Cartan Geometries with model the lightlike cone

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Second Carroll Workshop,
12 September – 16 September 2022
Mons, Belgium

OUTLINE

1. Carroll geometries
2. Cartan connections
3. Cartan geometries with model the lightlike cone

1. Carroll geometries

1. A general definition, examples and something more
2. The lightlike cone.

General Carrollian geometries (Levy-Leblond, 1965)

A triple (N^{m+1}, h, Z) is a Carrollian geometry when

1. h is a symmetric $(0,2)$-tensor on the manifold N^{m+1} which is positive but non-definite and whose radical

$$\mathrm{Rad}\, h = \{v \in TN : h(v, -) = 0\}$$

defines a 1-dimensional distribution on N^{m+1}.

2. $Z \in \Gamma(\mathrm{Rad}\, h) \subset \mathfrak{X}(N)$ spans the radical distribution at every point.

- $\boxed{\mathrm{Aut}(N^{m+1}, h, Z)}$ may be infinite dimensional.

Examples I

- Lightlike hypersurfaces of time-oriented Lorentzian manifolds.
- A General Relativity space-time $(\widetilde{M}, \widetilde{g})$ is called asymtotically simple, if there is another Lorentzian manifold (M, g) such that
  1. $\widetilde{M}$ is an open subset of M with smooth boundary $\partial\widetilde{M} = \mathcal{I}$.
  2. There is $\Omega \in C^{\infty}(M)$ such that $g = \Omega^2 \widetilde{g}$ on $\widetilde{M}$ and $\Omega \mid_{\mathcal{I}} = 0$ but $d\Omega \neq 0$ on $\mathcal{I}$.
  3. Every inextendible lightlike geodesic of $(\widetilde{M}, \widetilde{g})$ has a future/past endpoint on $\mathcal{I}$.

  Then,

$$\left(\mathcal{I}, g \mid_{\mathcal{I}}, Z := (d\Omega)^{\sharp}\right) \text{ is a Carroll geometry.}$$

Examples II. The bundle of scales of a conformal Riemannian manifold (M, c)

1. $$\pi \colon \mathcal{L} = \{g_x : g \in c, x \in M\} \to M$$
   is a principal fiber bundle with structure group $\mathbb{R}_+$ ($g_x \cdot t := t^2 g_x$).
2. $$h(\xi, \eta) = g_x(T_{g_x}\pi \cdot \xi, T_{g_x}\pi \cdot \eta), \quad \xi, \eta \in T_{g_x}\mathcal{L},$$
3. $$Z_{g_x} = \frac{d}{dt}\,|_{t=0}\,(e^{2t} g_x).$$

$(\mathcal{L}, h, Z)$ is a Carrollian geometry.

Remarks I

- h induces a bundle-like Riemannian metric $\bar{h}$ on the quotient vector fiber bundle
$$\mathcal{E} := TN/\mathrm{Rad}h \to N.$$
Thus, we can define the endomorphism A_Z on the fiber vector bundle $\mathcal{E}$ by
$$\mathcal{L}_Z\bar{h}([u],[v]) = 2\bar{h}(A_Z[u],[v]), \quad u, v \in T_yN.$$
(N^{m+1}, h, Z) is generic when A_Z is an isomorphism on $\mathcal{E}$.
- The space orbit N^{m+1}/Z is typically a manifold M^m (the absolute space)
$$\pi\colon N^{m+1} \to M^m := N^{m+1}/Z.$$
- The vector field Z can be assumed to be complete and then (suppose that integral curves of Z are lines)
$$N^{m+1} \times \mathbb{R}_+ \to N^{m+1}, \quad (y,t) \mapsto y \cdot t = \mathrm{Fl}^Z_{\log t}(y)$$
and $\pi\colon N \to M$ becomes an $\mathbb{R}_+$-principal fiber bundle.

Remarks II

Carroll geometries as generalized bundles of scales

Assume $\pi \colon N^{m+1} \to M^m$.

$$\Big(\underbrace{T_y N/\mathrm{Rad}(h_y)}_{\mathcal{E}_y}, h_y\Big) \xrightarrow{T_y\pi} \Big(T_{\pi(y)}M, c(y)\Big) \quad \Longrightarrow \quad c(y) \in \mathrm{Sym}^+(T_{\pi(y)}M)$$

$$\begin{array}{ccc} N & \xrightarrow{c} & \mathrm{Sym}^+(TM) \\ \pi \searrow & & \swarrow \\ & M & \end{array}$$

- Every section of π gives a Riemannian metric on M.
- $\mathcal{L}_Z h = 0 \quad \Longrightarrow \quad h$ induces a Riemaniann metric on M.
  Carrollian geometry (Y. Herfray)
- $\mathcal{L}_Z h = 2\rho h \Longrightarrow \quad h$ induces a Riemaniann conformal geometry on M.
  Conformal Carrollian geometry (Y. Herfray)

Our model space

Let $\mathbb{L}^{m+2}$ be the Minkowski space-time with basis $(\ell, e_1, \cdots, e_m, \eta)$ such that the Lorentzian metric $\langle\ ,\ \rangle$ corresponding to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{I}_m & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

- The lightlike cone

$$\left(C^{m+1} := \left\{v \in \mathbb{L}^{m+2} : \langle v, v\rangle = 0, \quad v \neq 0\right\}, \langle\ ,\ \rangle, Z\right)$$

  is a Carrollian geometry where $Z \in \mathfrak{X}(C^{m+1})$ is the restriction of the position vector field.

- Our *model* for Carrollian geometries

$$\boxed{\left(\mathcal{N}^{m+1} := C^{m+1}/\mathbb{Z}_2,\ h,\ \mathcal{Z}\right)}$$

$\mathcal{N}^{m+1}$ as homogeneous space

- The action of Möbius group : $G = PO(m+1,1) := O(m+1,1)/\{\pm \mathrm{Id}\}$

$$G \times \mathcal{N}^{m+1} \to \mathcal{N}^{m+1}, \quad [\sigma] \cdot [v] := [\sigma \cdot v]$$

is also transitive, therefore

$$\mathcal{N}^{m+1} = G/H,$$

where $H \subset G$ is the isotropy group of $[\ell] = \{\pm \ell\} \in \mathcal{N}^{m+1}$ and

$$\boxed{G = \mathrm{Aut}(\mathcal{N}^{m+1}, h, \mathcal{Z}).}$$

The absolute space for $\mathcal{N}^{m+1}$

- $\pi\colon \mathcal{N}^{m+1} \to \mathcal{N}^{m+1}/\mathcal{Z} \simeq \mathbb{S}^m$, the space of lightlike lines in $\mathcal{N}^{m+1}$.
- $\mathbb{S}^m = G/P$ is the model for conformal geometry, $\boxed{\mathrm{Conf}(\mathbb{S}^m) = G.}$
  $P \subset G$ is the isotropy group of $\pi[\ell] \in \mathbb{S}^m$ (Poincaré conformal group)

$$P = \left\{ \begin{bmatrix} \lambda & -\lambda w^t g & -\frac{\lambda}{2}\langle w, w\rangle \\ 0 & g & w \\ 0 & 0 & \lambda^{-1} \end{bmatrix} : \lambda \in \mathbb{R}\setminus\{0\}, w \in \mathbb{R}^m, g \in O(m) \right\}$$

- The isotropy group of $[\ell] \in \mathcal{N}^{m+1}$ is

$$H = \{\sigma \in P : \lambda = \pm 1\} \cong \mathbb{R}^m \rtimes O(m) = \mathrm{Euc}(\mathbb{R}^m)$$

$$\boxed{G/H = \mathcal{N}^{m+1} \xrightarrow{\pi} \mathbb{S}^m = G/P}$$

$\mathcal{N}^{m+1} = G/H$ at Lie algebras level

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & 0 \\ X & A & -Z^t \\ 0 & -X^t & -a \end{pmatrix} : a \in \mathbb{R}, X \in \mathbb{R}^m, Z \in (\mathbb{R}^m)^*, A \in \mathfrak{so}(m) \right\}$$

$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-grading, that is, $\boxed{[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}}$

$\boxed{\mathfrak{g} = \underbrace{\mathfrak{g}_{-1} \oplus \mathfrak{z}(\mathfrak{g}_0)}_{\mathfrak{m}} \oplus \underbrace{[\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{g}_1}_{\mathfrak{h}}}$ and $\boxed{\mathfrak{h} = \underbrace{[\mathfrak{g}_0, \mathfrak{g}_0]}_{\mathfrak{so}(m)} \oplus \mathfrak{g}_1 \leq \mathfrak{p}}$ Carrollian model

$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is not a reductive decomposition!!

In fact, the Lie algebra $\mathfrak{h}$ does not admit any reductive complement in $\mathfrak{g}$.

$\boxed{\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1}$ and $\boxed{\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1}$ conformal model

2. Cartan connections

Let $p : \mathcal{P} \to M$ be an H-principal fiber bundle, for every $X \in \mathfrak{h}$, the fundamental vector field $\zeta_X \in \mathfrak{X}(\mathcal{P})$ is $\zeta_X(u) := \frac{d}{dt}\,|_{t=0}\,(u \cdot \exp(tX))$.

Cartan geometry of type (G, H) on M (Charles Ehresmann, 1950)

- An H-principal fiber bundle $p : \mathcal{P} \to M$.
- A one-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called the Cartan connection such that
  1. $\omega(u)(\zeta_X(u)) = X$ for each $X \in \mathfrak{h}$.
  2. $(r^h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$ for all $h \in H$.
  3. $\omega(u) : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.

$$\boxed{\dim(\mathcal{P}) = \dim(G), \quad \dim(M) = \dim(G/H)}$$

The curvature form

- $K \in \Omega^2(\mathcal{P}, \mathfrak{g}), \quad K = d\omega + \frac{1}{2}[\omega, \omega].$

- $(G \to M \cong G/H, \omega_G)$ is called the homogeneous model for Cartan geometries of type (G, H) and has zero curvature. (Maurer-Cartan equation $d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$.)
- The converse is locally true. *The curvature measures how far is our Cartan geometry from the homogeneous model.*

$$\mathrm{Aut}(\mathcal{P}, \omega) = \Big\{(F, f) : \text{automorphism of } p : \mathcal{P} \to M \text{ with } F^*\omega = \omega\Big\}$$

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{F} & \mathcal{P} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[f]{} & M \end{array} \qquad \begin{cases} F(u \cdot h) = F(u) \cdot h, \ u \in \mathcal{P}, h \in H. \\ F \text{ is a diffeomorphism.} \end{cases}$$

$\mathrm{Aut}(\mathcal{P}, \omega)$ is a Lie group and $\dim \mathrm{Aut}(\mathcal{P}, \omega) \leq \dim G$.

3. Cartan geometries with model the lightlike cone

What kind of geometry structure does correspond with $\mathcal{N}^{m+1}$ as (G, H)?

Theorem I (P*, 21)

Every Cartan geometry $(p : \mathcal{G} \to N^{m+1}, \omega)$ of type $\mathcal{N}^{m+1} = G/H$ determines a Carrollian geometry $(N^{m+1}, h^{\omega}, Z^{\omega})$.
Moreover, $\mathcal{G} \simeq \Big\{(Z^{\omega}_x, e_1, \cdots, e_m) \in \mathcal{P}^1 N^{m+1} : h^{\omega}(e_i, e_j) = \delta_{ij}\Big\}$.

($\mathcal{G}$ gives a G-structure on N^{m+1} with structure group H.)

$$\mathcal{H} := \omega^{-1}(\mathfrak{g}_{-1} \oplus \mathfrak{z}(\mathfrak{g}_0)) \subset T\mathcal{G}$$

defines general connection (horizontal distribution) on $p \colon \mathcal{G} \to N^{m+1}$.

Theorem II (P*, 21)

$$\mathrm{Aut}(\mathcal{G}, \omega) = \Big\{f \in \mathrm{Aut}(N^{m+1}, h^{\omega}, Z^{\omega}) : F = Tf \text{ preserves the distribution } \mathcal{H}\Big\}.$$

That is, $T_u F \cdot \mathcal{H}(u) = \mathcal{H}(F(u))$.

If we write the elements $h \in H$ as follows

$$\begin{bmatrix} 1 & -w^t g & -\frac{1}{2}\langle w, w\rangle \\ 0 & g & w \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -w^t & -\frac{1}{2}\langle w, w\rangle \\ 0 & \mathrm{I}_m & w \\ 0 & 0 & 1 \end{bmatrix}}_{\sigma(w) \text{ translation}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\sigma(g) \text{ rotation}}$$

Theorem III (P*, 21)

A Cartan geometry $(p : \mathcal{G} \to N^{m+1}, \omega)$ of type $\mathcal{N}^{m+1} = G/H$ is equivalent to

1. a Carrollian geometry $(N^{m+1}, h^\omega, Z^\omega)$ and
2. a general connection ($\simeq$ horizontal distribution) $\mathcal{H} \subset T\mathcal{G}$ such that
    1. $T_u r^{\sigma(g)} \cdot \mathcal{H}(u) = \mathcal{H}(u \cdot \sigma(g))$ and
    2. $\omega(u \cdot \sigma(w))\Big(T_u r^{\sigma(w)} \cdot \mathcal{H}(u)\Big) = \mathrm{Ad}(\sigma(w)^{-1})(\mathfrak{m})$,

for all $u \in \mathcal{P}$, $\sigma(g), \sigma(w) \in H$.

An application to Lightlike hypersurfaces

- Let (M^{m+2}, g) be a timelike-oriented Lorentzian manifold with

$$O^{+}(M^{m+2}) = \{(\ell^{+}, e_1, \cdots, e_m, \ell^{-}) \in \mathcal{P}^1 M\}$$

  the $O^{+}(m+1,1)$-principal fiber bundle of g-admissible frames and
- $\psi \colon (N^{m+1}, h) \to (M^{m+2}, g)$ be a lightlike hypersurface.
- (N^{m+1}, h, Z) is a Carroll geometry and $\Psi := T\psi \colon \mathcal{G} \to O^{+}(M^{m+2})$.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\ \Psi\ } & O^{+}(M^{m+2}) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ N^{m+1} & \xrightarrow[\ \psi\ ]{} & M^{m+2} \end{array}$$

Theorem IV (P*, 21)

Assume $\gamma \in \Omega^1(\mathcal{O}^+(M), \mathfrak{g})$ is the Levi-Civita (principal) connection of (M^{m+2}, g) with corresponding linear connection ∇^g. The following assertions are equivalent

1. $\omega = \Psi^*(\gamma)$ is a Cartan connection on N^{m+1} with model $\mathcal{N}^{m+1} = G/H$.
2. $$\begin{array}{ccc} TN/\mathrm{Rad}\, h & \xrightarrow{\nabla^g Z} & TN/\mathrm{Rad}\, h \\ \pi \searrow & & \swarrow \\ & N & \end{array} , \qquad [v] \mapsto [\nabla^g_v Z]$$
   is an N-isomorphism of vector fiber bundles and the Z-expansion function λ ($\nabla^g_Z Z = \lambda Z$ on N) is a non-vanishing function.

Examples: Warped product space-times with two dimensional base

(B, g_B) a Lorentz surface, (F^m, g_B) a Riemann manifold and $f \in C^\infty(B)$.

$$(M^{m+2}, g) = (B \times_f F, g := g_B + f^2 g_f)$$

Assume $Z \in \mathfrak{X}(B)$ is lightlike, then

- $Z^\perp$ is an integrable distribution and M^{m+2} is foliated by a family of Carrollian geometries: the integral hypersurfaces N of $Z^\perp$.
- $\nabla^g Z \colon TN/\mathrm{Rad}\, h \to TN/\mathrm{Rad}\, h, \quad [\xi] \mapsto [\nabla^g_V Z] = \frac{Zf}{f} \cdot [V], \quad \xi = (X, V).$
- Hence, when Zf and λ are non-vanishing functions, the pull-back of the Levi-Civita connection ∇^g is a Cartan connection ω on every (N, h, Z) with model $\mathcal{N}^{m+1} = G/H$ such that

$$h^\omega = \left(\frac{Zf}{f}\right)^2 h \quad \text{and} \quad Z^\omega = \frac{1}{\lambda} Z$$

(i.e., Schwarzschild exterior and interior and Reissner-Nordström)

Remaining questions

1. Describe $\mathrm{Aut}(\mathcal{G},\omega)$ in terms of the base manifold N^{m+1}.

$$\mathrm{Aut}(\mathcal{G},\omega) = \mathrm{Aut}(N^{m+1}, h^{\omega}, Z^{\omega}, \underbrace{\cdots\cdots ???}_{\text{additional tensors...}})$$

2. Characterize those Carroll geometries locally equivalent to bundles of scales of conformal Riemannian manifolds.
(correspondence spaces by Čap and Slovák)
3. Develop the general theory of Cartan connections with model the lightlike cone $\mathcal{N}^{m+1} = G/H$.

Thank you very much for your attention!!

Some references

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