Aspects of Relativistic, Galilean and Carrollian fluids

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Introduction and motivation

- Relativistic fluid, hydrodynamic frame invariance is a crucial property.
- Fluid/gravity correspondence, reconstruction of Einstein's spaces with $\Lambda \neq 0$
  M. Haack & A. Yarom 08'; S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S.P. Trivedi & S.R. Wadia 08'; S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla & A. Sharma 08'; M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli & KS 12'; A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos, V. Pozzoli & K.S. 13'
- Carrollian group is a contraction of the Poincare group, where $c \longrightarrow 0$
  Lévy-Leblond 65', Sen Gupta 66'
- Conformal Carroll group & BMS group
  C. Duval, G. W. Gibbons & P. A. Horváthy 14'
- Carrollian dynamics emerges in asymptotically flat spacetimes
  L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos & KS '18; Penna '18; A. Bagchi, S. Chakrabortty, D. Grumiller, B. Radhakrishnan, M. Riegler & A. Sinha '21
- Carroll symmetry can be used in inflationary cosmology $\mathcal{E} + P = 0$
  J, de Boer, J. Hartong, N. A. Obers, W. Sybesma, S. Vandoren 21'

Focal Points

In this talk we will focus on:

1. Revisit of the relativistic hydrodynamics.
2. Galilean fluids using covariance and as a non-relativistic limit $c \to \infty$.
3. Carroll fluids using covariance and as a non-relativistic limit $c \to 0$.
4. Aristotelian structures.
5. Conclusions and Outlook

Plan of the talk

Revisit of the relativistic hydrodynamics

Galilean fluid dynamics

Carrolian fluid dynamics

Aristotelian fluid dynamics

Energy–momentum tensor and matter current

Without external forces the fluid equations are:

$$\nabla_\mu T^{\mu\nu} = 0\,, \quad \nabla_\mu J^\mu = 0\,, \quad \mu = 0, 1, \ldots, d$$

and are accompanied by a metric $g_{\mu\nu}$.

They are decomposed along a velocity field u^μ with $u_\mu u^\mu = -c^2$ as

$$T^{\mu\nu} = \frac{\varepsilon + p}{c^2} u^\mu u^\nu + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{1}{c^2}(u^\mu q^\nu + u^\nu q^\mu)\,,$$
$$J^\mu = \rho_0 u^\mu + j^\mu$$

where the viscous tensor $\tau^{\mu\nu}$ & the heat current q^μ (non-perfect e-m) obey

$$u^\mu q_\mu = 0\,, \quad u^\mu \tau_{\mu\nu} = 0\,, \quad u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu\,, \quad \varepsilon = \frac{1}{c^2} T_{\mu\nu} u^\mu u^\nu$$

and also

$$u^\mu j_\mu = 0\,, \quad \rho_0 = -\frac{1}{c^2} u^\mu J_\mu$$

The entropy current equals to

$$S^\mu = \frac{1}{T}\left(p u^\mu - T^{\mu\nu} u_\nu - \mu_0 J^\mu\right) + R^\mu$$

Let us now derive the e-m tensor and the matter current

ENERGY–MOMENTUM AND MATTER CONSERVATION

Let's start from the action

$$S = \int d^{d+1}x\sqrt{-g}\mathcal{L}$$

the energy–momentum tensor and the current can be defined as usual

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\,, \quad J^{\mu} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta B_{\mu}}$$

Demanding invariance under diffeomorphisms $x^{\mu} \to x^{\mu} - \xi^{\mu}$

$$\begin{aligned}\delta_{\xi} S = \int d^{d+1}x\sqrt{-g}\,\delta_{\xi} g_{\mu\nu} T^{\mu\nu} &= -2\int d^{d+1}x\sqrt{-g}\,\nabla_{\mu}\xi_{\nu}T^{\mu\nu} \\ &= -2\int d^{d+1}x\sqrt{-g}\,\left(\nabla_{\mu}(\xi_{\nu}T^{\mu\nu}) - \xi_{\nu}\nabla_{\mu}T^{\mu\nu}\right) = 0\end{aligned}$$

implying the conservation of $T^{\mu\nu}$. Similarly, J^{μ} is conserved under $B_{\mu} \to B_{\mu} + \partial_{\mu}\Lambda$
If ξ^{μ} is a Killing vector we can also define the conserved current

$$\nabla_{\mu}(\xi_{\nu}T^{\mu\nu}) = 0$$

and also for a conformal Killing vector provided that $T_{\mu}{}^{\mu} = 0$

WEYL INVARIANCE

The system may be invariant under Weyl transformations

$$ds^2 \to \Omega^{-2} ds^2\,, \quad u^\mu \to \Omega u^\mu$$

defining a Weyl connection

$$A_\mu = \frac{1}{c^2}\left(a_\mu - \frac{\Theta}{d} u_\mu\right)\,, \quad a_\mu = u^\nu \nabla_\nu u_\mu\,, \quad \Theta = \nabla_\mu u^\mu$$

The Weyl covariant derivative is metric compatible

$$\mathcal{D}_\rho g_{\mu\nu} = 0\,, \quad \mathcal{D}_\kappa f = (\partial_\kappa + w A_\kappa) f\,, \quad [\mathcal{D}_\kappa, \mathcal{D}_\lambda] f = w\, F_{\kappa\lambda} f\,, \quad F_{\kappa\lambda} = \partial_\kappa A_\lambda - \partial_\lambda A_\kappa$$

The fluid dynamics is Weyl invariant provided that

$$\nabla_\mu T^{\mu\nu} = \mathcal{D}_\mu T^{\mu\nu}\,, \quad \nabla_\mu J^\mu = \mathcal{D}_\mu J^\mu$$

where $T^{\mu\nu}$ and J^μ have conformal weights $d-1$ and $T^\mu{}_\mu = 0$.

This leads to $\varepsilon = d\,p + \tau^\mu{}_\mu$ and conformal weights

weight	**observables**
$d+1$	ε, p
d	q_μ, ρ_0
$d-1$	$\tau_{\mu\nu}, j_\mu$

ZERMELO COORDINATES

In a $d+1$ dimensional pseudo-Riemannian manifold we can write it in the form

$$ds^2 = -c^2\Omega^2 dt^2 + a_{ij}(dx^i - w^i dt)(dx^j - w^j dt)\,,$$

with (Ω, w^i, a_{ij}) functions of $(t, \mathbf{x})$.

These coordinates are well adapted for the Galilean limit $c \to \infty$, since

$$t' = t'(t)\,, \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$$

reduce to

$$\Omega' = \frac{\Omega}{J}\,, \quad w'^i = \frac{1}{J}\left(J^i{}_j w^j + j^i\right)\,, \quad a'_{ij} = a_{kl}\, J^{-1k}{}_i\, J^{-1l}{}_j$$

where

$$J = \frac{\partial t'}{\partial t}\,, \quad j^i = \frac{\partial x'^i}{\partial t}\,, \quad J^i{}_j = \frac{\partial x'^i}{\partial x^j}$$

An example is the $d+1$ vector u^μ

$$u'^0 = J\, u^0\,, \quad u'_i = J^{-1j}{}_i\, u_j$$

Galilean diffeomorphisms are generated by $\xi = \xi^t(t)\partial_t + \xi^i(t, \mathbf{x})\partial_i$

RANDERS–PAPAPETROU COORDINATES

In a $d+1$ dimensional pseudo-Riemannian manifold we can write it in the form

$$ds^2 = -c^2(\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j$$

with (Ω, b_i, a_{ij}) functions of $(t, \mathbf{x})$.

These coordinates are well adapted for the Carrollian limit $c \to 0$, since

$$t' = t'(t, \mathbf{x}), \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$$

reduce to

$$\Omega' = \frac{\Omega}{J}, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}{}_k, \quad a'^{ij} = J^i{}_k J^j{}_l a^{kl}$$

An example is the $d+1$ vector u^μ

$$u'_0 = \frac{u_0}{J}, \quad u'^i = J^i{}_j u^j$$

Carrollian diffeomorphisms are generated by $\xi = \xi^t(t, \mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i$

Plan of the talk

Revisit of the relativistic hydrodynamics

Galilean fluid dynamics

Carrolian fluid dynamics

Aristotelian fluid dynamics

Galilean covariance

Let us consider the manifold $\mathcal{M} = \mathbb{R} \times S$ with coordinates $(t, \mathbf{x})$ and the cometric

$$\partial_a^2 = a^{ij}\partial_i\partial_j\,, \quad i = 1, \ldots, d$$

along with the clock form and its dual/field of observers

$$\theta^{\hat{t}} = \Omega dt\,, \quad e_{\hat{t}} = \frac{1}{\Omega}\left(\partial_t + w^i\partial_i\right)$$

where we choose a^{ij} and w^i to be functions of $(t, \mathbf{x})$, where we choose $\Omega = \Omega(t)$.

- The clock form is exact, torsionless Newton–Cartan manifold.
- Invariance of ds^2, $\theta^{\hat{t}}$ and $e_{\hat{t}}$ under Galilean diffs: $t' = t'(t)\,, \mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$.
- We also introduce the positive-definite (degenerate) metric

$$ds^2 = a_{ij}(t, \mathbf{x})dx^i dx^j$$

and the corresponding metric compatible torsionless connection

$$\hat{\nabla}_i a_{jk} = 0\,, \quad \gamma^i{}_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right)$$

We can define Galilean tensors from objects transforming as connections

$$A'^i = \frac{1}{J}\left(J^i{}_j A^j + j^i\right)$$

as follows

$$\frac{1}{\Omega}\hat{\nabla}^{(k}A^{l)} - \frac{1}{2\Omega}\partial_t a^{kl} = -\frac{1}{2\Omega}\left(\mathcal{L}_{\mathbf{A}} a^{ij} + \partial_t a^{ij}\right)$$
$$\frac{1}{\Omega}\hat{\nabla}_{(k}A_{l)} + \frac{1}{2\Omega}\partial_t a_{kl} = \frac{1}{2\Omega}\left(\mathcal{L}_{\mathbf{A}} a_{ij} + \partial_t a_{ij}\right)$$

We also define the connection

$$\hat{\gamma}^w{}_{ij} = \frac{1}{\Omega}\left(\hat{\nabla}_{(i}w_{j)} + \frac{1}{2}\partial_t a_{ij}\right)$$

as well as the Galilean shear and expansion

$$\xi^w{}_{ij} = \frac{1}{\Omega}\left(\hat{\nabla}_{(i}w_{j)} + \frac{1}{2}\partial_t a_{ij}\right) - \frac{1}{d}a_{ij}\theta^w\,, \quad \theta^w = \frac{1}{\Omega}\left(\partial_t \ln\sqrt{a} + \hat{\nabla}_i w^i\right)$$

Finally, we also introduce a time, metric-compatible covariant derivative

$$\frac{1}{\Omega}\frac{\hat{\mathrm{D}}\Phi}{\mathrm{d}t} = \frac{1}{\Omega}(\partial_t + w^i\partial_i)\Phi\,, \quad \frac{1}{\Omega}\frac{\hat{\mathrm{D}}V^i}{\mathrm{d}t} = \frac{1}{\Omega}\left(\partial_t V^i + \mathcal{L}_w V^i\right) + \hat{\gamma}^{wi}{}_j V^j$$

GALILEAN DIFFEOMORPHISMS-I

Let us consider the action functional of Ω, w^i and a^{ij}

$$S = \int dt\Omega \int d^d x\, \sqrt{a}\mathcal{L}$$

and define the Galilean momenta (energy, current and stress-tensor)

$$\Pi = -\frac{1}{\Omega\sqrt{a}}\left(\Omega\frac{\delta S}{\delta\Omega} - \frac{w^i}{\Omega}\frac{\delta S}{\delta\frac{w^i}{\Omega}}\right), \quad P_i = -\frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta\frac{w^i}{\Omega}}, \quad \Pi_{ij} = -\frac{2}{\Omega\sqrt{a}}\frac{\delta S}{\delta a^{ij}}$$

leading to the variation

$$\delta S = -\int \mathrm{d}t\,\Omega \int \mathrm{d}^d x\sqrt{a}\left(\frac{1}{2}\Pi_{ij}\delta a^{ij} + P_i\delta\frac{w^i}{\Omega} + \left(\Pi + \frac{w^i}{\Omega}P_i\right)\delta\ln\Omega\right)$$

We would like to compute δS under Galilean diffeomorphisms

$$\xi = \xi^t(t)\partial_t + \xi^i(t,\mathbf{x})\partial_i = \xi^t\Omega\,\frac{1}{\Omega}(\partial_t + w^i\partial_i) + (\xi^i - \xi^t w^i)\partial_i = \xi^{\hat{t}}\mathrm{e}_{\hat{t}} + \xi^{\hat{\imath}}\partial_i$$

GALILEAN DIFFEOMORPHISMS-II

The Galilean diffeomorphisms $x^\mu \to x^\mu - \xi^\mu$ act infinitesimally on Ω, w^i and a^{ij} as

$$\mathcal{L}_\xi \Omega = -\partial_t \xi^{\hat{t}} - \mathcal{L}_w \xi^{\hat{t}}, \quad \mathcal{L}_\xi w^i = -\partial_t \xi^{\hat{\imath}} - \mathcal{L}_w \xi^{\hat{\imath}}, \quad \mathcal{L}_\xi a^{ij} = 2\left(\hat{\nabla}^{(i}\xi^{\hat{\jmath})} + \hat{\gamma}^{wij}\xi^{\hat{t}}\right)$$

Also on the clock form $\theta^{\hat{t}}$ and on the field of observers $e_{\hat{t}}$

$$\mathcal{L}_\xi \theta^{\hat{t}} = \frac{1}{\Omega}\left(\partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}}\right)\theta^{\hat{t}}, \quad \mathcal{L}_\xi e_{\hat{t}} = -\frac{1}{\Omega}\left(\partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}}\right) e_{\hat{t}} - \frac{1}{\Omega}\left(\partial_t \xi^{\hat{\imath}} + \mathcal{L}_w \xi^{\hat{\imath}}\right)\partial_i$$

Employing the above we find the energy and momentum equations

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$$\left(\frac{1}{\Omega}\frac{\hat{D}}{dt} + \theta^w\right)\Pi + \Pi_{ij}\hat{\gamma}^{wij} = -\hat{\nabla}_i \Pi^i, \quad \left(\frac{1}{\Omega}\frac{\hat{D}}{dt} + \theta^w\right)P_i + P_j \hat{\gamma}^{wj}{}_i + \hat{\nabla}^j \Pi_{ij} = 0$$

where Π^i is not determined through the variation; results as a boundary term.

Similarly invariance under gauge transformations leads to the continuity equation

$$\left(\frac{1}{\Omega}\frac{\hat{D}}{dt} + \theta^w\right)\rho + \hat{\nabla}_i N^i = 0, \quad \rho = \frac{1}{\sqrt{a}}\frac{\delta S}{\delta B}, \quad N^i = \frac{1}{\Omega\sqrt{a}}\left(-w^i\frac{\delta S}{\delta B} + \frac{\delta S}{\delta B_i}\right)$$

ISOMETRIES AND THE (NON)-CONSERVATION

Killing fields of the Galilean type satisfy

$$\mathcal{L}_\xi a^{ij} = 0\,, \quad \mathcal{L}_\xi \theta^{\hat{t}} = 0 \quad \Longrightarrow \quad \hat{\nabla}^{(i}\xi^{\hat{\jmath})} + \hat{\gamma}^{wij}\xi^{\hat{t}} = 0, \quad \frac{1}{\Omega}\frac{\hat{\mathrm{D}}\xi^{\hat{t}}}{\mathrm{d}t} = 0$$

whereas the field of observers $\mathrm{e}_{\hat{t}}$ is not apriori invariant.

An example $a_{ij} = \delta_{ij}, \Omega = 1$ & w^i = constant with Galilean algebra $\mathfrak{gal}(d+1)$

Duval 09'

$$\xi = T\partial_t + \left(\Omega_i{}^j x^i + V^j t + X^j\right)\partial_j \quad \Longrightarrow \quad \mathcal{L}_\xi \mathrm{e}_{\hat{t}} = -\left(V^i + w^j \Omega_j{}^i\right)\partial_i \neq 0$$

Assuming an isometry, we have on-shell vanishing scalar (continuity equation)

$$\mathcal{K} = \left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^w\right)\kappa + \hat{\nabla}_i K^i\,, \quad \kappa = \xi^{\hat{\imath}} P_i - \xi^{\hat{t}}\Pi\,, \quad K_i = \xi^{\hat{\jmath}}\Pi_{ij} - \xi^{\hat{t}}\Pi_i$$

Using the energy & momentum on-shell conservation $\mathcal{K} = \frac{P_i}{\Omega}\left(\partial_t \xi^{\hat{\imath}} + \mathcal{L}_{\mathbf{w}}\xi^{\hat{\imath}}\right) \neq 0$

Comments:

- Even in flat space $\mathcal{K} = P_i\left(V^i + w^k \Omega_k{}^i\right) \neq 0$, is not associated with a bnr term.
- The above construction extends for conformal isometries

$$\mathcal{L}_\xi a^{ij} = \lambda\, a^{ij}\,, \quad \mathcal{L}_\xi \theta^{\hat{t}} = \mu\theta^{\hat{t}}\,, \quad 2\mu + \lambda = 0$$

Galilean hydro as a non-relativistic limit

The energy–momentum tensor admits a large-c expansion (Zermelo frame)

$$\begin{cases} \Omega^2 T^{00} = \varepsilon_{\mathrm{r}} = \Pi + \mathcal{O}\left(1/c^2\right), \\ c\Omega T^0{}_i = q_{\mathrm{r}i} = c^2 P_i + \Pi_i + \mathcal{O}\left(1/c^2\right), \\ T_{ij} = p_{\mathrm{r}} a_{ij} + \tau_{\mathrm{r}ij} = \Pi_{ij} + \mathcal{O}\left(1/c^2\right). \end{cases}$$

Inserting the above into the conservation equations $\nabla_\mu T^{\mu\nu} = 0$ leads to

$$\begin{cases} c\Omega \nabla_\mu T^{\mu 0} = c^2 \hat{\nabla}^i P_i + \mathcal{E} + \mathcal{O}\left(1/c^2\right) = 0 \\ \nabla_\mu T^\mu{}_i = \mathcal{M}_i + \mathcal{O}\left(1/c^2\right) = 0, \end{cases}$$

where

$$\mathcal{E} = \left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^w\right)\Pi + \Pi_{ij}\hat{\gamma}^{wij} + \hat{\nabla}_i \Pi^i, \quad \mathcal{M}_i = \left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^w\right)P_i + P_j \hat{\gamma}^{wj}{}_i + \hat{\nabla}^j \Pi_{ij}$$

including the constraint on the current P_i, bnr term from diffeomorphism perspective.

Comments on the limit $c \to \infty$:

- Continuity equation emerges by adding $c^2\rho$ in ε_r: $\left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^w\right)\rho + \hat{\nabla}_i P^i = 0$
- The conservation equations $\nabla_\mu T^{\mu\nu} = 0$ are on-shell Galilean boost invariant.
- The (non)-conservation conditions emerges as a limit through $\nabla_\mu\left(T^{\mu\nu}\xi_\nu\right) = 0$
- The limit is richer in comparison with invariance under Galilean diffs.

More abstract equations - Galilean

Let us expand the energy–momentum tensor as

$$\begin{cases} \Omega^2 T^{00} = \varepsilon_{\mathrm{r}} = c^2 \rho + \Pi + \mathcal{O}\left({}^{1}\!/_{c^2}\right) \\ c\Omega T^{0}{}_{i} = q_{\mathrm{r}i} = c^4 \tilde{P}_i + c^2 P_i + \Pi_i + \mathcal{O}\left({}^{1}\!/_{c^2}\right) \\ T_{ij} = p_{\mathrm{r}} a_{ij} + \tau_{\mathrm{r}ij} = c^2 \tilde{\Pi}_{ij} + \Pi_{ij} + \mathcal{O}\left({}^{1}\!/_{c^2}\right) \end{cases}$$

yielding the equations

$$\begin{cases} \left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^{w}\right)\Pi + \Pi_{ij}\hat{\gamma}^{wij} + \hat{\nabla}_i \Pi^i = 0 \\ \left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^{w}\right)\rho + \tilde{\Pi}_{ij}\hat{\gamma}^{wij} + \hat{\nabla}_i P^i = 0 \\ \hat{\nabla}_j \tilde{P}^j = 0 \\ \left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^{w}\right)P_i + P_j \hat{\gamma}^{wj}{}_{i} + \hat{\nabla}^j \Pi_{ij} = 0 \\ \left(\frac{1}{\Omega}\frac{\hat{\mathrm{D}}}{\mathrm{d}t} + \theta^{w}\right)\tilde{P}_i + \tilde{P}_j \hat{\gamma}^{wj}{}_{i} + \hat{\nabla}^j \tilde{\Pi}_{ij} = 0. \end{cases}$$

Comments:

- The degrees of freedom are multiplied.
- These equations can be derived using diffs by incorporating additional fields.
- Again the conservation laws are no conservation laws, except if

$$\frac{P_i}{\Omega}\left(\partial_t \xi^{\hat{\imath}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{\imath}}\right) = 0 \quad \text{and} \quad \frac{\tilde{P}_i}{\Omega}\left(\partial_t \xi^{\hat{\imath}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{\imath}}\right) = 0$$

Plan of the talk

Revisit of the relativistic hydrodynamics

Galilean fluid dynamics

Carrolian fluid dynamics

Aristotelian fluid dynamics

CARROLIAN COVARIANCE

Let us again define a manifold $\mathcal{M} = \mathbb{R} \times S$ with coordinates $(t, \mathbf{x})$ equipped with

$$ds^2 = a_{ij} dx^i dx^j , \quad i = 1, \ldots, d$$

the field of observers $\mathrm{e}_{\hat{t}}$ and the clock form $\theta^{\hat{t}}$ (dual Ehresmann connection)

$$\mathrm{e}_{\hat{t}} = \frac{1}{\Omega} \partial_t \, , \quad \theta^{\hat{t}} = \Omega \mathrm{d}t - b_i \mathrm{d}x^i$$

where Ω, b_i and a_{ij} are functions of $(t, \mathbf{x})$. Properties:

- Invariance of $\mathrm{e}_{\hat{t}}, ds^2$ and $\theta^{\hat{t}}$ under Carrollian diffs: $t' = t'(t, \mathbf{x}) \, , \ \mathbf{x}' = \mathbf{x}'(\mathbf{x})$.
- Additional transformations

$$\partial'_t = \frac{1}{J} \partial_t \, , \quad \partial'_i = J^{-1j}{}_i \left(\partial_j - \frac{j_j}{J} \partial_t \right)$$

- Defining a new partial derivative

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t \, , \quad \hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j$$

We can define a torsionless and metric-compatible spatial connection

$$\hat{\nabla}_i a_{jk} = 0\,, \quad \hat{\gamma}^i{}_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right)$$

and the Carrollian vorticity and acceleration through

$$\left[\hat{\partial}_i, \hat{\partial}_j\right] = \frac{2}{\Omega}\varpi_{ij}\partial_t\,, \quad \varpi_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]}\,, \quad \varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right)$$

In addition, we also define the metric-compatible temporal connection

$$\hat{D}_t \Phi = \partial_t \Phi\,, \quad \frac{1}{\Omega}\hat{D}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j, \quad \hat{\gamma}_{ij} = \frac{1}{2\Omega}\partial_t a_{ij}$$

as well as the Carrollian expansion

$$\theta = a^{ij}\hat{\gamma}_{ij} = \frac{1}{\Omega}\partial_t \ln\sqrt{a}$$

CARROLLIAN DIFFEOMORPHISMS-I

Let us consider the action functional of Ω, w^i and a^{ij}

$$S = \int d^{d+1}x\,\Omega\sqrt{a}\mathcal{L}$$

and define the Carrollian momenta (energy, current and stress-tensor)

$$\Pi = -\frac{1}{\Omega\sqrt{a}}\left(\Omega\frac{\delta S}{\delta\Omega} + b_i\frac{\delta S}{\delta b_i}\right)$$

$$\Pi_i = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta b_i}, \quad \Pi^{ij} = \frac{2}{\Omega\sqrt{a}}\frac{\delta S}{\delta a_{ij}}$$

We would like to compute the variation

$$\delta_\xi S = \int dt\, d^d x \Omega\sqrt{a}\left(\frac{1}{2}\Pi^{ij}\delta_\xi a_{ij} + \Pi^i\delta_\xi b_i - \frac{1}{\Omega}\left(\Pi + b_i\Pi^i\right)\delta_\xi\Omega\right).$$

under Carrollian diffeomorphisms

$$\xi = \xi^t(t,\mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i = \left(\xi^t - \xi^i\frac{b_i}{\Omega}\right)\partial_t + \xi^i\left(\partial_i + \frac{b_i}{\Omega}\partial_t\right) = \xi^{\hat{t}}\frac{1}{\Omega}\partial_t + \xi^i\hat{\partial}_i$$

CARROLLIAN DIFFEOMORPHISMS-II

The Carrollian diffeomorphisms act infinitesimally on Ω, b_i and a_{ij} as

$$\mathcal{L}_\xi \ln\Omega = \frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i, \quad \mathcal{L}_\xi b_i = b_i \left(\frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_j \xi^j\right) - \left(\left(\hat{\partial}_i - \varphi_i\right) \xi^{\hat{t}} - 2\xi^j \varpi_{ji}\right)$$

$$\mathcal{L}_\xi a_{ij} = 2\hat{\nabla}_{(i}\xi^k a_{j)k} + 2\xi^{\hat{t}}\hat{\gamma}_{ij}$$

also on the field of observers $\mathrm{e}_{\hat{t}}$ and on the clock form $\theta^{\hat{t}}$

$$-\mathcal{L}_\xi \mathrm{e}_{\hat{t}} = \left(\frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i\right) \mathrm{e}_{\hat{t}}, \ \mathcal{L}_\xi \theta^{\hat{t}} = \left(\frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i\right) \theta^{\hat{t}} - \left(\left(\hat{\partial}_i - \varphi_i\right) \xi^{\hat{t}} - 2\xi^j \varpi_{ji}\right) \mathrm{d}x^i$$

Employing the above we find the energy and momentum equations

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$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\Pi + \left(\hat{\nabla}_i + 2\varphi_i\right)\Pi^i + \Pi^{ij}\hat{\gamma}_{ij} = 0\,,$$

$$\left(\hat{\nabla}_j + \varphi_j\right)\Pi^j{}_i + 2\Pi^j \varpi_{ji} + \Pi\varphi_i = -\left(\frac{1}{\Omega}\partial_t + \theta\right) P_i$$

where P_i is not defined throughout the variation – resulting from a boundary term.

Similarly invariance under gauge transformations: $\left(\frac{1}{\Omega}\partial_t + \theta\right)\rho + \left(\hat{\nabla}_i + \varphi_i\right) N^i = 0$

Isometries and the (non)-conservation

Killing fields of the Carrollian type satisfy

$$\mathcal{L}_\xi a_{ij} = 0, \quad \mathcal{L}_\xi e_{\hat{t}} = 0 \quad \Longrightarrow \quad \hat{\nabla}_{(i}\xi^k a_{j)k} + \xi^{\hat{t}}\hat{\gamma}_{ij} = 0, \quad \frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0$$

whereas the clock form $\theta^{\hat{t}}$ is not invariant.

An example $a^{ij} = \delta^{ij}, \Omega = 1$ and $b_i =$ constant with Carroll algebra $\mathfrak{carr}(d+1)$

$$\xi = \left(\Omega_i{}^j x^i + X^j\right)\partial_j + (T - B_i x^i)\partial_t \quad \Longrightarrow \quad \delta_\xi \theta^{\hat{t}} = \left(B_i + \Omega_i{}^j b_j\right) \mathrm{d}x^i \neq 0$$

Assuming an isometry, we have on-shell vanishing scalar (continuity equation)

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\kappa + \left(\hat{\nabla}_i + \varphi_i\right) K^i = 0\,, \quad \kappa = \xi^i P_i - \xi^{\hat{t}}\Pi\,, \quad K^i = \xi^j \Pi_j{}^i - \xi^{\hat{t}}\Pi^i$$

Using the energy & momentum on-shell conservation we find

$$\mathcal{K} = -\Pi^i \left(\left(\hat{\partial}_i - \varphi_i\right)\xi^{\hat{t}} - 2\xi^j \varpi_{ji}\right)$$

Comments:

- Even in flat space $\mathcal{K} = \Pi^i\left(B_i + \Omega_i{}^j b_j\right) \neq 0$, is not associated with a bnr term.
- The above construction extends for conformal isometries

$$\mathcal{L}_\xi a_{ij} = \lambda\, a_{ij}\,, \quad \mathcal{L}_\xi e_{\hat{t}} = \mu\, e_{\hat{t}}\,, \quad 2\mu + \lambda = 0$$

Carrollian hydro as a non-relativistic limit

Energy–momentum tensor admits a small-c expansion (Randers–Papapetrou frame)

$$\begin{cases} \frac{1}{\Omega^2} T_{00} = \varepsilon_{\mathrm{r}} = \Pi + \mathcal{O}\left(c^2\right), \\ -\frac{c}{\Omega} T_0{}^i = q_{\mathrm{r}}^i = \Pi^i + c^2 P^i + \mathcal{O}\left(c^4\right), \\ T^{ij} = p_{\mathrm{r}} a^{ij} + \tau_{\mathrm{r}}^{ij} = \Pi^{ij} + \mathcal{O}\left(c^2\right). \end{cases}$$

Inserting the above into the conservation equations $\nabla_\mu T^{\mu\nu} = 0$, leads to

$$\begin{cases} \frac{c}{\Omega} \nabla_\mu T^\mu{}_0 = \mathcal{E} + \mathcal{O}\left(c^2\right) = 0, \\ \nabla_\mu T^{\mu i} = \frac{1}{c^2}\left(\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\Pi^i + \Pi^j \hat{\gamma}_j{}^i\right) + \mathcal{G}^i + \mathcal{O}\left(c^2\right) = 0, \end{cases}$$

where

$$\mathcal{E} = -\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\Pi - \left(\hat{\nabla}_i + 2\varphi_i\right)\Pi^i - \Pi^{ij}\hat{\gamma}_{ij},$$

$$\mathcal{G}_j = \left(\hat{\nabla}_i + \varphi_i\right)\Pi^i{}_j + 2\Pi^i \varpi_{ij} + \Pi\varphi_j + \left(\frac{1}{\Omega}\hat{D}_t + \theta\right)P_j + P^i\hat{\gamma}_{ij},$$

including the constraint on the current Π^i, which is a bnr term from diff perspective.
Comments on $c \to 0$:

- The conservation equations $\nabla_\mu T^{\mu\nu} = 0$ are on-shell Carrollian boost invariant.
- The (non)-conservation conditions emerges as a limit $\nabla_\mu (T^{\mu\nu}\xi_\nu) = 0$.
- The limit is richer in comparison with invariance under Carrollian diffs.

More abstract equations - Carrollian

Let us expand the energy–momentum tensor as

$$\begin{cases} \frac{1}{\Omega^2} T_{00} = \varepsilon_{\mathrm{r}} = \frac{\tilde{\Pi}}{c^2} + \Pi + \mathcal{O}\left(c^2\right), \\ -\frac{c}{\Omega} T_0{}^i = q_{\mathrm{r}}^i = \frac{\tilde{\Pi}^i}{c^2} + \Pi^i + c^2 P^i + \mathcal{O}\left(c^4\right), \\ T^{ij} = p_{\mathrm{r}} a^{ij} + \tau_{\mathrm{r}}^{ij} = \frac{\tilde{\Pi}^{ij}}{c^2} + \Pi^{ij} + \mathcal{O}\left(c^2\right) \end{cases}$$

yielding the additional equations

$$-\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\tilde{\Pi} - \left(\hat{\nabla}_i + 2\varphi_i\right)\tilde{\Pi}^i - \tilde{\Pi}^{ij}\hat{\gamma}_{ij} = 0\,,$$

$$\left(\hat{\nabla}_i + \varphi_i\right)\tilde{\Pi}^i_j + 2\tilde{\Pi}^i\varpi_{ij} + \tilde{\Pi}\varphi_j + \left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\Pi_j + \Pi^i\hat{\gamma}_{ij} = 0\,,$$

$$\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\tilde{\Pi}_j + \tilde{\Pi}^i\hat{\gamma}_{ij} = 0\,.$$

Comments:

1. The degrees of freedom are multiplied.
2. These equations can be derived using diffs by incorporating additional fields.
3. Again the conservation equations do not imply conservation, except if

$$\Pi^i\left(\left(\hat{\partial}_i - \varphi_i\right)\xi^{\hat{t}} - 2\xi^j\varpi_{ji}\right) = 0 \quad \text{and} \quad \tilde{\Pi}^i\left(\left(\hat{\partial}_i - \varphi_i\right)\xi^{\hat{t}} - 2\xi^j\varpi_{ji}\right) = 0$$

Hydrodynamic frame invariance

In the relativistic case the frame transformations (local Lorentz) are given through

$$\delta\varepsilon = -2\frac{q^i\delta\beta_i}{\sqrt{1-c^2\boldsymbol{\beta}^2}},$$

$$\delta q^i = \frac{c^2\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}}\left(\frac{q^k\beta^i}{\sqrt{1-c^2\boldsymbol{\beta}^2}} - wh^{ki} - \tau^{ki}\right),$$

$$\delta\left(ph^{ij}+\tau^{ij}\right) = \frac{c^2\delta\beta_k}{1-c^2\boldsymbol{\beta}^2}\left(\beta^i\left(ph^{jk}+\tau^{jk}\right)+\beta^j\left(ph^{ik}+\tau^{ik}\right)\right) - \frac{\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}}\left(q^ih^{jk}+q^jh^{ik}\right).$$

Leaving $T_{\mu\nu}$ invariant.

In the Carrollian case we find

$$\varepsilon = \eta + \mathcal{O}\left(c^2\right), \quad p = \varpi + \mathcal{O}\left(c^2\right), \quad q^i = Q^i + c^2\pi^i + \mathcal{O}\left(c^4\right), \quad \tau^{ij} = -\Xi^{ij} + \mathcal{O}\left(c^2\right),$$

with transformations

$$\delta\eta = -2\delta\beta_iQ^i, \quad \delta Q^i = 0,, \quad \delta\pi^i = \delta\beta_j\left(\Xi^{ij} - (\eta+\varpi)a^{ij} + \beta^iQ^j\right), \quad \delta\left(\Xi^{ij} - \varpi a^{ij}\right) = \delta\beta_k\left(Q^ia^{jk}+Q^ja^{ik}\right).$$

Leaving Π, P_i, Π^i and Π^{ij} invariant.

In the Galilean case the hydrodynamic invariance is broken in the massive case. The velocity field and the fluid density are physical and observable quantities.

Comments

Electric and magnetic: C. Duval, G. W. Gibbons, P. A. Horváthy & P. M. Zhang '14

- Using the Hamiltonian approach M. Henneaux & P. Salgado-Rebolledo '21
- The magnetic Carrollian scalar field has a non-vanishing energy flux $\Pi^i_m \neq 0$

  D. Rivera-Betancour & M. Vilatte '22 (see Mathieu's talk)

  See also: S. Baiguera, G. Oling, W. Sybesma & B. T. Søgaard '22

Chern–Simons action and the Cotton tensor:

$$S_{\mathrm{CS}} = \frac{1}{2w_{\mathrm{CS}}} \int \mathrm{Tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right),$$

Its metric variation yields the Cotton tensor

$$C^{\mu}{}_{\nu} = \frac{\varepsilon^{\rho\lambda\mu}}{\sqrt{g}} \nabla_{\rho}\left(R_{\nu\lambda} - \frac{1}{4} R g_{\nu\lambda}\right)$$

having $C^{i}{}_{0} \neq 0$, in the limit of $c \longrightarrow 0$ in the Randers–Papapetrou frame.

Specific example: The Robinson–Trautman case for $k \longrightarrow 0$:

L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos & KS '18

$$ds^2 = -k^2 dt^2 + \frac{2}{P^2} d\zeta d\bar{\zeta}, \quad P = P(t, \zeta, \bar{\zeta})$$

where

$$C_{i0}\, dx^i = \frac{i}{2}\left(\partial_{\zeta} K d\zeta - \partial_{\bar{\zeta}} K d\bar{\zeta}\right) \neq 0, \quad K = 2P^2 \partial_{\zeta}\partial_{\bar{\zeta}} \ln P$$

Plan of the talk

Revisit of the relativistic hydrodynamics

Galilean fluid dynamics

Carrolian fluid dynamics

Aristotelian fluid dynamics

ARISTOTELIAN COVARIANCE

Let us again define a manifold $\mathcal{M} = \mathbb{R} \times S$ with coordinates $(t, \mathbf{x})$ equipped with
R. Penrose 68'

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i \, dx^j$$

along with the field of observers $e_{\hat{t}}$ and the clock form $\theta^{\hat{t}}$

$$e_{\hat{t}} = \frac{1}{\Omega} \partial_t \,, \quad \theta^{\hat{t}} = \Omega dt$$

The Aristotelian diffeomorphisms act as

$$t' = t'(t) \,, \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$$

We can define a temporal and a spatial metric-compatible covariant derivatives

$$\gamma_{ij} = \frac{1}{2\Omega} \partial_t a_{ij} \,, \quad \gamma^i{}_{jk} = \frac{a^{il}}{2} \left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk} \right)$$

as well as the expansion and the acceleration form

$$\theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a} \,, \quad \varphi_i = \partial_i \ln \Omega \,.$$

ARISTOTELIAN DIFFEOMORPHISMS

Take the action $S = \int d^{d+1}x\, \Omega\sqrt{a}\mathcal{L}$ and define the Aristotelian momenta

$$\Pi = -\frac{1}{\sqrt{a}}\frac{\delta S}{\delta\Omega}\,, \quad \Pi^{ij} = \frac{2}{\Omega\sqrt{a}}\frac{\delta S}{\delta a_{ij}}$$

Varying the action

$$\delta_\xi S = -\int dt\, \Omega \int d^d x \sqrt{a}\left(\frac{1}{2}\Pi^{ij}\delta_\xi a_{ij} + \Pi\delta_\xi \ln\Omega\right)$$

with respect to Aristotelian diffeomorphisms $\xi = \xi^t(t)\partial_t + \xi^i(\mathbf{x})\partial_i = \Omega\xi^t \frac{1}{\Omega}\partial_t + \xi^i\partial_i$ leads to the energy and momentum equations

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\Pi + \Pi^{ij}\gamma_{ij} = -\left(\nabla_i + 2\varphi_i\right)\Pi^i\,, \quad \left(\nabla_j + \varphi_j\right)\Pi^j{}_i + \Pi\varphi_i = -\left(\frac{1}{\Omega}\partial_t + \theta\right)P_i$$

where Π^i and P_i are not determined through the variation – boundary terms.

Comments on Aristotelian fluids:

- Introduced by J. de Boer, J. Hartong, N. Obers, W. Sybesma & S. Vandoren 17'
- ”Self-dual” resulting from Galilean or Carrollian with $w^i = 0$ or $b_i = 0$.

  L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos & KS ’18

ISOMETRIES AND THE (NON)-CONSERVATION

Similarly, invariance under gauge transformation leads to

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\rho + (\nabla_i + \varphi_i)\,N^i = 0$$

Killing fields of the Aristotelian type satisfy

$$\mathcal{L}_\xi a_{ij} = 0\,, \quad \mathcal{L}_\xi \mu = 0 \quad \Longrightarrow \quad \nabla_{(i}\xi^k a_{j)k} + \xi^{\hat{t}}\gamma_{ij} = 0, \quad \frac{1}{\Omega}\partial_t\xi^{\hat{t}} + \varphi_i\xi^i = 0$$

Assuming an isometry, we have on-shell vanishing scalar (continuity equation)

$$\mathcal{K} = \left(\frac{1}{\Omega}\partial_t + \theta\right)\kappa + (\nabla_i + \varphi_i)K^i\,, \quad \kappa = \xi^i P_i - \xi^{\hat{t}}\Pi, \quad K^i = \xi^j \Pi_j^{\ i} - \xi^{\hat{t}}\Pi^i$$

Using energy & momentum on-shell conservation – $\mathcal{K} = 0$ (no-extra constraints).

Conclusion & Outlook

We studied Galilean & Carrollian hydrodynamics on arbitrary backgrounds:

- Our approach was based on covariance and diffeomorphism invariance.
- Killing vectors do not guarantee an on-shell conservation.
- In agreement with the $c \longrightarrow \infty$ and $c \longrightarrow 0$ limits of $\nabla_\mu (T^\mu{}_\nu \xi^\nu) = 0$
- Limiting procedure is richer, further variables and equations.
- Compatible with diffeomorphism invariance, conjugate to new momenta.
- Richer structure is needed, connection with flat holography – flux balance Eqs.
  See Romain's talk

Hydrodynamic frame invariance:

- Relativistic fluid: Important property in reconstructing Einstein's spaces $\Lambda \neq 0$
  Bulk diffs: bnr diffs, Weyl transformations and local Lorentz transformations.
- Similarly for the Carrollian fluid for reconstructing Ricci-flat spaces $\Lambda = 0$
  Bulk diffs: bnr diffs, Weyl transformations and Local Carroll transformations.

A. Campoleoni, L. Ciambelli, C. Marteau, P. M. Petropoulos, KS 18';

L. Ciambelli, C. Marteau, P. M. Petropoulos & R. Ruzziconi 20'; A. Campoleoni, L. Ciambelli,, A. Delfante, C. Marteau, P. M. Petropoulos, R. Ruzziconi 22'

We also studied Aristotelian fluids, a limiting case of Galilean and Carrollian:

- Our approach was based again on covariance and diffeomorphism invariance.
- Killings guarantee an on-shell conservation.