BMS charges at the critical sets of null infinity

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Introduction

(Asymptopia: "a far away country from which we know little", J M Stewart).

- There is a vast literature on the "asymptotics of the gravitational field".

- Builds on Penrose's characterisation of isolated systems in GR using the notion of asymptotic simplicity.

- Most of it formal: it makes a number of assumptions which may or not be generic.

There exists a vast body of work aimed at setting the "asymptotics of GR" on solid foundations:

- H. Friedrich, ...
- P. Chrusciel, R. Beig & B.G. Schmidt, ...
- D. Christodoulou & S. Klainerman, ...
- P. Hintz & A. Vasy, ...
- L. Kehrberger, ...

Although great progress has occurred in recent years, there is still some cleaning to do.

Some (historical) context:

Asymptotic simplicity (AS).

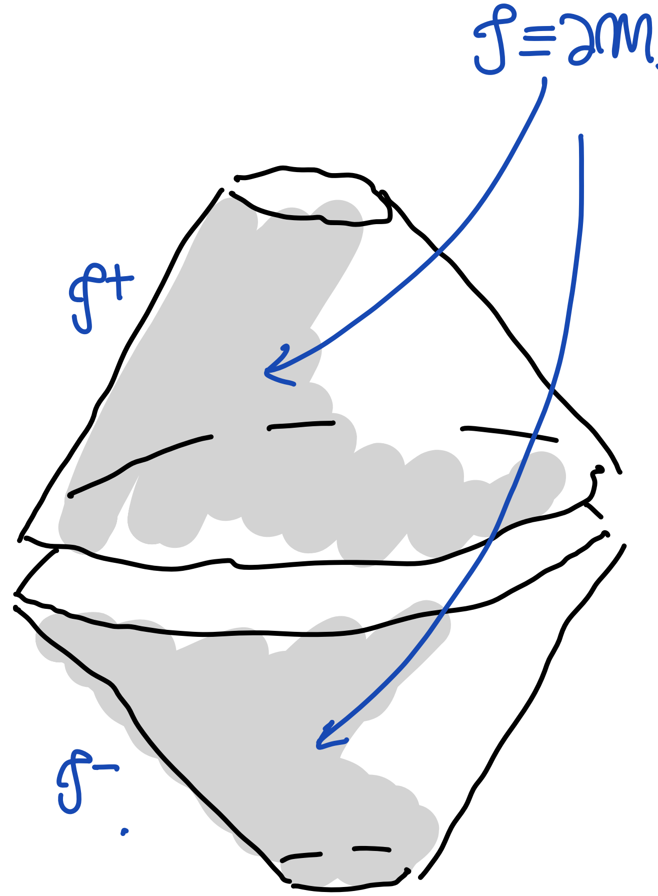
Given $(\tilde{\mathcal{M}}, \tilde{g})$ with $Ric[\tilde{g}] = 0$, $\exists$ smooth $(\mathcal{M}, g)$, Ω on $\mathcal{M}$ such that:

i) $\mathcal{M}$ is a manifold with $\mathscr{I} \equiv \partial\mathcal{M}$.

ii) $\Omega > 0$ on $\mathcal{M} \setminus \mathscr{I}$ and $\Omega = 0$, $d\Omega \neq 0$ on $\mathscr{I}$.

iii) $\exists\, \varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ such that $\varphi(\tilde{\mathcal{M}}) = \mathcal{M} \setminus \mathscr{I}$, $\varphi^* g = \Omega^2 \tilde{g}$.

iv) Each null geodesic of $(\tilde{\mathcal{M}}, \tilde{g})$ acquires two distinct endpoints on $\mathscr{I}$.



$\Rightarrow$ Provides a geometric framework to study the asymptotics of the gravitational field!

Key aspect: smoothness of $\mathscr{I}$.

$$\boxed{\text{Smoothness at } \mathscr{I}^{\pm} \iff \text{decay of fields (peeling).}}$$

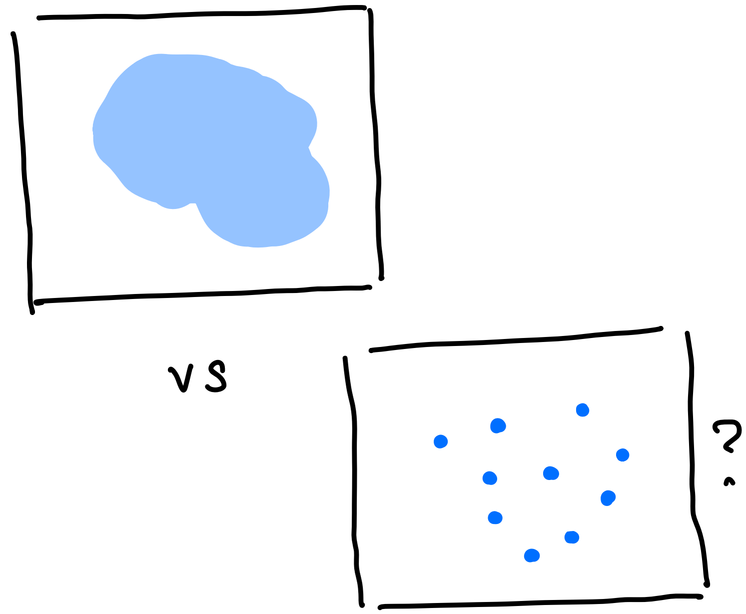
Corollary:

restricted smoothness $\iff$ modified decay.

(also valid for linear fields).

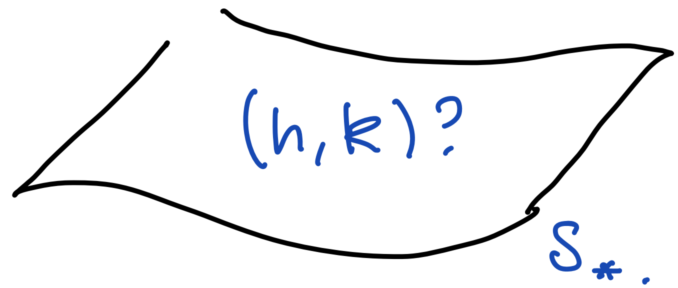
Natural questions:

a.) "How large" is the class of spacetimes with smooth Penrose compactification?

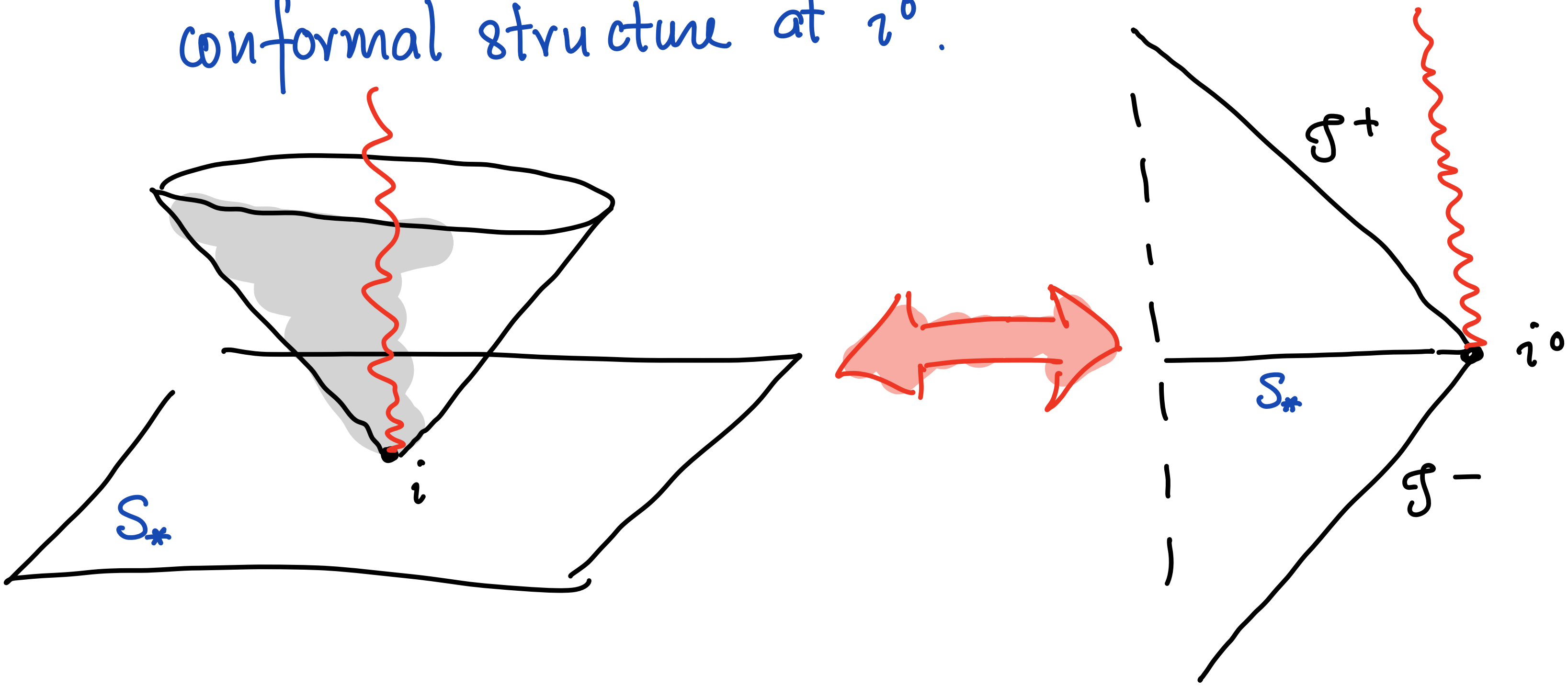


b). How to construct the spacetimes from e.g. Cauchy initial data?

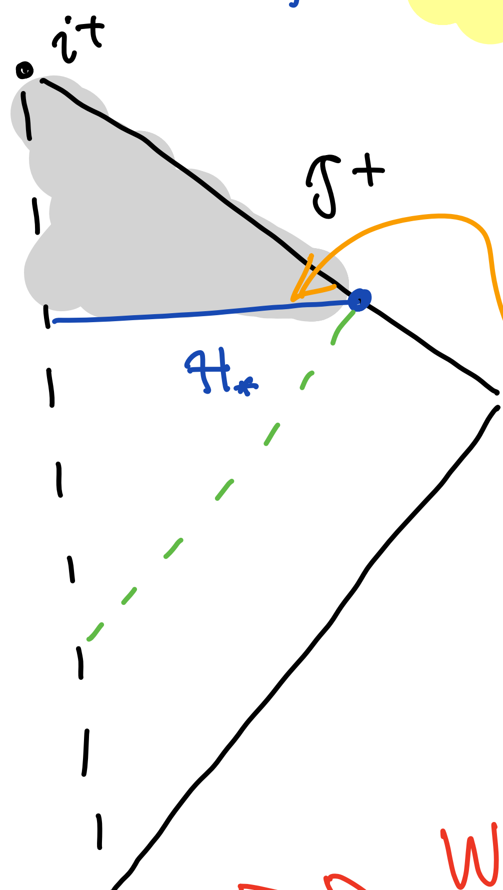
What extra conditions are required?



(Penrose ~'65). observed that the presence of ADM mass produces a singularity of the conformal structure at i^0.



⇒ (Friedrich, '86). Semiglobal stability of the Minkowski spacetime from hyperboloidal data.



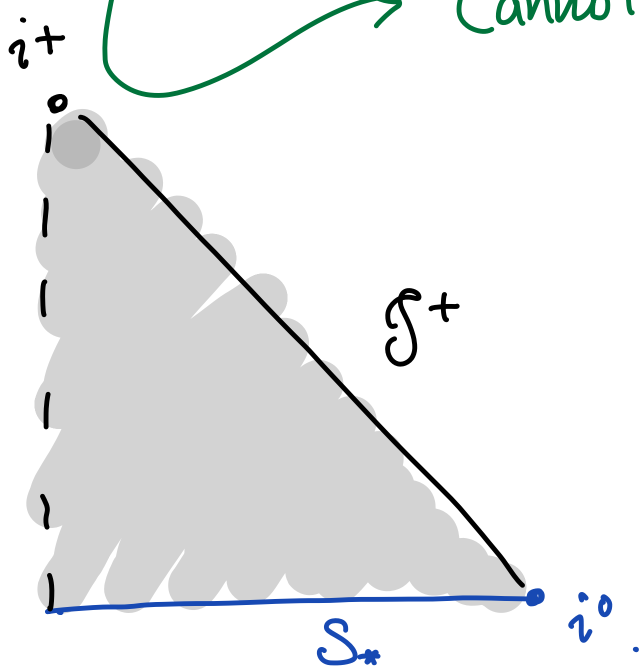
For suitable data one can recover a smooth $\mathscr{I}^+$ on $D^+(\mathcal{H}_*)$.

⇒ What about Cauchy data?

(Christodoulou & Klainerman '90).

Global non-linear stability of the Minkowski spacetime

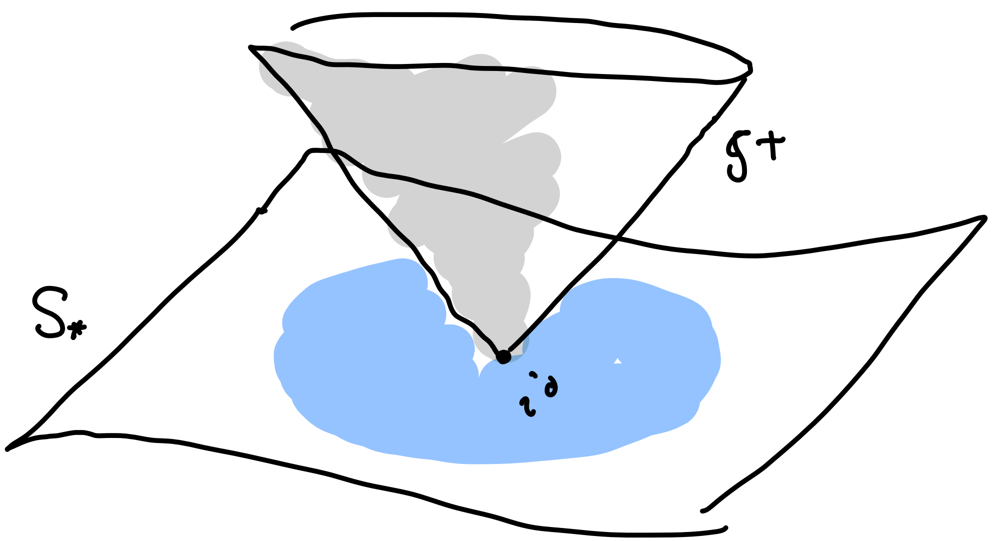
Cannot recover peeling!



Technical or fundamental obstructions?

(Friedrich '98). "Regular asymptotic initial value problem at spatial infinity"

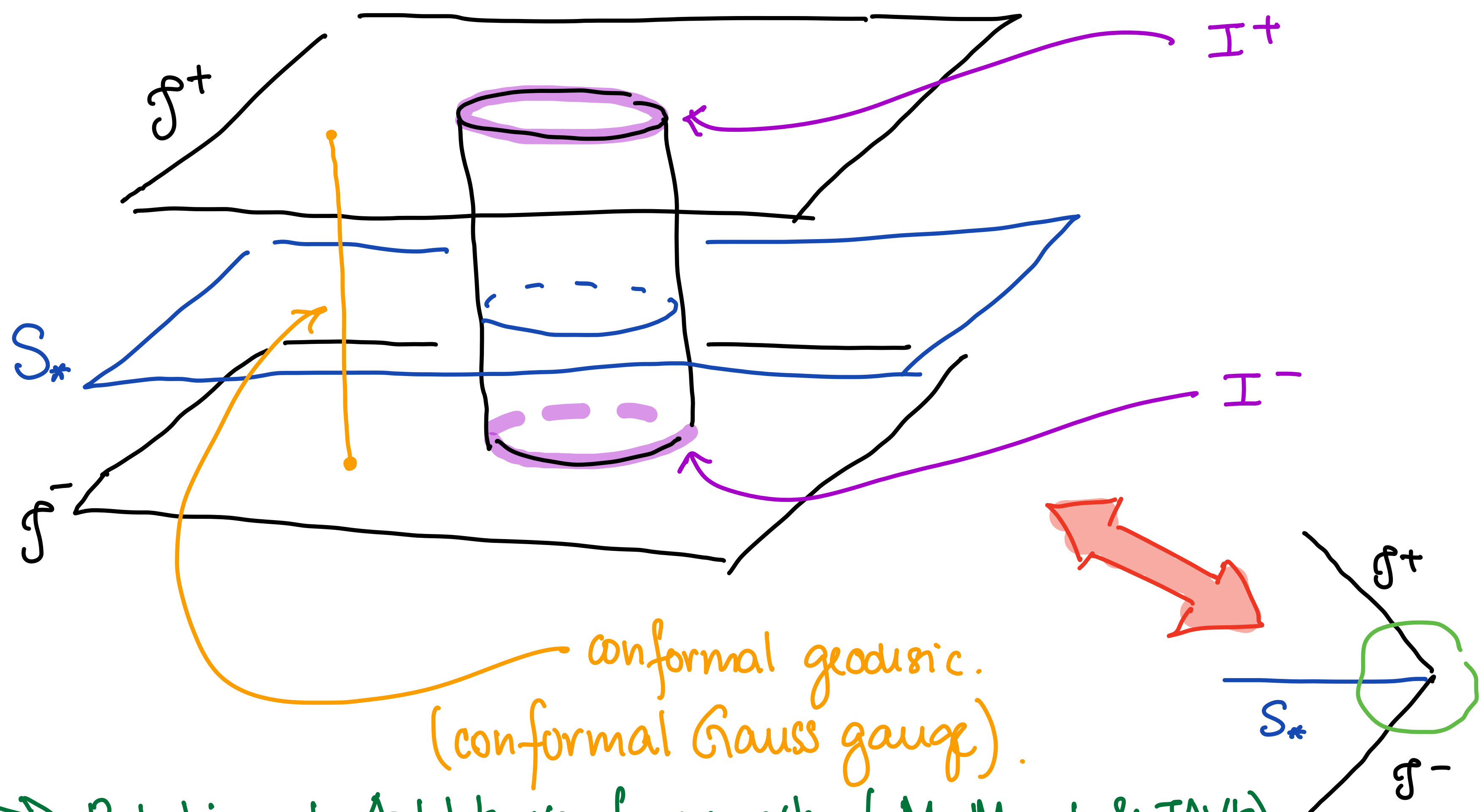
Detailed study of the structure of spatial infinity from the point of view of the initial value problem.



Equations and data regular at spatial infinity.

Cylinder at spatial infinity

(See also Ashtekar & Hansen, Beig & Schmidt).

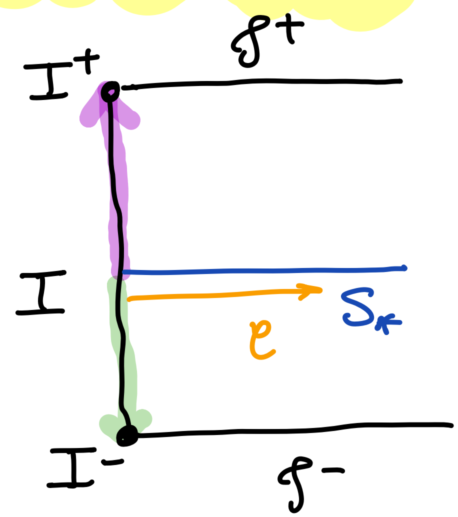


Relation to Ashtekar's framework (M. Magdy & JAVK).

Key technical aspect:

the cylinder is a total characteristic of the (conformal) Einstein field equations.

All the evolution eqns reduce to transport eqns on the cylinder I.



Data on $I_\star$ $\Longrightarrow$ Solutions at $I^\pm$

Solution jets

$$J[\phi^{(p)}] = \left\{ (\partial_\rho^p \phi)\big|_{\rho=0} \right\}$$

Weyl tensor

Obstructions to the smoothness of $\mathscr{I}$.

The regularity of the $\phi^{(p)}$'s can be explicitly computed (modulo computational complexities).

logarithmic divergences at $\mathscr{I}^{\pm}$ (H. Friedrich, JAVK).

Data needs to be fine-tuned to obtain suitably regular solutions.

(Hintz & Vasy, 2018).

Global non-linear stability of Minkowski with polyhomogeneous expansions

logs.

Not yet sharp but an important step forward!

Why care?

The conclusions from the analysis of i^0 are generic.

— i.e. independent from the stability set up —

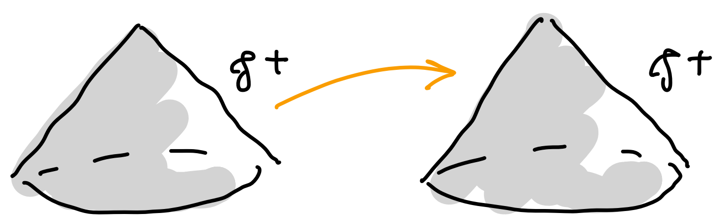
Friedrich's cylinder at spatial infinity provides a framework for the study of asymptotic charges and their relation to initial data!

BMS charges.

The symmetry group of $\mathscr{I}$ of AS. spacetime is the BMS (Bondi-Metzner-Sachs) group.

Asymptotic symmetries

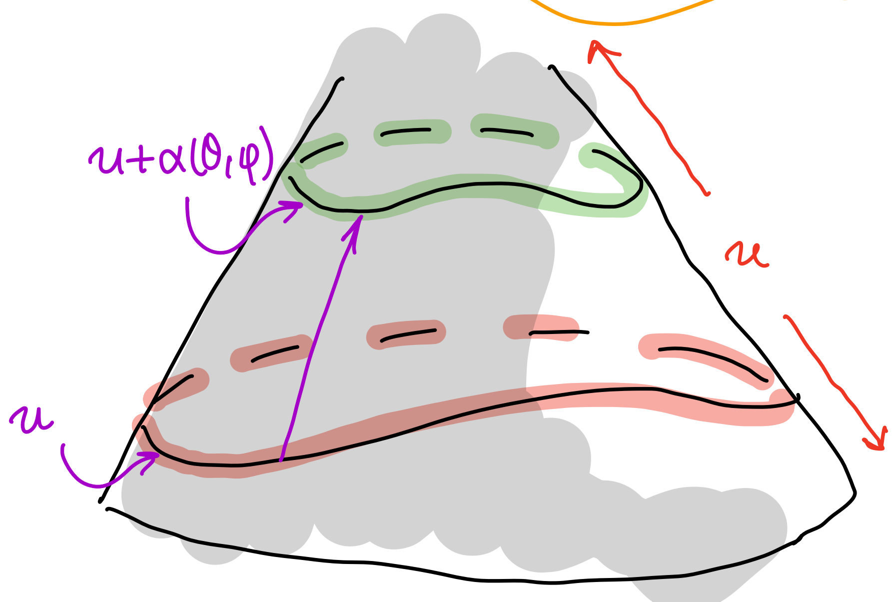
- Solutions to the asymptotic killing eqns.
- Transformations of $\mathscr{I}$ onto itself preserving structure



↪ Of particular interest are supertranslations (reparametrisations of cuts of $\mathscr{I}^+$).

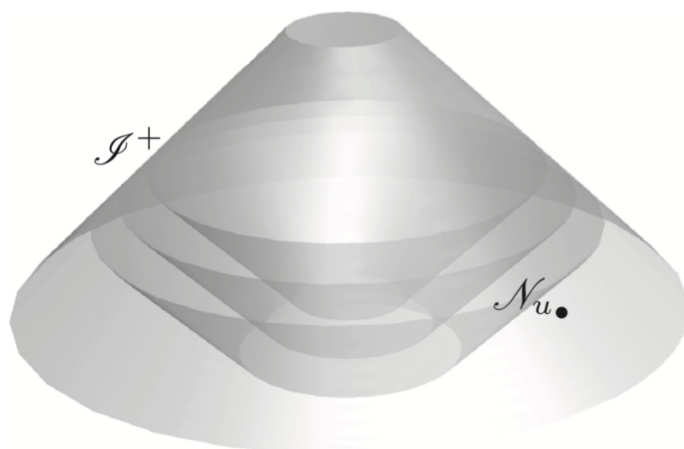
$$u \longrightarrow u + \alpha(\theta, \varphi).$$

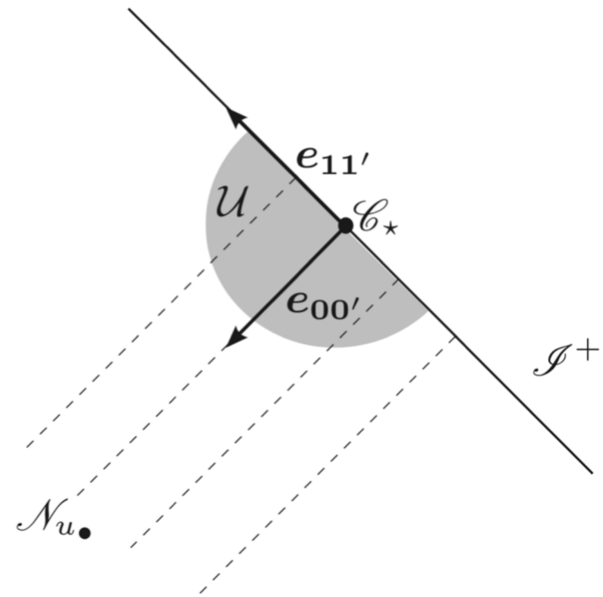
smooth function on $\mathcal{C} \approx S^2$.



Newman-Penrose gauge

A choice of coordinates, frame and conformal scaling adapted to the geometry in $\mathscr{I}^+$.





Work in the unphysical spacetime $(\mathcal{M}, g)$ with $g = \Omega^2 \tilde{g}$.

Coordinates: $\bar{x} = (x^\mu) = (u, r, x^A)$ (Bondi coords).

Frame: $\{\vec{l}', \vec{n}', \vec{m}', \vec{\bar{m}}'\} = \{\vec{e}'_{00'}, \vec{e}'_{11'}, \vec{e}'_{01'}, \vec{e}'_{10'}\}$

- $\vec{e}'_{00'}$ $(\vec{l}')$ tangent to $\mathscr{I}^+$ and $\nabla_{11'}\vec{e}'_{11'} = 0$
- $\vec{e}'_{11'}(u) = 1$ on $\mathscr{I}^+$
- $\vec{e}'_{00'} = (\,\mathrm{d}u)^\sharp$

Conformal freedom and residual freedom in the frame can be used to set some components of the connection and Ricci tensor.

The spin-2 equations.

$$\nabla^{A}{}_{A'}\phi_{ABCD} = 0 \,, \qquad \phi_{ABCD} = \phi_{(ABCD)}$$

(NB. can also do Maxwell!).

The BMS charge associated to supertranslations on a cut $\mathcal{C}$ of $\mathscr{I}^+$ is given by

$$\phi_{ABCD} = \begin{cases} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{cases}$$

$$Q = 2\oint_{\mathcal{C}} \lambda \bar{\Phi}_2 \, dS$$

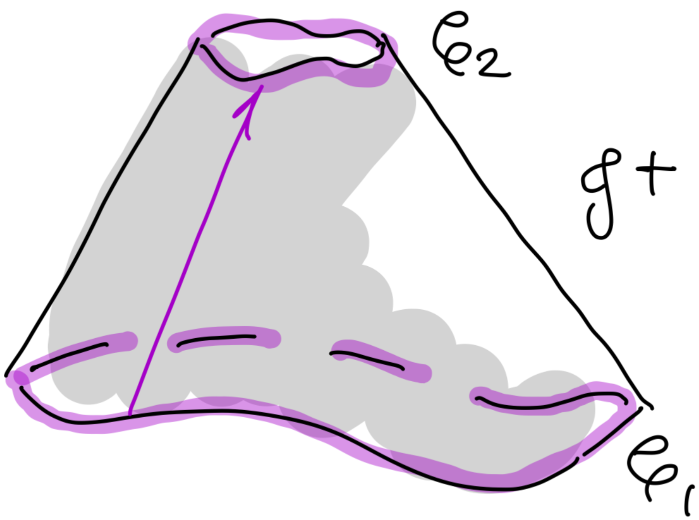
(K. Prabhu).

Coulomb field.

λ: smooth function on S^2

e.g. $Y_{\ell m}$ (spherical harmonic)

Note: the charges are not conserved!



$$Q_2 - Q_1 \neq 0$$

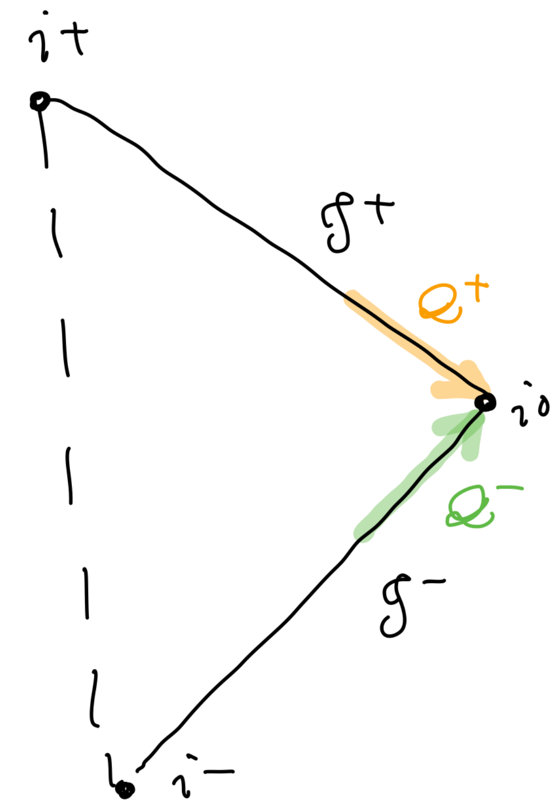
for two cuts $\mathcal{C}_2, \mathcal{C}_1$.

NB: an analogous computation can be carried out on $\mathcal{I}^-$.

What happens if one considers the limit of $\mathcal{E}$ approaching the "critical sets" where $\mathscr{I}^\pm$ meets i^0?

Under which conditions are the limits well defined?

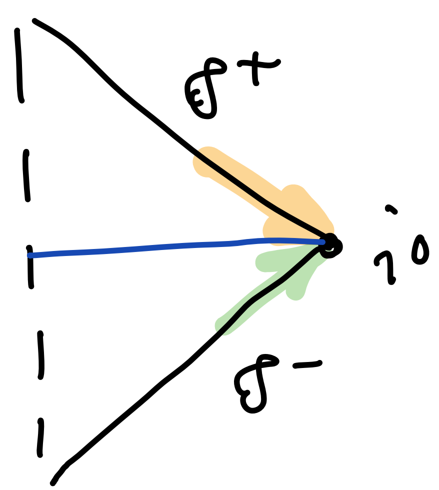
Are the charges on $\mathscr{I}^+$ and $\mathscr{I}^-$ related in some way?



Matching "problem"

Strategy: study the matching problem for BMS charges in terms of the initial value problem.

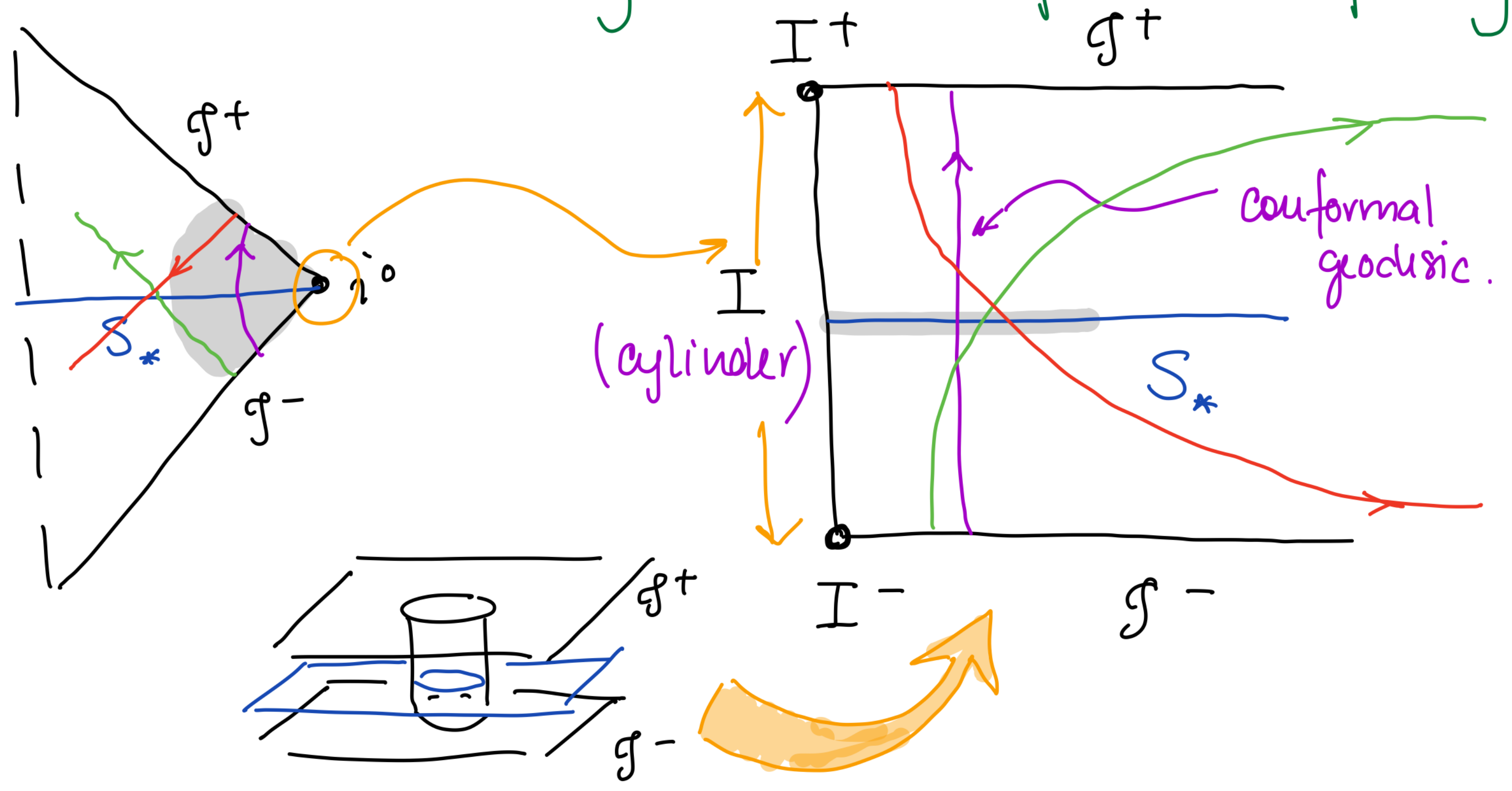
Use H. Friedrich's formulation of spatial infinity.



Assumptions controlled in terms of initial data

Write the charges in terms of free data.

Friedrich's cylinder at spatial infinity



The F (Friedrich)-gauge

Based on a non-intersecting congruence of conformal geodesics in a neighbourhood of i^0.

Coords: (τ, ρ, x^A).

The c. geodesics start at $\mathscr{I}^- \equiv \{\tau = -1\}$,

end at $\mathscr{I}^+ \equiv \{\tau = 1\}$.

Frame: $\{\vec{e}_{AA'}\} = \{\vec{l}, \vec{n}, \vec{m}, \vec{\bar{m}}\}$ Weyl propagated along the conformal geodesics.

How are the NP and F-gauge related?
(Friedrich & Kánnár 2000).

computation of NP constants

$\in SL(2,\mathbb{C})$.

Scaling $g \mapsto g' = \theta^2 g$

Lorenz transf. $\vec{e}\,'_{AA'} = \theta^{-1} \Lambda^{B}{}_{A} \bar{\Lambda}^{B'}{}_{A'} \vec{e}_{BB'}$.

Minkowski: the transf. is known explicitly, e.g. on $\mathscr{I}^+$

$$\Lambda^1{}_0 = \frac{2}{\sqrt{\rho}(1+\tau)}, \quad \Lambda^0{}_1 = \frac{\sqrt{\rho}(1+\tau)}{2}$$

$$\Lambda^1{}_1 = \Lambda^0{}_0 = 0.$$

The BMS charges at $\mathcal{I}^{\pm}$

Assumption: near $\mathcal{I}$ one has a solution of the form

Boosted!

Usually $a_{00} Y_{00} + O(\rho)$

$$\phi_2 = \sum_{l=0} \sum_{m=-l}^{l} a_{lm}(\tau) Y_{lm} + O(\rho)$$

Solutions of this form can be constructed using certain estimates by H. Friedrich (2003).

The subleading terms can be controlled.

The coeffs. $a_{l,m}(\tau)$ can be explicitly computed from transport equs on $\mathcal{I}$. total characteristic

e.g. $(1-\tau^2)\ddot{a}_{lm} - 2\tau\dot{a}_{2;l,m} + l(l+1)a_{2;l,m} = 0.$

Solutions: $a_{lm}(\tau) = A_{l,m} P_l(\tau) + B_{l,m} Q_l(\tau).$

Legendre polynomial

Legendre funct. of second kind.

$$Q_l = c_l \ln(1 \pm \tau) + \mathcal{O}(1).$$

The leading terms in the Ansatz are singular unless one fine-tunes the data!

Proposition: $a_{2;\ell,m}(0) = 0$ for ℓ odd,

$\mathring{a}_{2;\ell,m}(0) = 0$ for ℓ even.

The above conditions can be written in terms of free data for the spin-2.

Gauss constraint: $D^{AB}\phi_{ABCD} = 0$

$$\phi_{ABCD} = (\mathcal{L}\xi)_{ABCD}$$

third order operator (Andersson, Bäckdahl & Joudioux)

One can find a choice of ξ_{ABCD} (free data) to satisfy the reg. conditions

If the solutions are well-defined at $\mathcal{I}^{\pm}$ one finds that:

$$\begin{cases} Q\big|_{\mathcal{I}^+} = -2\,\bar{a}_{2;\ell,m}(1) \\ Q\big|_{\mathcal{I}^-} = -2\,\bar{a}_{2;\ell,m}(-1). \end{cases}$$

The limits are generically not well defined unless one fine-tunes the data!

When the charges are defined at $\mathcal{I}^\pm$ one gets

$$\begin{cases} Q^+ = Q^- & \text{for } l \text{ even} \\ Q^+ = (-1)^l Q^- & \text{for } l \text{ odd} \end{cases}$$

No need to put any identif. by hand!

BMS charges in G.R. (in progress)

(with MMAM and K. Prabhu).

In this case the charges are given by

$$Q = \oint_{\mathcal{C}} \varepsilon_2 \, \lambda \left(\phi_2 + \frac{1}{2} \sigma^{ab} N_{ab} \right).$$

Coulomb component of the rescaled Weyl spinor $\phi_{ABCD} = \Xi^{-1} \Psi_{ABCD}$.

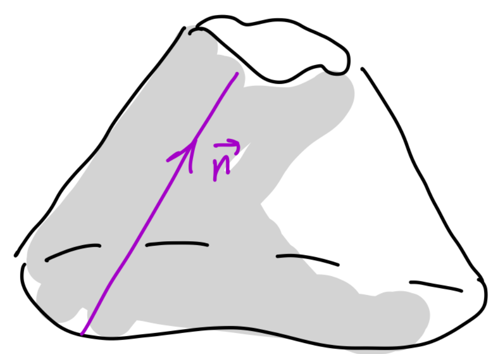
smooth function over S^2.

with σ_{ab} (shear tensor on $\mathscr{I}^+$)

$$N_{ab} \equiv 2 \left(\mathcal{L}_n - \Phi \right) \sigma_{ab}$$

$$\Phi \equiv \frac{1}{4} \nabla_a n^a \big|_{\mathscr{I}^+}$$

Null geodesic generator of $\mathscr{I}^+$.



What is a good class of initial data to consider?

L.-H. Huang "Solutions of special asymptotics to the Einstein constraint equs" Class. Quantum Grav **27**, 245002 (2010).

Proposition. For any $\alpha, \beta \in C^\infty(S^2)$ there exists an initial data set with the asymptotics

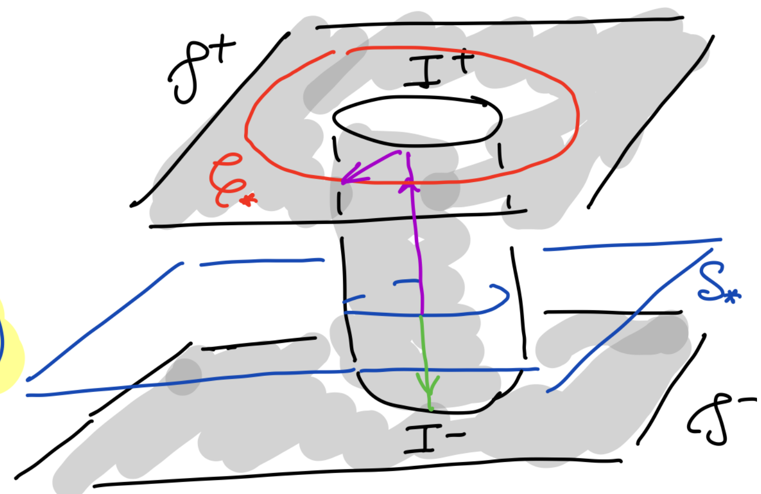
$$\begin{cases} h_{ij} = \left(1 + \frac{A}{r}\right)\delta_{ij} + \frac{\alpha}{r}\left(\frac{x_i x_j}{r^2} - \frac{1}{2}\delta_{ij}\right) + O(r^{-2}) \\ \pi_{ij} = \beta \frac{x_i x_j}{r^4} + \frac{1}{r^3}\left(B_\ell x^\ell \delta_{ij} - B_i x_j - B_j x_i\right) + O(r^{-3}). \end{cases}$$

$\pi_{ij} \equiv k_{ij} - k h_{ij}$.

Constants

(Proof uses gluing).

For the above class of initial data one can make use the properties of the cylinder at spatial infinity (total charact.) to compute asymptotic expansions of all the relevant fields:



$$\phi_{ABCD}, \quad \sigma_{ab}, \quad \Lambda^A{}_B, \quad \vartheta$$

$$N_{ab}.$$

give the transf. between gauges

Caveat: the expansions are formal!

One needs to adapt F-estimates to full GR (or do Hintz-Vasy for the CFE).

The leading behaviour of ϕ_2 is given by

$$\phi_2 = \sum_{l=0}^{\infty} \sum_{l=-m}^{m} a_{2;l,m}(\tau) Y_{l,m} + O(\rho).$$

Again $a_{2;l,m}(\tau) = A_{lm} P_l(\tau) + B_{lm} Q_l(\tau)$

logarithmic divergences

Take away: regularity of solns at $\mathcal{I}^{\pm}$ is obtained by imposing conds on the multipolar structure of α and β.

Also, can show that $\sigma_{ab}\big|_{\mathcal{I}^{\pm}} \rightarrow 0$ as one approaches $\mathcal{I}^{\pm}$.

The BMS charges at $\mathcal{I}^{\pm}$ (when defined) correspond to the multipolar structure of α, β.

Can establish the correspondence between Q^{+} and Q^{-}.

Conclusions:

- H. Friedrich's formalism can be used to study the assumptions behind asymptopia.
- Some of the standard assumptions are not generic.
- Assumptions on free data are ok and a way to avoid difficulties.

Outlook: wrap up H. Friedrich's programme with rigorous statements about the relation between F-expansions and solutions to the CFE.
(cf. Hintz-Vasy).

(Under construction)