Conformal scattering theory

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Two main methods.

1. **Stationary method.** Fourier transform. Chandrasekhar.
2. **Time dependent method.** Spectral analysis.

$$\partial_t \phi = i(H+P)\phi$$

$H+P$ simplifies asymptotically to H

$$\lim_{t\to\pm\infty} \| e^{it(H+P)}\phi_0 - e^{itH}\phi_\pm \| = 0\,.$$

Alternative geometric approach: conformal scattering.

Origins : Penrose, Lax-Phillips, Friedlander

1. **Penrose :** Zero rest-mass fields including gravitation : asymptotic behaviour (1965)
2. **Lax-Phillips :** Scattering theory (1967)
3. **Friedlander :** study of radiation fields (1970's), link with the Lax-Phillips theory (1980) this is the first construction of a conformal scattering theory.

Lax-Phillips theory

Wave equation on flat spacetime

$$\partial_t^2 \phi - \Delta \phi = 0 \,.$$

As a Schrödinger equation

$$\partial_t U = iAU \,,\ \ U := \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix} \,,\ \ A = -i \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \,.$$

A self-adjoint on $\mathcal{H} = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) = \overline{\mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)}$ in

$$\|U\|^2 := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla_x u_1|^2 + |u_2|^2) \mathrm{d}^3 x \,,$$

$$\sigma(A) = \sigma_{ac}(A) = \mathbb{R} \,.$$

Generalized eigenfunctions for $\sigma \in \mathbb{R}$ (plane waves)

$$e_{\sigma,\omega}(x) = \begin{pmatrix} e^{-i\sigma x.\omega} \\ i\sigma e^{-i\sigma x.\omega} \end{pmatrix} \,,\ \ \omega \in S^2 \,.$$

Lax-Phillips theory

We "project" the elements of $\mathcal{H}$ onto the plane waves $e_{\sigma,\omega}$

$$U \in \mathcal{H} \; : \; \text{put} \;\; \tilde{U}(\sigma,\omega) := \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\langle U, e_{\sigma,\omega}\rangle_{\mathcal{H}}$$

Ill defined! However, for $U \in \mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}
\tilde{U}(\sigma,\omega) &:= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\langle U, e_{\sigma,\omega}\rangle_{\mathcal{H}} \\
&= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\left(\nabla u_1 \overline{\nabla e^{-i\sigma x.\omega}} + u_2 \overline{i\sigma e^{-i\sigma x.\omega}}\right)\mathrm{d}^3x \\
&= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\left(u_1 (\overline{-\Delta e^{-i\sigma x.\omega}}) + u_2 \overline{i\sigma e^{-i\sigma x.\omega}}\right)\mathrm{d}^3x \\
&= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}(\sigma^2 u_1 - i\sigma u_2)e^{i\sigma x.\omega}\mathrm{d}^3x \\
&= \frac{1}{2}\left(\sigma^2 \hat{u}_1(-\sigma\omega) - i\sigma\hat{u}_2(-\sigma\omega)\right).
\end{aligned}$$

Now $\tilde{U}(\sigma, \omega)$ well defined on $\mathcal{H}$ and

$$\widetilde{AU} = \sigma \tilde{U}\,, \;\; \widetilde{e^{itA}U} = e^{it\sigma}\tilde{U}\,.$$

This is a spectral representation of the Hamiltonian and propagator.
Then, define

$$\mathcal{R}U(r, \omega) := \mathcal{F}_\sigma(\tilde{U}(., \omega))(r)\,.$$

$$\mathcal{R}(e^{itA}U)(r, \omega) = (\mathcal{R}U)(r - t, \omega)\,.$$

This is a translation representation of the propagator.

$\mathcal{R}$ is an isometry from $\mathcal{H}$ onto $L^2(\mathbb{R} \times S^2)$.

Lax-Phillips theory

Translation representation related to the Radon transform. Allows to recover the solution ϕ from its translation representer k

$$\phi(t,x) = \frac{1}{2\pi}\int_{S^2} k(x.\omega - t, \omega)\mathrm{d}^2\omega\,, \tag{1}$$

and to prove

Theorem (Asymptotic profiles)

Assuming that the data ϕ_0, ϕ_1 are smooth and compactly supported, denoting

$$k = \mathcal{R}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}\,,$$

we have

$$k(s,\omega) = -\lim_{r\to+\infty} r\partial_t\phi(r,(r+s)\omega)\,. \tag{2}$$

The first conformal scattering construction

- Formula (2) shows that the translation representer is a radiation field.
- Therefore can be recovered by conformal compactification of the geometry.
- Friedlander 1980, "Radiation fields and hyperbolic scattering theory". The framework is static.

Some more history

- Baez, Segal, Zhou, 1989-1990, extension to nonlinear equations (static).
- Hörmander, 1990, "A remark on the characteristic Cauchy problem" (general).
- Mason, Nicolas, 2002, "Conformal scattering and the Goursat problem" (general).
- Joudioux, 2012, 2020, extension to nonlinear equations (general).
- Nicolas, 2016, extension to black hole spacetimes (static).
- Mokdad, 2019, 2022, black holes with positive cosmological constant including interior (static or general)
- Taujanskas, 2019, nonlinear gauge fields
- Pham, 2020, 2022, linearized gravity on Schwarzschild (static) and Dirac on Kerr (stationnary)
- Coudray, in progress, waves on Vaidya (dynamic)

Conformal compactification of Minkowski spacetime

Minkowski spacetime : $\mathbb{M} = \mathbb{R}^4$ with the Minkowski metric

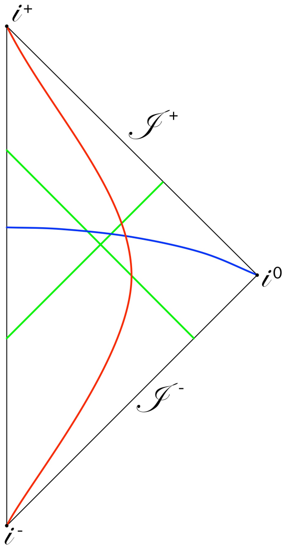
$$\eta := \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\omega^2\,,\ \mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\,.$$

Change of coordinates : $u = t - r$, $v = t + r$;

$$\tau = \arctan(t-r) + \arctan(t+r)\,,$$
$$\zeta = \arctan(t+r) - \arctan(t-r)\,,$$

$$\eta = \frac{(1+u^2)(1+v^2)}{4}\left(\mathrm{d}\tau^2 - \mathrm{d}\zeta^2\right) - \frac{(v-u)^2}{4}\mathrm{d}\omega^2\,.$$

Conformal compactification of Minkowski spacetime



- $i^{\pm}$ $(\tau = \pm\pi,\ \zeta = 0)$
- i^{0} $(\tau = 0,\ \zeta = \pi)$
- green = null
- red = timelike
- blue = spacelike

Conformal compactification of Minkowski spacetime

Conformal factor : $\Omega^2 = \dfrac{4}{(1+u^2)(1+v^2)}$.

$$\hat{\eta} := \Omega^2 \eta = \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - (\sin\zeta)^2\,\mathrm{d}\omega^2 = \mathrm{d}\tau^2 - \sigma^2_{S^3}\,,$$

analytic over $\mathcal{E} = \mathbb{R} \times S^3$.

$\mathbb{M}$ open subset of the Einstein cylinder $\mathcal{E}$.

$\hat{\eta}$ is the Einstein metric.

Conformal compactification of Minkowski spacetime

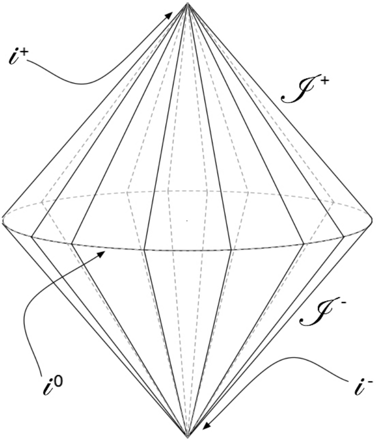


Figure: Compactified Minkowski spacetime.

- $\mathscr{I}^{\pm}$ = future and past end-points of null geodesics,
- $i^{\pm}$ = future (resp. past) endpoint of timelike geodesics,
- i^{0} = endpoint of spacelike geodesics.
- $i^{\pm}$ and i^{0} are points.

Conformal scattering for the wave equation

Conformally invariant wave equation : $(\Box_g + \frac{1}{6}\mathrm{Scal}_g)\phi = 0$

$$\Box_\eta \phi = 0 \Leftrightarrow \Box_{\hat\eta}\hat\phi + \hat\phi = 0\,,\ \ \hat\phi = \Omega^{-1}\phi\,.$$

where

$$\Box_\eta = \partial_t^2 - \Delta_{\mathbb{R}^3}\,,\ \ \Box_{\hat\eta} = \partial_\tau^2 - \Delta_{S^3}\,.$$

Definition (Conformal scattering theory for the wave equation)

$$\mathbb{T}^\pm \ :\ (\phi|_{t=0}, \partial_t\phi|_{t=0}) \longmapsto \hat\phi|_{\mathscr{I}^\pm}$$

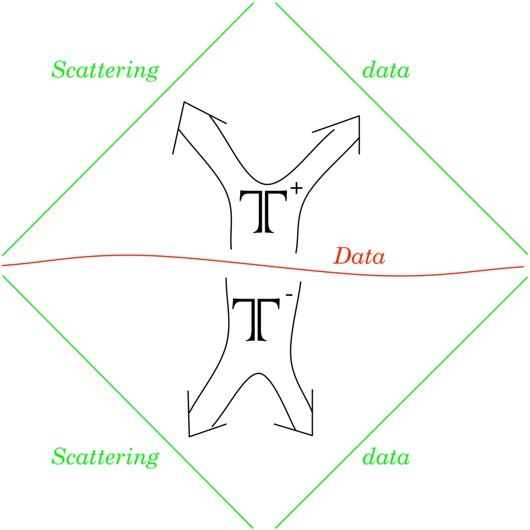
are isomorphisms between well-chosen function spaces.
Scattering operator : $\mathbb{S} := \mathbb{T}^+(\mathbb{T}^-)^{-1}$.

Conformal scattering construction

- Step 1. Define operators $\mathbb{T}^{\pm}$ for $\mathcal{C}_0^\infty$ data (Leray).
- Step 2. Energy estimates both ways between a Cauchy hypersurface and $\mathscr{I}^{\pm}$. Allows to extend $\mathbb{T}^{\pm}$ as bounded one-to-one operators with closed range.
- Step 3. Prove surjectivity of $\mathbb{T}^{\pm}$, i.e. density of $\mathrm{Ran}\mathbb{T}^{\pm}$ (by solving the Goursat problem for a dense class of data, follows Hörmander 1990).
- Step 4. Forget about the spacelike slice : $\mathbb{S} = \mathbb{T}^{+}(\mathbb{T}^{-})^{-1}$.

Construction insensitive to time dependence.

Schematically



Some details on step 2

- Define an energy current.
  - Stress-energy tensor T_{ab}
    On the Einstein cylinder exactly conserved (Klein-Gordon)
  - Timelike vector field (observer) T^a
  - Energy current $J_a := T^b T_{ab}$
- Energy estimates obtained using the divergence theorem.
- Error terms (if present) controlled by Gronwall-type estimates.
  Two possible origins:
  - Killing form of T^a, $T_{ab}\nabla^{(a}T^{b)}$;
  - non conservation of T_{ab}, $T^b\nabla^a T_{ab}$.

Choice of observer

On Minkowski spacetime, two natural choices:

1. Time translation along the Einstein cylinder
2. Timelike Killing vector field of the physical spacetime.

Give two inequivalent scattering theories.

1. Energy on $\mathscr{I}^+$ in the first case

$$\frac{1}{\sqrt{2}}\int_{\mathscr{I}^+}\left(\left|\hat{\phi}_\tau-\hat{\phi}_\zeta\right|^2+\frac{1}{\sin^2\zeta}\left|\nabla_{S^2}\hat{\phi}\right|^2+\hat{\phi}^2\right)\mathrm{d}\mu_{S^3}\,;$$

2. Energy on $\mathscr{I}^+$ in the second case

$$\frac{1}{\sqrt{2}}\int_{\mathscr{I}^+}\left|\hat{\phi}_\tau-\hat{\phi}_\zeta\right|^2\mathrm{d}\mu_{S^3}\,.$$

Different weights in the energy on the initial slice.

Extension to asymptotically simple spacetimes

Corvino-Schoen / Chrusciel-Delay spacetimes:

- smooth $\mathscr{I}^\pm$ and $i^\pm$;
- $\simeq$ Schwarzschild near i^0.
- Observer: timelike Killing vector near i^0, tangent to $\mathscr{I}^\pm$ everywhere.
- Energy estimates: use special features of $\mathscr{I}$.
- Difficulty with the singularity at i^0.

We take advantage of the finite propagation speed.

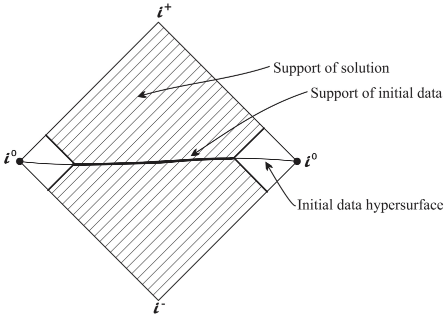


Figure: Avoiding the singularity at i^0

Extension to black holes

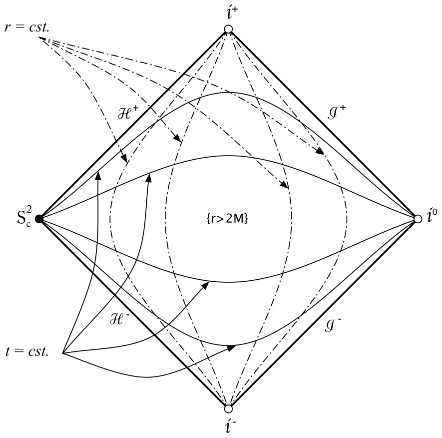


Figure: Penrose diagram of the compactified exterior of a Schwarzschild black hole

Extension to black holes

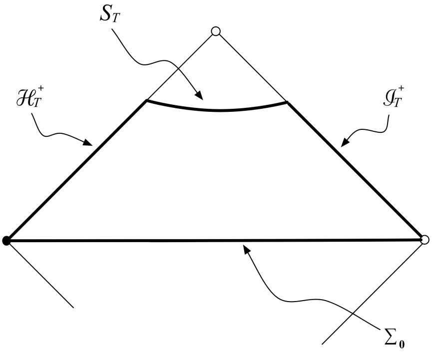


Figure: Strategy for obtaining the energy estimates

Projects

- More general spacetimes (radiation)
- Behaviour inside black holes (stability of charged black holes)
- Nonlinear gauge systems and their charges

The end

THANK YOU!

"The time has come," the walrus said, "to talk of many things: of shoes and ships - and sealing wax - of cabbages and kings."