} = 32\pi M^2 \gamma_{ABCD}\delta(v-v')\delta^2(z-z')$$

where $\gamma_{ABCD} = \gamma_{AC}\gamma_{BD} + \gamma_{AD}\gamma_{BC} - \gamma_{AB}\gamma_{CD}$.

- Horizon supertranslation charge

$$\delta Q^{\mathcal{H}^+}[f] = \int_{\mathcal{H}^+} dv\, d^2z \sqrt{\gamma}\left(D^A D^B f\right)\partial_v h_{AB}$$

[Hawking, Perry, Strominger]

Dual Supertranslation

- There are asymptotic symmetry transformations on the metric that are not diffeomorphisms, referred to as the **dual supertranslations**.

$$\widetilde{\delta}_f C_{zz} = -2iD_z^2 f$$

- These are associated to the complex Weyl scalar Ψ_2

$$\text{Supertranslation charge:} \qquad Q^+[f] = \int_{\mathcal{I}^+_-} d^2z\sqrt{\gamma}\, f(z,\bar{z})\, \mathrm{Re}\,\Psi_2^0$$

$$\text{Dual supertranslation charge:} \qquad \widetilde{Q}^+[f] = \int_{\mathcal{I}^+_-} d^2z\sqrt{\gamma}\, f(z,\bar{z})\, \mathrm{Im}\,\Psi_2^0$$

- Dual supertranslation is the “magnetic” dual of supertranslation.

[M Godazgar, H Godazgar, Kol, Pope, Porrati]

Dual Supertranslation

- An example of a spacetime with non-zero global dual supertranslation charge is the Taub-NUT metric.
- The complex Weyl scalar of a Taub-NUT spacetime has the leading asymptotic form

$$\Psi_2^0 = -M + i\ell,$$

where M is the mass and and ℓ is the NUT parameter.

- The global dual charge of an asymptotically flat spacetime is zero, but there can be non-zero charges for higher spherical harmonics.
- There are also bulk dust configurations that have non-zero dual supertranslation charge. [Satishchandran, Wald]

[Kol, Porrati]

Dual Supertranslation

- Standard/dual supertranslations are *asymptotic* symmetries:

$$\langle \text{out}|[Q[f], S]|\text{in}\rangle = 0 \quad \rightarrow \quad \text{Weinberg soft theorem}$$

$$\langle \text{out}|[\widetilde{Q}[f], S]|\text{in}\rangle = 0 \quad \rightarrow \quad \textit{magnetic} \text{ soft theorem, satisfied trivially}$$

- For smooth functions we have an abelian algebra,

$$\{Q[f_1], Q[f_2]\} = 0, \qquad \{Q[f_1], \widetilde{Q}[f_2]\} = 0, \qquad \{\widetilde{Q}[f_1], \widetilde{Q}[f_2]\} = 0$$

- Asymptotic scattering states can be organized into simultaneous eigenstates of Q and $\widetilde{Q}$.

Diagonalization of Q → Faddeev-Kulish states (Wilson line dressings)

Diagonalization of $\widetilde{Q}$ → 't Hooft line dressings

You get mileage out of diagonalizing asymptotic symmetry charges: Faddeev-Kulish states cure infrared divergence in scattering amplitude.

[Akhoury, Choi]

Dual Supertranslation

- Can dual charges be obtained by covariant phase space formalism?

- This can be done in the first-order formalism of gravity using the Holst action

$$\begin{aligned} L_{\text{Palatini}} &= \epsilon_{abcd}\mathcal{R}^{ab}(\omega) \wedge e^c \wedge e^d \\ L_{\text{Holst}} &= (\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})\mathcal{R}^{ab}(\omega) \wedge e^c \wedge e^d \end{aligned}$$

  where $\mathcal{R}^a{}_b(\omega) = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ is the Riemann curvature 2-form.

- The Holst action corresponds to the "magnetic" dual of the Palatini action. The "θ-term" of gravity:

$$\mathcal{L}_{\text{Palatini}} + \lambda\mathcal{L}_{\text{Holst}} \quad \leftrightarrow \quad -\frac{1}{4}F \wedge (*F) + \frac{\theta}{32\pi^2}F \wedge F$$

  Addition of $\mathcal{L}_{\text{Holst}}$ does not alter the equations of motion.

- Dual supertranslations are *diffeomorphisms* in the gravity defined by Holst action.

[Holst, H Godazgar, M Godazgar, Perry]

Roadmap

1. Review

- BMS supertranslation
- Supertranslation with a black hole
- Dual supertranslation

2. Main story

- Singular supertranslation
- Boundary Chern-Simons theory

Singular Supertranslation

- Oftentimes $f(z,\bar{z})$ is taken to be a smooth function on the sphere, but it can also take on singular functions (ex. charge conservation $\rightarrow$ soft theorems).
- Singular supertranslations arise naturally in the BMS algebra with extended phase space, with meromorphic superrotations. [Barnich, Troessaert]
- In electromagnetism, singular large gauge transformations are associated with Dirac string configurations in the bulk. [Freidel, Pranzetti]
- Large gauge transformations have electric and magnetic charges, which commute for smooth parameters. In the presence of singularities, the charge algebra exhibits central terms [Hosseinzadeh, Seraj, Sheikh-Jabbari, Freidel, Pranzetti]

$$\{Q^M(\widetilde{\alpha}), Q^E(\alpha)\} = -\frac{1}{2\pi}\sum_p \oint_p \alpha \, \mathrm{d}\widetilde{\alpha}$$

Dual Supertranslation with a Black Hole

- Linear metric perturbation h_{AB} (analogous to shear C_{AB} on $\mathcal{I}^+$)

$$\{\partial_v h_{AB}(v,z,\bar{z}), h_{CD}(v',z',\bar{z}')\} = 32\pi M^2 \gamma_{ABCD}\delta(v-v')\delta^2(z-z')$$

  where $\gamma_{ABCD} = \gamma_{AC}\gamma_{BD} + \gamma_{AD}\gamma_{BC} - \gamma_{AB}\gamma_{CD}$.

- Using covariant phase space formalism with the Holst action, we find the dual supertranslation charge

$$\delta Q^{\mathcal{H}^+}[f] = \int_{\mathcal{H}^+} dv\, d^2z\sqrt{\gamma}\left(D^A D^B f\right)\partial_v h_{AB}$$
$$\delta \widetilde{Q}^{\mathcal{H}^+}[f] = \int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\left(D^A D^B f\right)\epsilon_{(A}{}^{C} h_{B)C}$$

- Non-integrable terms appear in the presence of singularities, but they don't contribute to the algebra.

Singularity Induces Central Term

- In the absence of singularities, the charges commute.

- In the presence of singularities, the standard and dual supertranslation generators exhibit a central term. For example, if one parameter has a pole $\frac{1}{z-w}$,

$$\left\{Q^{\mathcal{H}^+}[\tfrac{1}{z-w}], \widetilde{Q}^{\mathcal{H}^+}[f]\right\} = -\frac{1}{4}(D^z D_z^2 f)|_{z=w}.$$

Singularity Induces Central Term

- In the absence of singularities, the charges commute.

- In the presence of singularities, the standard and dual supertranslation generators exhibit a central term. For example, if one parameter has a pole $\frac{1}{z-w}$,

$$\left\{Q^{\mathcal{H}^+}[\tfrac{1}{z-w}], \widetilde{Q}^{\mathcal{H}^+}[f]\right\} = -\frac{1}{4}(D^z D_z^2 f)|_{z=w}.$$

- The asymptotic charges no longer commute.
  How can we make sense of the central term?

- We study a scenario where we can remove the central term.
  $\rightarrow$ Add extra degrees of freedom on the horizon.

BMS Supertranslation & Dual Supertranslation

What gravitational theory do we put on the horizon?

- The horizon theory should not disturb the gravitational field.
- The horizon is a null surface — metric dependence is a no-go.

$$\implies \text{Topological theory}$$

- Chern-Simons theory

$$I_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{H}^+} \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

with gauge group $G = \mathrm{SL}(2, \mathbb{C})$ is a three-dimensional theory of gravity. [Witten]

$$A_i = e_i^a P_a + \omega_i^a J_a$$

Gravitational Chern-Simons Theory

- We can compute the asymptotic symmetry generators from the Chern-Simons action using covariant phase space formalism.

$$Q_{\rm CS}[\rho, \tau] = -\frac{k}{\pi} \int_{\partial S} (\tau^a e_a + \rho^a \omega_a)$$

  S is a section of the horizon.

  A peculiarity is that these generators exist only in the presence of singularities.

- There are two generating vector fields ρ^a and τ^a:
  ∘ ρ generates diffeomorphism (BMS supertranslation).
  ∘ τ generates local Lorentz transformation.

- What is the dual asymptotic symmetry in Chern-Simons theory?

Gravitational Chern-Simons Theory

- Looking for a dual symmetry boils down to constructing a Chern-Simons analog of the Holst action in 4D.

- The $SL(2,\mathbb{C})$ Chern-Simons theory does have a magnetic dual. (Not every gravitational Chern-Simons theory does.) The existence of a dual action has to do with the curvature of a section of the horizon being positive. [Witten]

- Accordingly, the theory has four asymptotic symmetries:
  - Standard and dual diffeomorphisms
  - Standard and dual local Lorentz transformations

$$Q_{\mathrm{CS}}[\rho,\tau] = -\frac{k}{\pi}\int_{\partial S}(\tau^a e_a + \rho^a \omega_a), \quad \widetilde{Q}_{\mathrm{CS}}[\rho,\tau] = -\frac{kM}{\pi}\int_{\partial S}\left(\tau^a \omega_a - \frac{\rho^a e_a}{4M^2}\right).$$

Gravitational Chern-Simons Theory

- The 3D diffeomorphisms ρ^a have a natural action $\rho^a[f]$ under supertranslation f inherited from 4D:

$$\rho^0[f] = 0, \qquad \rho^1[f] = \frac{D^z f}{1 + z\bar{z}} + \text{c.c.}, \qquad \rho^2[f] = \frac{iD^z f}{1 + z\bar{z}} + \text{c.c.}$$

  In addition to diffeomorphisms, there is a standard/dual pair of compensating local Lorentz transformations τ^a.

- By choosing an appropriate compensating Lorentz transformation $\tau^a[f]$, the Chern-Simons charge algebra cancels the central term in the 4D algebra.

$$\tau^0[f] = \frac{(D^2 + 2)f}{8kM}, \qquad \tau^1[f] = \frac{\rho^2[f]}{2M}, \qquad \tau^2[f] = -\frac{\rho^1[f]}{2M}.$$

- For a supertranslation with a pole at $z = w$, we have

$$\left\{ Q_{\rm CS}[\tfrac{1}{z-w}], \widetilde{Q}_{\rm CS}[f] \right\} = \widetilde{Q}_{\rm CS}[\tau'', \rho''] + \frac{1}{4}(D^z D_z^2 f)|_{z=w}.$$

  The central term cancels that of the supertranslation algebra.

Summary and Outlook

- The algebra of standard and dual supertranslations on the black hole horizon exhibits a central term in the presence of singularities.
- We can close the charge algebra by introducing new degrees of freedom on the horizon: an $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory.
- Since the structures of the horizon and the null infinity are similar, can we do the same on the future null infinity $\mathcal{I}^+$?
- Is there a topological theory that sits on $\mathcal{I}^+$?
  Is there a way to measure this via scattering?

Thank you for your attention!