Magnetic Carrollian gravity from the Carroll algebra

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based on [2207.14167] with A. Campoleoni, M. Henneaux, A. Pérez and P. Salgado-Rebolledo
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PREVIOUSLY ON

CARROLL WORKSHOP

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Carrollian theories of gravity (Hamiltonian)

Electric Carrollian limit of gravity

Again, the limits are most conveniently taken in the Hamiltonian formulation.

The (Dirac-ADM) Hamiltonian action for Einstein gravity reads

$$S[g_{ij}, \pi^{ij}, N, N^i] = \int dx^0 \int d^d x (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i)$$

(where we do not write explicitly the surface terms, which depend on the boundary conditions).

Here, $\mathcal{H} \approx 0$ is the Hamiltonian constraint and $\mathcal{H}_i \approx 0$ is the momentum constraint with the following explicit expressions (in appropriate units and with appropriate rescalings, see below)

$$\mathcal{H} = G_{ijkm} \pi^{ij} \pi^{mn} - c^6 R \sqrt{g}, \qquad \mathcal{H}_i = -2\pi_i{}^j{}_{|j}.$$

M. Henneaux "Carroll-invariant field theories" @ Carroll Workshop, Vienna

Carrollian expansion of gravity (metric)

LO and NLO action

- expand R for c→0; LO action:

$$\overset{(2)}{\mathcal{L}}_{\rm LO} = \frac{e}{16\pi G_N}\left[K^{\mu\nu}K_{\mu\nu} - K^2\right],$$

Henneaux,1979

$$\delta\overset{(2)}{\mathcal{L}}_{\rm LO} = \frac{e}{8\pi G_N}\left[\overset{(2)}{G}{}^v_\mu \delta v^\mu + \frac{1}{2}\overset{(2)}{G}{}^h_{\mu\nu}\delta h^{\mu\nu}\right],$$

$$\overset{(2)}{G}{}^v_\mu = -\frac{1}{2}\tau_\mu(K^{\rho\sigma}K_{\rho\sigma} - K^2) + h^{\gamma\lambda}\tilde{\nabla}_\lambda(K_{\mu\gamma} - K h_{\mu\gamma}),$$

$$\overset{(2)}{G}{}^h_{\mu\nu} = -\frac{1}{2}h_{\mu\nu}(K^{\rho\sigma}K_{\rho\sigma} - K^2) + K(K_{\mu\nu} - K h_{\mu\nu}) - v^\rho\tilde{\nabla}_\rho(K_{\mu\nu} - K h_{\mu\nu}).$$

- NLO action:

$$\overset{(4)}{\mathcal{L}}_{\rm NLO} = \frac{e}{8\pi G_N}\left[\frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu} + \overset{(2)}{G}{}^v_\mu M^\mu + \frac{1}{2}\overset{(2)}{G}{}^h_{\mu\nu}\Phi^{\mu\nu}\right].$$

see also: Bergshoeff et al,2017

N. Obers "Carroll symmetry in field theory and gravity" @ Carroll Workshop, Vienna
see also G. Oling

Electric and Magnetic theories

- The $c \to 0$ ("zero-signature" or "strong coupling") limit of Einstein relativity was first written in Hamiltonian form [Isham '76], [Teitelboim '78, '82] then in Lagrangian form [Henneaux '79].

- At least two ways to define a Carrollian limit: Electric and Magnetic
Recently, the previous theory was realized as an Electric contraction of GR in Hamiltonian form, and a new Magnetic contraction was constructed [Henneaux, Salgado-Rebolledo '21].

$$S[g_{ij}, \pi^{ij}, N, N^i] = \int dx^0 \int d^d x (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i) \qquad \mathcal{H}_i = -2\pi_i{}^j{}_{|j}.$$

$$\mathcal{H}^E = G_{ijkm}\pi^{ij}\pi^{mn} \qquad \mathcal{H}^M = -R\sqrt{g}$$

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$$\boxed{\mathcal{H}^E = G_{ijkm}\pi^{ij}\pi^{mn}} \qquad \boxed{\mathcal{H}^M = -R\sqrt{g}}$$

- Electric and Magnetic theories are qualitatively different:
  - in the Electric case, π^{ij} can be eliminated and becomes the extrinsic curvature but in the Magnetic case, it is a Lagrange multiplier and cannot be eliminated
  - different space of solutions [Hansen, Obers, Oling, Søgaard '22] and asymptotic symmetries [Pérez '21, '22], [Fuentealba, Henneaux, Salgado-Rebolledo, Salzer '22]
  - Hamiltonian analysis performed in [Sengupta '22].

Theories of Carrollian gravity

In metric and Hamiltonian form: more than one theory of Carrollian gravity

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What about gauge theories?

Gauging procedure and the Carrollian case

- "Gauging" an algebra $\mathfrak{g}$ is not a straightforward procedure (see José's talk!)

- Tentative gauging procedure (for the purpose of this talk only):
  - select a Klein pair $(\mathfrak{g}, \mathfrak{h})$ modeling spatially isotropic homogeneous space $M \simeq \mathfrak{g}/\mathfrak{h}$
  - write a gauge connection taking values in $\mathfrak{g}$, gauge fields in $\mathfrak{g}/\mathfrak{h}$ is a (co)frame
  - identify curvatures of gauge fields in $\mathfrak{h}$ as "curvatures" and in $\mathfrak{g}/\mathfrak{h}$ as "torsions"
  - impose zero torsions, build an action from $\mathfrak{h}$-gauge-invariant objects (curvature, metric).

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- Leftover ambiguity, e.g. in Carroll spacetime, the vanishing of the torsion two-form is not enough to entirely fix the connection $\Gamma^{\rho}{}_{\mu\nu}$
  - $\Gamma^{\rho}{}_{\mu\nu}$ is "torsion-free" (symmetric) iff the second fundamental form vanishes
  - when this is the case, $\Gamma^{\rho}{}_{\mu\nu}$ is only defined up to a shift by $n^{\rho} S_{\mu\nu}$ for any symmetric $S_{\mu\nu}$.

- Unsurprisingly, there is no unique proposition for a gauge theory of Carroll transformations.

Comparing gauge theories with metric ones

$$S = \int d^3x e \left[C \left(K_{\mu\nu} K_{\rho\sigma} \hat{h}^{\mu\rho} \hat{h}^{\nu\sigma} - \lambda \left(\hat{h}^{\mu\nu} K_{\mu\nu} \right)^2 \right) - \mathcal{V} \right]$$

[Hartong '15]

$$S_{\text{Car}} = -\frac{1}{16\pi G_C} \int e \left(2\tau^\mu e_a^\nu R(G)_{\mu\nu}{}^a + e_a^\mu e_b^\nu R(J)_{\mu\nu}{}^{ab} \right)$$

[Bergshoeff, Gomis, Rollier, Rosseel and ter Veldhuis '17]

- Choosing $\mathfrak{g} = \text{span}\left\{ J_{ab}, C_a, P_a, H \right\}$ and $\mathfrak{h} = \text{span}\left\{ J_{ab}, C_a \right\}$ → action of [Bergshoeff et al.]. Similar to the Magnetic theory! But is it really the same?

- On the other hand, the action of [Hartong] looks more Electric theory (but it is also more general, depending on the choice of potential $\mathscr{V}$). It doesn't follow the previous procedure.

The (A)dS Carroll algebra

- Start with the $(D+1)$-dimensional (A)dS Carroll algebra, with $\lambda^2 = -\,\sigma \dfrac{2\Lambda}{D(D-1)}$

$$
\begin{aligned}
[J_{ab}\,,\,J_{cd}] &= \delta_{ac}\,J_{db} + \delta_{bd}\,J_{ca} - \delta_{ad}\,J_{cb} - \delta_{bc}\,J_{da}\,, & [J_{ab}\,,\,P_c] &= \delta_{cb}\,P_a - \delta_{ca}\,P_b\,,\\
[J_{ab}\,,\,C_c] &= \delta_{cb}\,C_a - \delta_{ca}\,C_b\,, & [C_a\,,\,P_b] &= \delta_{ab}\,H\,,\\
[H\,,\,P_a] &= \sigma\lambda^2\,C_a\,, & [P_a\,,\,P_b] &= \sigma\lambda^2\,J_{ab}\,.
\end{aligned}
$$

- Also written in a compact way

$$
\begin{aligned}
[J_{AB}\,,\,J_{CD}] &= \zeta_{AC}\,J_{DB} + \zeta_{BD}\,J_{CA} - \zeta_{AD}\,J_{CB} - \zeta_{BC}\,J_{DA}\,,\\
[J_{AB}\,,\,P_C] &= \zeta_{CB}\,P_A - \zeta_{CA}\,P_B\,,\\
[P_A\,,\,P_B] &= \sigma\lambda^2\,J_{AB}\,,
\end{aligned}
$$

using the $c \to 0$ limit of the Minkowski metric $\zeta_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ab} \end{pmatrix}$ degenerate along $n^A = \delta^A_0$.

Carrollian connection

- Connection one-form

$$A_\mu = \frac{1}{2}\,\omega_\mu{}^{ab}\,J_{ab} + e_\mu{}^a\,P_a + \omega_\mu{}^a\,C_a + \tau_\mu\,H\,.$$

- Non-degenerate soldering form $E_\mu{}^A = \big(e_\mu{}^a\,,\,\tau_\mu\big) \;\leftrightarrow\;$ vielbein $E^\mu{}_A = \big(e^\mu{}_a\,,\,n^\mu\big)$

$$e_\mu{}^a\,e^\mu{}_b = \delta^a_b\,,\quad \tau_\mu\,e^\mu{}_a = 0\,,\quad n^\mu\,e_\mu{}^a = 0\,,\quad \tau_\mu\,n^\mu = 0\,,\quad e_\mu{}^a\,e^\nu{}_b + \tau_\mu\,n^\nu = \delta^\nu_\mu\,.$$

- Degenerate metric $g_{\mu\nu} = e_\mu{}^a\,e_\nu{}^b\,\delta_{ab}$ along the null n^μ-direction: $n^\mu\,g_{\mu\nu} = 0$ and non-vanishing density $E = \det(E_\mu{}^A) \;\rightarrow\; (g_{\mu\nu}\,,\,n^\mu\,,\,E)$ "minimal" Carrollian geometry [Henneaux '79].

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- They are gauge-invariant under *local Carroll rotations and boosts*

$$\delta\,E = 0\,,\quad \delta\,n^\mu = 0\,,\quad \delta\,g_{\mu\nu} = 0\,.$$

- Extrinsic curvature (second fundamental form) $\quad K_{\mu\nu} = -\dfrac{1}{2}\mathscr{L}_n\,g_{\mu\nu}\,.$

Structure equations and action

- Decomposing $F = d\,A + \frac{1}{2}[A, A]$ on the generators of the Carroll algebra H, P_a, C_a and J_{ab}

$$
\begin{aligned}
F_{\mu\nu} &= \partial_{[\mu}\tau_{\nu]} + \omega_{[\mu}{}^{a} e_{\nu]a} && = T_{\mu\nu}\,,\\
F_{\mu\nu}{}^{a} &= \partial_{[\mu} e_{\nu]}{}^{a} + \omega_{[\mu}{}^{ab} e_{\nu]b} && = T_{\mu\nu}{}^{a}\,,\\
F_{\mu\nu}{}^{a} &= \partial_{[\mu}\omega_{\nu]}{}^{a} + \omega_{[\mu}{}^{ab}\omega_{\nu]b} + \sigma\lambda^2\tau_{[\mu} e_{\nu]}{}^{a} && = R_{\mu\nu}{}^{a} + \sigma\lambda^2\tau_{[\mu} e_{\nu]}{}^{a}\,,\\
F_{\mu\nu}{}^{ab} &= \partial_{[\mu}\omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b} + \sigma\lambda^2 e_{[\mu}{}^{a} e_{\nu]}{}^{b} && = R_{\mu\nu}{}^{ab} + \sigma\lambda^2 e_{[\mu}{}^{a} e_{\nu]}{}^{b}\,.
\end{aligned}
$$

- Impose $T_{\mu\nu} = 0$ and $T_{\mu\nu}{}^{a} = 0$, corresponding to $\mathfrak{g} = \mathfrak{carr}(1, D)$ and $\mathfrak{h} = \mathfrak{so}(D) \ltimes \mathbb{R}^D$.

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- Obvious candidate action is the $c \to 0$ limit of Einstein-Cartan [Bergshoeff et al. '17]

$$
I_{Car}[E_{\mu}{}^{A}, \omega_{\mu}{}^{AB}] = \frac{1}{16\pi G_M}\int dt\, d^{D}x\; E\left(e^{\mu}{}_{a}e^{\nu}{}_{b}R_{\mu\nu}{}^{ab} + 2\,n^{\mu}e^{\nu}{}_{a}R_{\mu\nu}{}^{a} - 2\,\Lambda\right).
$$

- Variation with respect to $\omega_{\mu}{}^{a}$ and $\omega_{\mu}{}^{ab}$ gives rise to the torsion constraints $T_{\mu\nu} \approx 0$ and $T_{\mu\nu}{}^{a} \approx 0$.

From first-order to Hamiltonian

- Zero torsion → all components of the spin-connections are determined *except* $S_{(ab)} \equiv \omega_{(ab)}{}^0$, which is a Lagrange multiplier for the first-class constraint $K_{ab} = e^\mu{}_a e^\nu{}_b K_{\mu\nu} \approx 0$.

- Local Carroll boost-invariance → fix "time gauge" $\tau_i = 0$. Remaining degrees of freedom

$$n^\mu = \left(\frac{1}{N}, \, -\frac{N^i}{N}\right), \quad e^\mu{}_a = \left(0\,,\, \mathbf{e}^i{}_a\right)\,, \quad \tau_\mu = (N\,,\,0)\,, \quad e_\mu{}^a = \left(\mathbf{e}_i{}^a N^i\,,\, \mathbf{e}_i{}^a\right)\,,$$

- The action reads

$$I_{Car}\,[E_\mu{}^A, \omega_\mu{}^{AB}] = \frac{1}{16\pi G_M}\int dt\, d^Dx \;\mathbf{e}\left(2\,\mathbf{e}^j{}_a R_{tj}{}^a - 2\,N^i\,\mathbf{e}^j{}_a R_{ij}{}^a + N\,\mathbf{e}^i{}_a\,\mathbf{e}^j{}_b R_{ij}{}^{ab} - 2\,N\,\Lambda\right).$$

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  - kinetic term

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  - kinetic term
  - spatial Hamiltonian density

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  - kinetic term
  - spatial Hamiltonian density
  - transverse Hamiltonian density.

From first-order to Hamiltonian

- Eliminating the spatial spin connection $\omega_i{}^{jk}$, the curvature term takes a familiar form

$$\mathbf{e}^i{}_a\, \mathbf{e}^j{}_b\, R_{ij}{}^{ab} = R\ ,$$

where $\quad R = h^{ij}\left(\partial_k \gamma^k{}_{ij} - \partial_i \gamma^k{}_{jk} + \gamma^l{}_{ij}\gamma^k{}_{kl} - \gamma^l{}_{ik}\gamma^k{}_{jl}\right) \quad$ and $\quad \gamma^k{}_{ij} = \frac{1}{2} h^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$.

- Eliminating the rest of ω, the action only depends on the gauge-invariant metric fields g_{ij} , N^i , N as well as $S_{ij} = \mathbf{e}_i{}^a\, \mathbf{e}_j{}^b\, S_{ab}$ as a Lagrange multiplier

$$I_{Car}[g_{ij}, N, N^i, \pi^{ij}] = \int dt\, d^D x\, \left(\pi^{ij}\dot{g}_{ij} + 2\, N_i\, \nabla_j\, \pi^{ij} - N\, \mathscr{H}_M\right)\ ,$$

where we defined $\pi^{ij} = \frac{\sqrt{g}}{16\pi G_M}\left(S^{ij} - h^{ij} S'\right)$ and $\mathscr{H}_M = -\frac{\sqrt{g}}{16\pi G_M}(R - 2\,\Lambda)$.

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- It is an action which is 1st order in time derivative and 2nd order in spatial derivatives … and it is exactly the action of the Magnetic theory in Hamiltonian form!

Carrollian rewriting of Einstein-Cartan

- One can play a similar game, starting from the Einstein-Cartan action gauging the (A)dS algebra

$$I_{EC} = \frac{c^3}{16\pi G_N} \int dt\, d^D x\, \mathcal{E} \left(\mathcal{E}^{\mu}{}_{A}\, \mathcal{E}^{\nu}{}_{B}\, \mathcal{R}_{\mu\nu}{}^{AB} - 2\,\Lambda \right) .$$

- We use $c = \varepsilon\,\hat{c}$ and set the geometrical units $\hat{c} = 1$ so that the Carrollian limit is $\varepsilon \to 0$.

- Parametrising the vielbein as in [Pilati '78], [Castellani, van Nieuwenhuizen, Pilati '82]

$$\mathcal{E}^{\mu}{}_{A} = \left(-\frac{1}{\varepsilon N}\, n_A\, ,\, \frac{N^i}{\varepsilon N}\, n_A + e^{i}{}_{A} \right) , \quad \mathcal{E}_{\mu}{}^{A} = \left(e_i{}^{A}\, N^i + \varepsilon N\, n^A\, ,\, e_i{}^{A} \right) ,$$

  reproduces the standard metric of [Arnowitt, Deser, Misner '59]

$$g_{\mu\nu} = \begin{pmatrix} N^i N_i - \varepsilon^2 N^2 & N_i \\ N_i & g_{ij} \end{pmatrix} .$$

Partial solving of the torsion constraints

- Full torsion constraint allows to eliminate completely the spin connection $\Omega_\mu{}^{AB}$

$$\mathscr{T}_{\mu\nu}{}^A = \partial_{[\mu}\,\mathscr{E}_{\nu]}{}^A + \Omega_{[\mu}{}^{AB}\,\mathscr{E}_{\nu]}{}^C\,\eta_{BC} \approx 0\,.$$

- One can choose instead to eliminate **almost all** components of the spin connection

$$\mathscr{T}_{ij\perp} \approx 0\,,\quad \mathscr{T}_{ijk} \approx 0\,,\quad \mathscr{T}_{ti\perp} \approx 0\,,\quad \mathscr{T}_{t[ij]} \approx 0\,. \qquad (X_\perp \equiv X^A\,n_A)$$

Can be done consistently by varying with respect to $\left\{\Omega_{[ij]\perp}\,,\,\Omega_{ij}{}^k\,,\,\Omega_{ti\perp}\,,\,\Omega_{tij}\right\}$.

- Remaining components $\mathscr{T}_{t(ij)}$ are singular when $\varepsilon \to 0$.

- Corresponding components of the spin connection $\Omega_{(ij)\perp}$ cannot be solved in the Carrollian limit and are related to the conjugate momenta to the metric.

Magnetic action from the limit

- Choosing the time gauge once again so that $n^A = \delta^A_0$ and performing a Legendre transform

$$p^i{}_A \equiv \frac{\delta \mathscr{L}_{EC}}{\delta \dot{e}_i{}^A} \quad \text{and defining} \quad \pi^{ij} \equiv \frac{1}{2} p^{(i}{}_A\, e^{j)A} = -\frac{\sqrt{g}}{16\pi G_M} \left(\Omega^{(ij)}{}_\perp - h^{ij}\, \Omega_k{}^k{}_\perp\right) ,$$

we recover directly the Hamiltonian formulation of general relativity

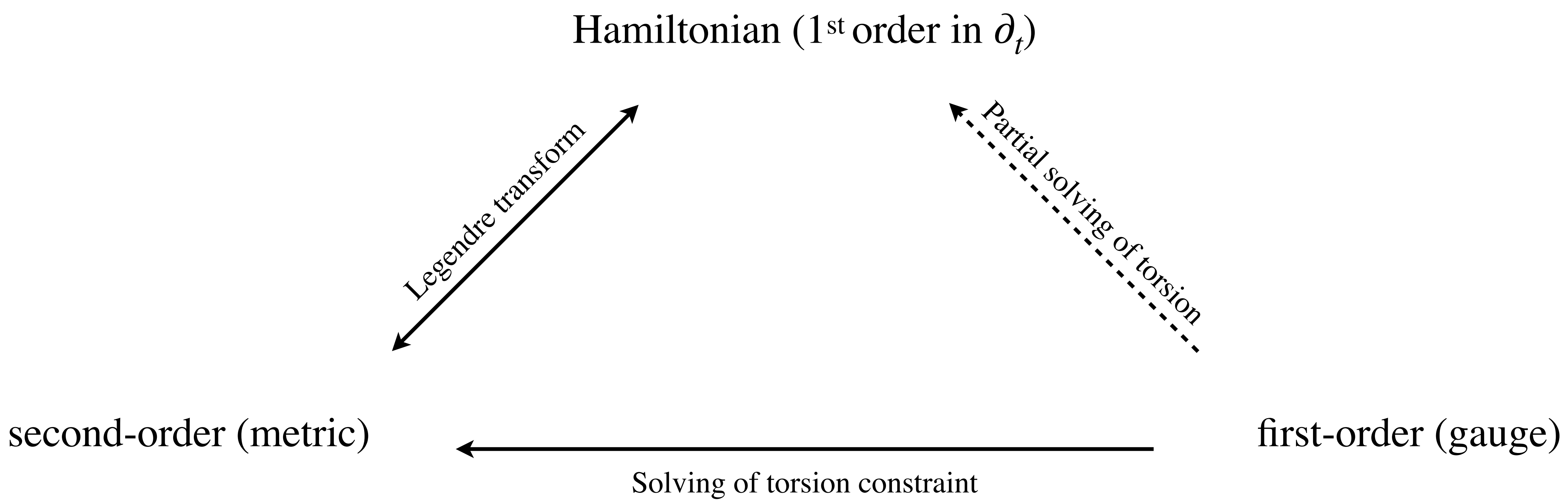
$$I_{ADM}[g_{ij}, N, N^i, \pi^{ij}] = \int dt\, d^D x \left(\pi^{ij} \dot{g}_{ij} + 2\, N_i\, \nabla_j\, \pi^{ij} - N\, \mathscr{H}_\perp\right) ,$$

$$\text{with} \quad \mathscr{H}_\perp = \mathscr{H}_M + \varepsilon^2 \left[\frac{16\pi G_M}{\sqrt{g}} \left(g_{il}\, g_{jk} - \frac{1}{D-1}\, g_{ij}\, g_{kl}\right) \pi^{ij}\, \pi^{kl}\right] .$$

- Magnetic theory is obtained by sending $\varepsilon \to 0$ (Electric potential term subleading in ε) provided one rescales Newton's constant $G_N = \varepsilon^4\, G_M$.

Conclusion and outlook

- *Relativistic* theory of GR is the same in second-order and Hamiltonian formulations. The $\big(\mathfrak{g} = \mathfrak{iso}(1,D)\,,\ \mathfrak{h} = \mathfrak{so}(1,D)\big)$ Einstein-Cartan action is also equivalent.



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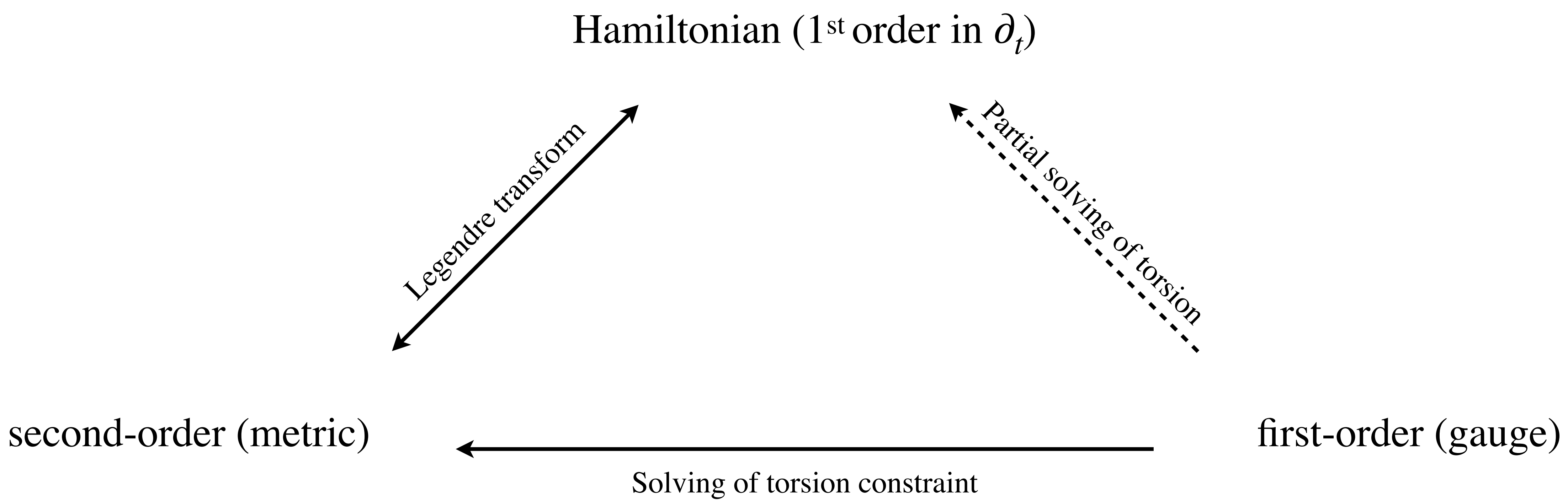

- This is not true in Carroll gravity: the theory emerging from the Carrollian limit is Electric in second-order formulation, but Magnetic in Hamiltonian formulation.

- The $\left(\mathfrak{g} = \mathfrak{carr}(1,D)\,,\ \mathfrak{h} = \mathfrak{so}(D) \ltimes \mathbb{R}^D\right)$ first-order action reproduces the Magnetic theory.

- ***Future work***: coupling of fermionic fields to Magnetic gravity in first-order formulation.

Looking for the Electric theory

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- Final thoughts:
  - in first-order formulation, one can always reverse the hierarchy by rescaling fields, however the action *is not invariant under local Carroll boosts* in the limit
  - in second-order formulation, the *full* Magnetic theory can be recovered using an auxiliary field, leading to the same hierarchy as in first-order.
  - ***Question***: can we apply this trick to get a first-order formulation of Electric gravity?

Carrollian wonderland