# Carroll Geometry and Superrotation Vacua

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Carroll Workshop, UMons

Based on 2008.03321, 2105.10526 w/ Kévin Nguyen

and 2206.14178 w/ Figueroa-O'Farrill, Have, Prohazka

# Motivation (i)

- The infrared structure of gravity in asymptotically flat (AF) spacetimes is remarkably rich.
- Gravity in AF spacetimes gives rise to an infinite-dimensional symmetry group (at null infinity):

$$\mathrm{BMS} \simeq \underbrace{SO(3,1)}_{\text{Lorentz group}} \ltimes \underbrace{\mathcal{T}}_{C^\infty(S^2)\text{ supertranslations}}$$

- But vacuum (= Minkowski space) only invariant under Poincaré group $\Longrightarrow$ spontaneous symmetry breaking

vacuum is infinitely degenerate! [Geroch '77, Ashtekar '81]

# Motivation (ii)

(original) $\mathrm{BMS} \simeq \underbrace{SO(3,1)}_{\substack{\text{global conformal} \\ \text{symmetries of } S^2}} \ltimes \mathcal{T}$

$\Downarrow$

extended $\mathrm{BMS} \simeq \underbrace{\mathrm{Vir} \otimes \mathrm{Vir}}_{\substack{\text{local conformal} \\ \text{symmetries of } S^2}} \ltimes \mathcal{T}$ [Barnich, Troessaert]

more recently: include $\mathrm{Diff}(S^2)$ superrotations and/or Weyl

[Campiglia, Laddha; Barnich, Lambert; Ciambelli, Leigh, ....]

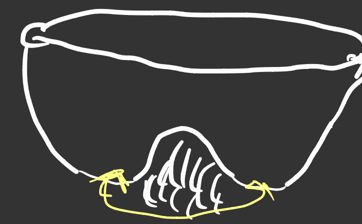
$\Longrightarrow$ vacuum highly degenerate $\Longrightarrow$ spontaneous symmetry breaking

$\Longrightarrow$ Goldstone bosons

# Motivation (iii)

Quick reminder:

spontaneously broken global $U(1)$ symmetry: scalar field w/ potential



$$\int (\partial\varphi)^2 \qquad \phi \to \phi + \alpha$$

$\varphi = \text{const} \Rightarrow$ zero modes

## This talk

1) find an effective action for superrotation vacua
2) find an action for Goldstone bosons of superrotation

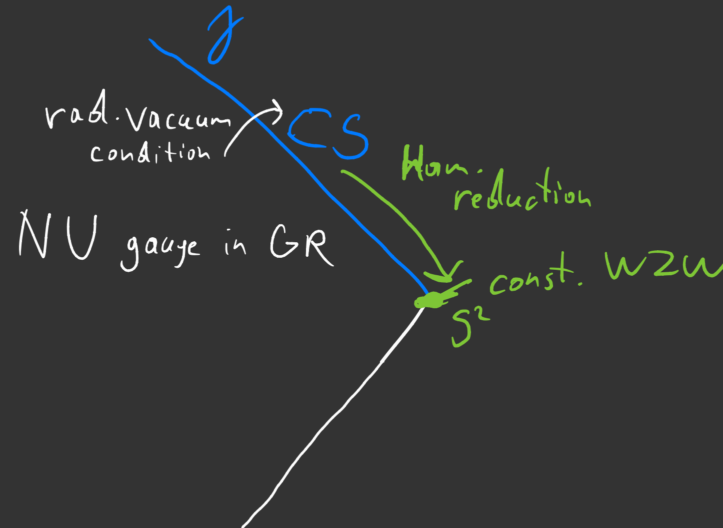
# Outline

1) Asymptotically flat gravity

2) The vacuum conditions & dynamical Carroll geometry on $\mathscr{I}^+$

3) The effective action of superrotation modes

4) Comments & Outlook



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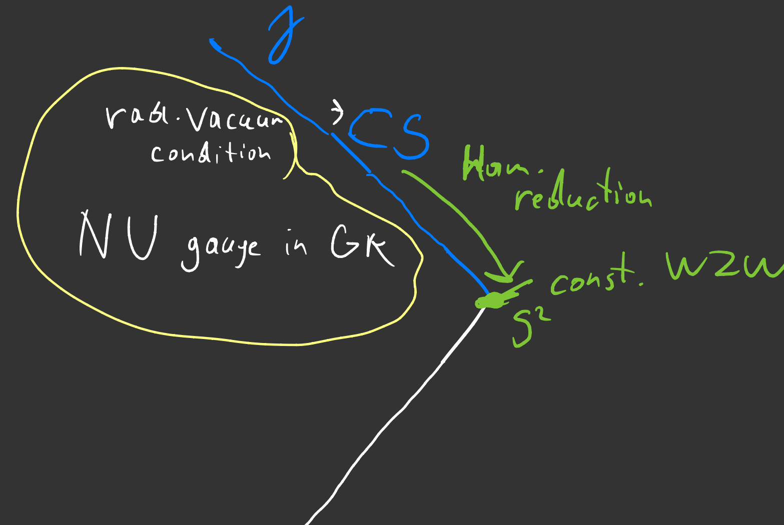
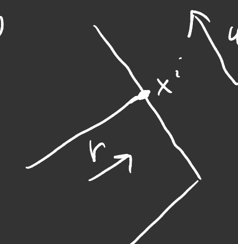


# Asymptotically flat gravity in Newman– Unti gauge (i)

$$ds^2 = g_{uu}\,du^2 - 2\,du\,dr + g_{ui}\,du\,dx^i + g_{ij}\,dx^i dx^j$$

$$g_{uu} = O(r) \qquad g_{ui} = O(1) \qquad g_{ij} = r^2\gamma_{ij} + r\,C_{ij} + \dots$$




work in first-order formulation

$$ds^2 = \eta_{IJ}\, E^I \otimes E^J$$

$$E^{\hat{u}} = du$$

$$E^{\hat{r}} = dr + E^{\hat{r}}_\mu\, dx^\mu$$

$$E^{a} = E^{a}_\mu\, dx^\mu$$

$$\eta_{IJ} = \begin{array}{c} \begin{array}{ccc} \hat{u} & \hat{r} & \overbrace{\quad}^{a,b} \end{array} \\ \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & \delta^a_b \end{pmatrix}_{IJ} \end{array}$$

# AF spacetimes in NU gauge (ii)

**tetrad expansion**

$$E^{\hat{u}} = du$$

$$E^{\hat{r}} = dr + (r\, h_\mu + h^{(0)}_\mu + \dots)\, dx^\mu$$

$$E^{a} = (r\, e^a_\mu + e^{a(0)}_\mu + \dots)\, dx^\mu$$

**spin connection expansion**

$$\Omega_\mu{}^{ab} = \omega_\mu \varepsilon^a{}_b + O(r^{-2})$$

$$\Omega_\mu{}^{\hat{r}a} = \sigma^a_\mu + O(r^{-1})$$

$$\Omega_r{}^{IJ} = 0$$

## Asymptotic Symmetries

**Local Lorentz $(\lambda, \lambda^a)$**

rotation of $e^a$: $\delta_\lambda e^a = \lambda \varepsilon^a{}_b e^b$

"null rotation": $\delta_{\lambda^a} h = \lambda_a e^a$

$\delta_{\lambda^a} e^a = \lambda^a h$

**asymp. diffeos $(\eta, \xi^\mu e^a_\mu)$**

Weyl rescalings: $\delta_\eta e^a = \eta e^a$

$\delta_\eta h = d\eta$

Diff$(S^2)$ superrot: $\delta_{\xi^i} e^a = \mathcal{L}_\xi e^a$

Supertranslations: do not act on leading order fields

# Radiative vacua of AF spacetimes

Solve first order Einstein equations order by order in r

- leading order torsion constraint

$$dh - e^a \wedge b_a = 0 \; ; \; de^a + \omega\, \varepsilon^a{}_b \wedge e^b - h \wedge e^a = 0$$

- scalar curvature $\sim \mathcal{O}(r^{-2})$ [true for most matter fields]

$$d\omega - \varepsilon_{ab}\, e^a \wedge b^b = 0$$

Additional condition: no radiation @ $\mathcal{J}^+$

$$db^a + h \wedge b^a + \omega\, \varepsilon^a{}_b \wedge b^b = 0$$

How to interpret these equations?

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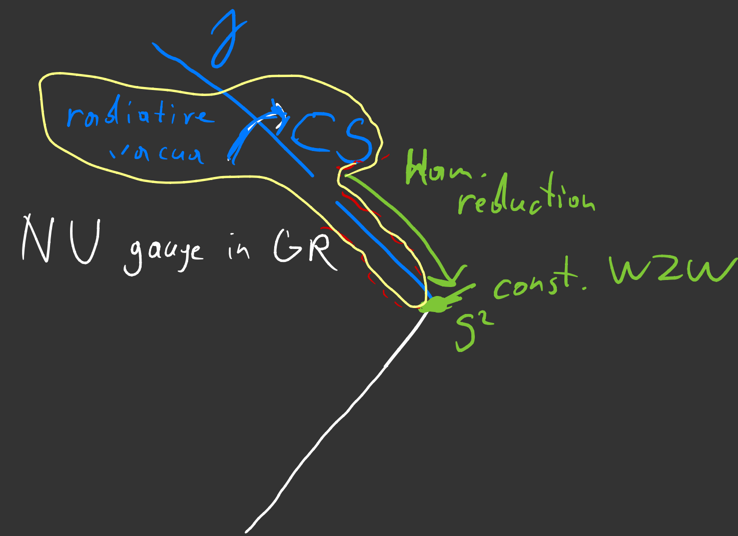

# Homogeneous Spaces of $ISO(3,1)$

$$\mathrm{Mink}_4 \simeq \frac{ISO(3,1)}{SO(3,1)}$$

choice of codimension 4 subgroup

Minkowski $\Longleftrightarrow$ $(\mathfrak{iso}(3,1), \mathfrak{so}(3,1))$ "Klein pair"

natural split of generators $T$ of $\mathfrak{g}$

"internal" $\Longleftrightarrow$ $T \in \mathfrak{so}(3,1)$

"external" $\Longleftrightarrow$ $T \in \mathfrak{g}/\mathfrak{so}(3,1)$

More generally: $(\mathfrak{iso}(3,1), \mathfrak{h}) \Longrightarrow (10 - \dim \mathfrak{h})$-dimensional homogeneous space of Poincaré

# Homogeneous Spaces of ISO(3,1)

$$\mathcal{J} = \frac{ISO(3,1)}{(ISO(2)\ltimes\mathbb{R})\ltimes\mathbb{R}^3}.$$

$$\text{light-cone}_3 \simeq \frac{SO(3,1)}{ISO(2)}$$ "$\mathcal{J}$ after forgetting about supertranslations"

$[J, B_a] = \varepsilon_a{}^b B_b \qquad [H, B_a] = B_a \qquad [B_a, P_b] = H\delta_{ab} - J\varepsilon_{ab}$

$[J, P_a] = \varepsilon_a{}^b P_b \qquad [H, P_a] = -P_a$

$\Longrightarrow$ "gauge the light cone"

[cf. 2112.03319 w/ Figueroa O'Farrill, Have, Prohazka
& Stefan's talk @ CW Vienna]

# Effective theory of radiative vacua [also Herfray; see also Hartong]

"Gauge the light-cone" [$\Longleftrightarrow$ construct Cartan geometry based on Klein pair]

$$A = h\,H + e^a P_a + b^a B_a + \omega\, J$$

Symmetry: $\delta_\Lambda A = d\Lambda + [A, \Lambda]$

$$\Lambda = \eta\, H + \pi^a P_a + \lambda^a B_a + \lambda\, J$$

act as: Weyl Diff$(S^2)$ "null rotation" spatial rotation

## Effective theory of radiative vacua [also: Herfray]

Define field strength: $F(A) = dA + A \wedge A$

$$\boxed{F(A) = 0 \quad \Longleftrightarrow \quad \text{rad. vacuum conditions}}$$

[NB: Until this point we could have "gauged" $\mathscr{I}$ to obtain also supertranslation vacua; cf. Herfray.]

However, for light-cone this follows from an action principle:

$$\boxed{S_{CS} = \frac{k}{2\pi} \int_{\mathscr{I}} \mathrm{tr}\left( A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A \right)}$$

Chern-Simons action

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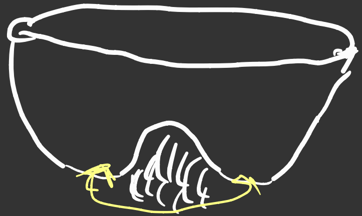

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$\Rightarrow$ CS action based on $(so(3,1), iso(2))$ ✓

2) find an action for Goldstone bosons of superrotation

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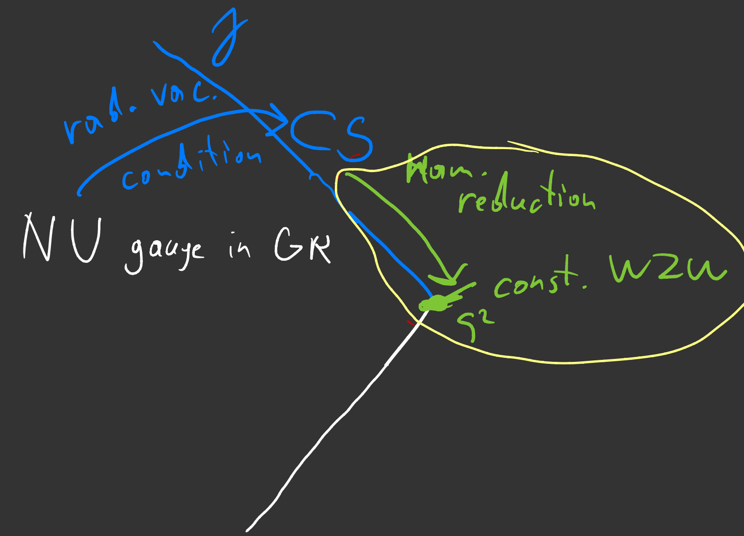
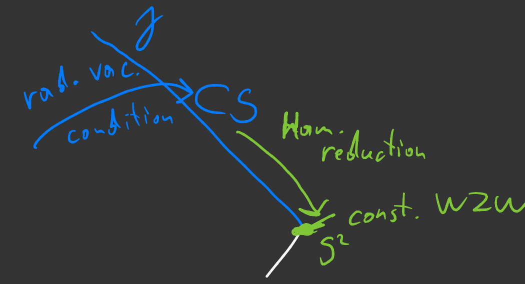


# Hamiltonian reduction of CS theory [Coussaert, Henneaux, van Driel; Cotler, Jensen]

- CS theory is topological $\Rightarrow$ dynamics happen at boundary




- impose bdy. conditions @ $\mathcal{J}^+_-$

- Hamiltonian reduction of CS:
  solve constraint $A_i = g^{-1}\partial_i g$

$$g = e^{B_a B^a}\, e^{\Lambda_w J + \Lambda_H H}\, e^{\Pi^a P_a}$$

$$\boxed{S[\Pi,\bar{\Pi}] = \frac{k}{2\pi}\int_{S^2} dz\, d\bar{z}\left[\frac{\partial\bar{\partial}\Pi\,\partial\Pi}{(\partial\Pi)^2} + \frac{\partial\bar{\partial}\bar{\Pi}\,\bar{\partial}\bar{\Pi}}{(\bar{\partial}\bar{\Pi})^2}\right]}$$

$\Pi = \Pi^1 + i\Pi^2$ ... Goldstone mode of Vir superrotations

2d effective action of superrotations

# The effective action of superrotation modes

$$\boxed{S[\Pi, \overline{\Pi}] = \frac{k}{2\pi} \int_{S^2} dz\, d\bar{z} \left[ \frac{\partial_z \partial_{\bar{z}} \Pi\, \partial_z \Pi}{(\partial_z \Pi)^2} + \frac{\partial_{\bar{z}} \partial_z \overline{\Pi}\, \partial_{\bar{z}} \overline{\Pi}}{(\partial_{\bar{z}} \overline{\Pi})^2} \right].}$$

zero modes: $\Pi(z,\bar{z}) = \Pi(z)$ , $\overline{\Pi}(z,\bar{z}) = \overline{\Pi}(\bar{z})$

labels $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$ Vacua

$SO(3,1)$ gauge symmetry: $\Pi \longmapsto \dfrac{a\Pi + b}{c\Pi + d}$ $\quad ad - bc = 1$

$\Pi, \overline{\Pi} \in \mathrm{Vir} \otimes \overline{\mathrm{Vir}} / PSL(2,\mathbb{C})$ $\Longrightarrow$ Goldstone modes of broken superrotation

# Conclusion

- Leading order fields in NU gauge obey equations of $SO(3,1)$ CS theory when restricted to radiative vacua

- Interpret $SO(3,1)$-CS theory as "gauging of light-cone"

- $SO(3,1)$ CS theory for vacuum sector leads to Alekseev-Shatashvili theory on boundary of $\mathcal{J}$ $\Longrightarrow$ effective 2d description of superrotation vacua

# Outlook

- "Gauge" Scri $\Longrightarrow$ include supertranslations
  [Herfray]
  write down action!

- physical significance? related to IR divergences? CCFT?
  [supertranslation Goldstone bosons related to
  IR divergences] [Himwich, Narayanan, Pate, Paul, Strominger; Nguyen JS]

- homogeneous space: AdS-Carroll $\simeq \frac{ISO(3,1)}{ISO(2,1)}$
  describes "time-like" infinity, similar "space-like" infinity
  "gauged" + Lagrangian in 2206.14178 w/ Figueroa-O'Farrill, Have, Prohazka

$\Longrightarrow$ relate to vacua at space-like infinity

Thank You !