Embedding Galilean and Carrollian geometries

Second Carroll Workshop

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Based on:

X. Bekaert, K.M. J. Math. Phys. **59**, no. 7, 072503 (2018)
K.M. Part I. J. Math. Phys. **61**, no. 8, 082502 (2020)
K.M. Part II. To appear

2022/09/12

Spacetime geometries

Spacetime geometries can be described along three different, complementary, approaches:

- **Intrinsic approach**
  Spacetime is a $(d+1)$-dimensional manifold endowed with a « metric structure » $\mathfrak{m}$ together with a compatible Koszul connection $\nabla\mathfrak{m} = 0$.
- **Cartan approach**
  Spacetime is the base manifold of a Cartan geometry modelled on an effective kinematical Klein pair.
- **Ambient approach**
  Spacetime arises as a quotient/hypersurface of a $(d+2)$-dimensional manifold.

Relating those three approaches yields different problems:

- **Cartan equivalence problem**:
  Solve the Cartan connection in terms of its invariants to make contact between the Cartan and intrinsic approaches
- **Lifting problem**:
  Given an intrinsic geometry $\bar{\mathscr{M}}$, find an ambient geometry $\mathscr{M}$ such that $\bar{\mathscr{M}}$ is isomorphic to the quotient $\mathscr{M}/\mathbb{R}$
- **Embedding problem**:
  Given an intrinsic geometry $\widetilde{\mathscr{M}}$, find an ambient geometry $\mathscr{M}$ such that $\widetilde{\mathscr{M}}$ is isomorphic to a hypersurface of $\mathscr{M}$

Bargmann–Eisenhart waves

Eisenhart 28', Duval, Burdet, Künzle, Perrin 85', Duval, Gibbons, Horvathy, Zhang 14'

- The paradigmatic example of ambient approach is given by the embedding of Galilean/Carrollian geometries inside Bargmann–Eisenhart waves (*a.k.a.* Brinkmann spacetimes or pp-waves).
- Bargmann–Eisenhart waves lie at the intersection of two interesting categories of structures:

$$\begin{array}{c} \mathbb{R} \\ \downarrow \\ \mathscr{M} \\ \downarrow \pi \\ \mathscr{M}/\mathbb{R} \end{array}$$

1. **Gravitational waves** $(\mathscr{M}, [\xi], g)$ *i.e.* Lorentzian manifolds $(\mathscr{M}, g)$ endowed with an equivalence class $[\xi]$ of nowhere vanishing vector fields:

$$\xi \sim \Omega\, \xi \quad \text{where} \quad \Omega \in \mathscr{C}^{\infty}_{\neq 0}(\mathscr{M})$$

   such that each representative $\xi \in [\xi]$ is:

   - lightlike *i.e.* $g(\xi, \xi) = 0$
   - hypersurface-orthogonal *i.e.* $d\psi \wedge \psi = 0$ where $\psi := g(\xi)$ (*i.e.* $\operatorname{Ker}\psi$ is involutive)

   Bargmann–Eisenhart waves are characterised among gravitational waves by the existence of a parallel representative $\nabla_g\, \xi = 0$ (or equivalently $\mathcal{L}_\xi\, g = 0 = d\psi$).

2. **Bargmannian manifolds** *i.e.* Cartan geometries for the reductive Bargmann algebra:

$$\Big[\mathbf{K}, H\Big] \sim \mathbf{P} \quad , \quad \Big[\mathbf{K}, \mathbf{P}\Big] \sim M \quad .$$

   After solving the Cartan equivalence problem, Bargmannian manifolds can be characterised as tuples $(\mathscr{M}, \xi, g, \nabla)$ *i.e.* Lorentzian manifolds $(\mathscr{M}, g)$ endowed with a lightlike vector field ξ together with a (possibly torsionful) connection ∇ preserving both the metric and the lightlike vector field, *i.e.* $\nabla \xi = 0$, $\nabla g = 0$.

   In this context, Bargmann–Eisenhart waves identify with torsionfree Bargmannian manifolds.

Lifting *à la* Duval *et al.*

$$\mathbb{R}$$

$$\downarrow$$

Bargmann–Eisenhart wave

$$(\mathscr{M}, \xi, g)$$

$$\mathcal{L}_\xi g = 0,\ d\psi = 0,\ \mathsf{Tor}(\nabla_g) = 0,\ \mathcal{L}_\xi \nabla_g = 0,\ R_{\mu\nu|\alpha\beta} = R_{\alpha\beta|\mu\nu}$$

$$\downarrow \pi$$

Newtonian manifold

$$(\bar{\mathscr{M}}, \bar{\psi}, \bar{\gamma}, \bar{\nabla})$$

$$d\bar{\psi} = 0,\ \mathsf{Tor}(\bar{\nabla}) = 0,\ \bar{R}^{\bar{\mu}}{}_{\bar{\alpha}}{}^{\bar{\nu}}{}_{\bar{\beta}} = \bar{R}^{\bar{\nu}}{}_{\bar{\beta}}{}^{\bar{\mu}}{}_{\bar{\alpha}}$$

Theorem Duval *et al.* 85'

The quotient manifold of a Bargmann–Eisenhart wave is a Newtonian manifold. Conversely, any Newtonian manifold can be lifted to a Bargmann–Eisenhart wave.

Embedding *à la* Duval *et al.*

Invariant torsionfree Carrollian manifold $\qquad\qquad$ Bargmann–Eisenhart wave

$$(\widetilde{\mathscr{M}}, \widetilde{\xi}, \widetilde{\gamma}, \widetilde{\nabla}) \hookrightarrow (\mathscr{M}, \xi, g)$$

$$\mathcal{L}_{\widetilde{\xi}}\widetilde{\gamma} = 0,\ \mathsf{Tor}(\widetilde{\nabla}) = 0,\ \mathcal{L}_{\widetilde{\xi}}\widetilde{\nabla} = 0 \qquad \mathcal{L}_{\xi}g = 0, \mathsf{Tor}(\nabla_g) = 0,\ \mathcal{L}_{\xi}\nabla_g = 0$$

Theorem Duval *et al.* 14'

Any Bargmann–Eisenhart wave admits a lightlike foliation by invariant torsionfree Carrollian manifolds. Conversely, any invariant torsionfree Carrollian manifold can be embedded into a B–E wave.

Example
Duval *et al.* 14'

Flat Carroll manifold $\qquad\qquad$ Minkowski wave

$$(\widetilde{\mathscr{M}}, \widetilde{\xi}, \widetilde{\gamma}, \widetilde{\nabla}) \hookrightarrow (\mathscr{M}, \xi, g)$$

$$\widetilde{\xi} = \partial_u,\ \widetilde{\gamma} = \delta_{ij}dx^i \otimes dx^j,\ \widetilde{\Gamma} = 0 \qquad \xi = \partial_u,\ g = du \otimes dt + dt \otimes du + \delta_{ij}dx^i \otimes dx^j$$

Question: What about the (A)dS Carroll manifold? Bergshoeff, Gomis, Parra 15'

Embedding (A)dS Carroll

No-Go: The (A)dS Carroll manifold $(\widetilde{\mathscr{M}}, \widetilde{\xi}, \widetilde{\gamma}, \widetilde{\nabla})$ reads:

$$\widetilde{\xi} = \partial_u \quad , \quad \widetilde{\gamma} = \gamma_{ij}\, dx^i \otimes dx^j \quad , \quad \widetilde{\Gamma}^u_{ij} = \epsilon\, u\, \gamma_{ij} \quad , \quad \widetilde{\Gamma}^i_{jk} = \Gamma_\gamma$$

where γ_{ij} is the d-dimensional hyperbolic (resp. spherical) metric whenever $\epsilon < 0$ (resp. $\epsilon > 0$).

The connection is *not* invariant, as:

$$\mathcal{L}_{\tilde{\xi}} \widetilde{\nabla} = \epsilon\, \widetilde{\xi} \otimes \widetilde{\gamma}$$

hence (A)dS Carroll *cannot* be embedded into a Bargmann–Eisenhart wave.

There are two possible directions to circumvent this issue:

1. **Gravitational waves**: K.M. Part I
   Allowing for the (more general) class of gravitational waves characterised by $\nabla\xi = \psi \otimes \chi$ allows to embed torsionfree Carrollian manifold satisfying the less restrictive (non)-invariance condition:

   $$\mathcal{L}_{\tilde{\xi}} \widetilde{\nabla}^{\tilde{\lambda}}_{\tilde{\mu}\tilde{\nu}} = -\widetilde{\xi}^{\tilde{\lambda}}\, \Omega^{-1} \widetilde{\nabla}_{\tilde{\mu}} \widetilde{\nabla}_{\tilde{\nu}}\, \Omega \quad \text{for some nowhere vanishing invariant function } \Omega$$

   In particular, one can show that the (Anti)-de Sitter wave is foliated by (A)dS Carroll manifolds (*cf.* also Figueroa-O'Farrill, Prohazka 18').

2. **Bargmannian manifolds**: Bekaert, K.M. 15', K.M. Part II
   Allowing for torsional Bargmannian geometries permits to lift/embed any Galilean/Carrollian manifold into a Bargmannian manifold.

Ambient problems

One can distinguish in a more refined way between three classes of ambient problems:

1. **Generic lifting problem**
   Given a Galilean geometry G, find an ambient geometry $(\mathscr{M}, \xi, \mathfrak{m})$ such that $G \simeq \mathscr{M}/\mathbb{R}$

2. **Torsionfree lifting problem**
   Given a *torsionfree* Galilean geometry G, find a *torsionfree* ambient geometry $(\mathscr{M}, \xi, \mathfrak{m})$ such that $G \simeq \mathscr{M}/\mathbb{R}$

3. **Homogeneous lifting problem**
   Given a *homogeneous* Galilean geometry G, find a *homogeneous* ambient geometry $(\mathscr{M}, \xi, \mathfrak{m})$ such that $G \simeq \mathscr{M}/\mathbb{R}$

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   Given a Carrollian geometry C, find an ambient geometry $(\mathscr{M}, \psi, \mathfrak{m})$ such that C is isomorphic to a hypersurface of the foliation $\mathrm{Ker}\ \psi$

2. **Torsionfree embedding problem**
   Given a *torsionfree* Carrollian geometry C, find a *torsionfree* ambient geometry $(\mathscr{M}, \psi, \mathfrak{m})$ such that C is isomorphic to a hypersurface of the foliation $\mathrm{Ker}\ \psi$

3. **Homogeneous embedding problem**
   Given a *homogeneous* Carrollian geometry C, find a *homogeneous* ambient geometry $(\mathscr{M}, \psi, \mathfrak{m})$ such that C is isomorphic to a hypersurface of the foliation $\mathrm{Ker}\ \psi$

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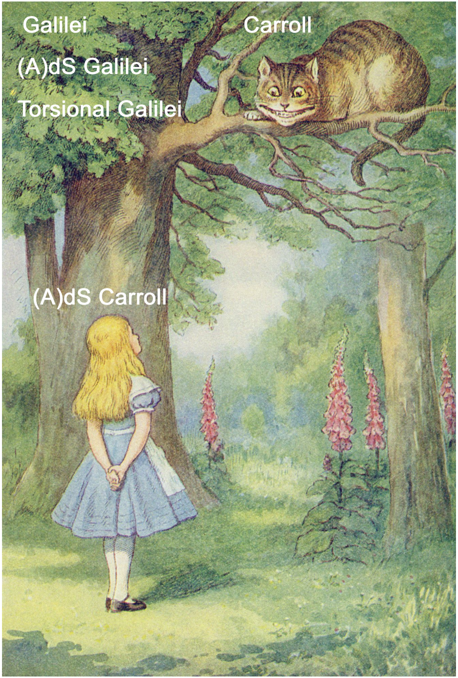
Bargmann scoreboard:

Ambient manifold	**Ambient problem**	**Lifting** (Galilean)	**Embedding** (Carrollian)
Bargmann	Generic	Full	Full
	Torsionfree	Partial	Partial
	Homogeneous	Full	Partial



Homogeneous problem

(Bargmann version)

Homogeneous lifting problem into Bargmann:

- Galilei: Duval, Burdet, Künzle, Perrin 85’
- (A)dS Galilei: Gibbons, Patricot 03’
- Torsional Galilei: Figueroa-O’Farrill, Grassie, Prohazka 22’

Homogeneous embedding problem into Bargmann:

- Carroll: Duval, Gibbons, Horvathy, Zhang 14’

No go: (A)dS Carroll does not admit an embedding into a homogeneous Bargmannian manifold.

Question: Does there exist other possible ambient geometries where to lift/embed all homogeneous Galilean and Carrollian manifolds?

Leibnizian geometry

Bekaert, K.M. 15'

Definition

A **Leibnizian structure** is a quadruplet $(\mathscr{M}, \xi, \psi, \gamma)$ where

- $\mathscr{M}$ is a smooth manifold of dimension $d + 2$.
- $\xi \in \Gamma(T\mathscr{M})$ is a nowhere vanishing vector field.
- $\psi \in \Omega(\mathscr{M})$ is a nowhere vanishing 1-form annihilating ξ *i.e.* $\psi(\xi) = 0$.
- $\gamma \in \Gamma\Big(\vee^2\ (\text{Ker}\ \psi)^*\Big)$ is a positive semi-definite rank d covariant metric acting on $\Gamma(\text{Ker}\ \psi)$ and whose radical is spanned by ξ *i.e.* $\gamma(X, \cdot) = 0 \Leftrightarrow X \sim \xi$.

A Leibnizian structure endowed with a compatible connection:

$$\nabla\xi = 0 \quad , \quad \nabla\psi = 0 \quad , \quad \nabla\gamma = 0$$

is called a **Leibnizian manifold** $(\mathscr{M}, \xi, \psi, \gamma, \nabla)$.

- The intrinsic torsion (Figueroa-O'Farrill 20') of a Leibnizian structure is given by $d\psi \oplus \mathcal{L}_\xi\gamma$.
- Any Bargmannian manifold $(\mathscr{M}, \xi, g, \nabla)$ induces a Leibnizian manifold $(\mathscr{M}, \xi, \psi, \gamma, \nabla)$ upon the identification:

$$\psi := g(\xi) \quad , \quad \gamma := g|_{\text{Ker}\ \psi}.$$

Leibnizian geometry

Bekaert, K.M. 15'

Example

Flat Leibniz structure

Let $\mathscr{M} \cong \mathbb{R}^{d+2}$ be a $(d+2)$-dimensional spacetime coordinatised by (u, t, x^i) where $i \in \{1, \ldots, d\}$.

The Leibniz metric structure is defined as the quadruplet $(\mathscr{M}, \xi, \psi, \gamma)$ where:

$$\xi = \partial_u \quad , \quad \psi = dt \quad \text{and where} \quad \gamma = \delta_{ij}\, dx^i \otimes dx^j \text{ acts on } \mathrm{Ker}\ \psi.$$

The quotient $\bar{\mathscr{M}} = \mathscr{M}/\mathbb{R}$ is isomorphic to the **flat Galilei structure**:

$$\psi = dt \quad \text{and where} \quad \gamma = \delta_{ij}\, dx^i \otimes dx^j \text{ acts on } \mathrm{Ker}\ \psi.$$

The hypersurfaces $\widetilde{\mathscr{M}}_t$ characterised by $t =$ const are isomorphic to the **flat Carroll structure**:

$$\xi = \partial_u \quad \text{and where} \quad \gamma = \delta_{ij}\, dx^i \otimes dx^j \text{ acts on the whole } T\widetilde{\mathscr{M}}_t.$$

More generally, any (torsionfree) Galilean/Carrollian manifold can be lifted/embedded into a (torsionfree) Leibnizian manifold (generic and torsionfree lifting/embedding problem).

Leibnizian geometry

Bekaert, K.M. 15'

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$$\xi = \partial_u \quad , \quad \psi = dt \quad \text{and where} \quad \gamma = \delta_{ij}\, dx^i \otimes dx^j \text{ acts on } \mathrm{Ker}\ \psi.$$

Flat Leibniz manifold

Endowing the Leibniz metric structure with the flat connection ∇ with $\Gamma = 0$ yields the flat Leibniz manifold whose isometry algebra:

$$\mathfrak{g} := \left\{ X \in \Gamma\Big(T\mathscr{M}\Big) \mid \mathcal{L}_X \xi = 0, \mathcal{L}_X \psi = 0, \mathcal{L}_X \gamma = 0 \text{ and } \mathcal{L}_X \nabla = 0 \right\}$$

is the **Leibniz algebra** $\mathfrak{g} = \mathrm{Span}\,\{M, H, \mathbf{P}, C, \mathbf{D}, \mathbf{K}, \mathbf{J}\}$:

$$\Big[\mathbf{D}, \mathbf{K}\Big] = C \quad , \quad \Big[\mathbf{K}, H\Big] = \mathbf{P} \quad , \quad \Big[\mathbf{D}, \mathbf{P}\Big] = M \quad , \quad \Big[C, H\Big] = M$$

spanned by:

$$M = \partial_u \quad , \quad H = \partial_t \quad , \quad P_i = \partial_i$$

$$C = -t\, \partial_u \quad , \quad D_i = -x_i \partial_u \quad , \quad K_i = -t\partial_i \quad , \quad J_{ij} = x_i \partial_j - x_j \partial_i .$$

Possible ambient kinematics

K.M. Part II

- The Leibniz algebra has maximal dimension in spacetime dimension $d+2$. However, it does not contain a $\mathfrak{so}(d+1)$ subalgebra hence it is *not* a kinematical algebra in $(d+2)$-dimensions.
- We define the concept of ambient kinematical algebra:

Definition

An **ambient kinematical algebra** in d spatial dimensions is a $\frac{(d+3)(d+2)}{2}$-dimensional Lie algebra

$$\mathfrak{g} = \mathrm{Span}\left\{M, H, \mathbf{P}, C, \mathbf{D}, \mathbf{K}, \mathbf{J}\right\}$$

and satisfying the following properties:

1. The generators $\mathbf{J}$ span a $\mathfrak{so}(d)$ Lie algebra.
2. The generator M, H and C are in the scalar representation of $\mathfrak{so}(d)$.
3. The generators $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{K}$ are in the vector representation of $\mathfrak{so}(d)$.

An **ambient Klein pair** is a pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h} = \mathrm{Span}\left\{C, \mathbf{D}, \mathbf{K}, \mathbf{J}\right\}$ is a subalgebra of $\mathfrak{g}$.

- Of particular interest for the ambient approach is the following subclass:

Definition

A **Leibnizian pair** is an ambient Klein pair satisfying:

$$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(M) = 0 \quad , \quad \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(H^*) = 0 \quad , \quad \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\gamma) = 0 \quad \text{on} \quad \mathrm{Ker}\ H^*.$$

Lifting kinematical algebras

K.M. Part II

- Any kinematical algebra in $(d+2)$ spacetime dimensions induces an ambient kinematical algebra in d spatial dimensions via:

$$H \mapsto H \quad , \quad P_0 \mapsto M \quad , \quad \mathbf{P} \mapsto \mathbf{P} \quad , \quad K_0 \mapsto C \quad , \quad \mathbf{K} \mapsto \mathbf{K} \quad , \quad J_{0i} \mapsto \mathbf{D} \quad , \quad \mathbf{J} \mapsto \mathbf{J}.$$

  Example: Galilei in ambient form

$$\left[\mathbf{K}, H\right] \sim \mathbf{P} \quad , \quad \left[C, H\right] \sim M$$

$$\left[\mathbf{D}, M\right] \sim -\mathbf{P} \quad , \quad \left[\mathbf{D}, \mathbf{P}\right] \sim M \quad , \quad \left[\mathbf{D}, C\right] \sim -\mathbf{K} \quad , \quad \left[\mathbf{D}, \mathbf{K}\right] \sim C \quad , \quad \left[\mathbf{D}, \mathbf{D}\right] \sim -\mathbf{J}.$$

- By construction, a kinematical algebra in ambient form possesses a canonical subalgebra (the original kinematical algebra in d spatial dimension) spanned by $\{H, \mathbf{P}, \mathbf{K}, \mathbf{J}\}$.
- Performing an İnönü-Wigner contraction along this subalgebra yields a new algebra for which $\mathfrak{i} = \{M, C, \mathbf{D}\}$ is a canonical ideal.
- Applying this procedure to the Galilei algebra yields the Leibniz algebra, which is ensured by construction to project on the Galilei algebra.

Lifting kinematical algebras

K.M. Part II

- Any kinematical algebra in $(d+2)$ spacetime dimensions induces an ambient kinematical algebra in d spatial dimensions via:

$$H \mapsto H \quad , \quad P_0 \mapsto M \quad , \quad \mathbf{P} \mapsto \mathbf{P} \quad , \quad K_0 \mapsto C \quad , \quad \mathbf{K} \mapsto \mathbf{K} \quad , \quad J_{0i} \mapsto \mathbf{D} \quad , \quad \mathbf{J} \mapsto \mathbf{J}.$$

  Example: Galilei in ambient form (after İnönü-Wigner contraction)

$$\left[\mathbf{K}, H\right] \sim \mathbf{P} \quad , \quad \left[C, H\right] \sim M$$

$$\cancel{\left[\mathbf{D}, M\right] \sim -\mathbf{P}} \quad , \quad \left[\mathbf{D}, \mathbf{P}\right] \sim M \quad , \quad \cancel{\left[\mathbf{D}, C\right] \sim -\mathbf{K}} \quad , \quad \left[\mathbf{D}, \mathbf{K}\right] \sim C \quad , \quad \cancel{\left[\mathbf{D}, \mathbf{D}\right] \sim -\mathbf{J}}.$$

- By construction, a kinematical algebra in ambient form possesses a canonical subalgebra (the original kinematical algebra in d spatial dimension) spanned by $\{H, \mathbf{P}, \mathbf{K}, \mathbf{J}\}$.
- Performing an İnönü-Wigner contraction along this subalgebra yields a new algebra for which $\mathfrak{i} = \{M, C, \mathbf{D}\}$ is a canonical ideal.
- Applying this procedure to the Galilei algebra yields the Leibniz algebra, which is ensured by construction to project on the Galilei algebra.

Lifting Galilean algebras

K.M. Part II

Repeating the procedure for all effective Galilean kinematical pairs (Figueroa-O'Farrill, Prohazka 18'):

Algebra	$[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$	$[\mathfrak{h},\mathfrak{p}]=\mathfrak{p}$	$[\mathfrak{p},\mathfrak{p}]=\mathfrak{h}$ (Curvature)	$[\mathfrak{p},\mathfrak{p}]=\mathfrak{p}$ (Torsion)
Galilei		$[\mathbf{K},H]=\mathbf{P}$		
		$[\mathbf{K},H]=\mathbf{P}$		$[H,\mathbf{P}]=\mathbf{P}$
$\alpha_+\geq 0$		$[\mathbf{K},H]=\mathbf{P}$	$[H,\mathbf{P}]=\mathbf{K}$	$[H,\mathbf{P}]=\alpha_+\mathbf{P}$
$\alpha_-\geq 0$		$[\mathbf{K},H]=\mathbf{P}$	$[H,\mathbf{P}]=-\mathbf{K}$	$[H,\mathbf{P}]=\alpha_-\mathbf{P}$

yields effective Leibnizian pairs:

Algebra	$[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$	$[\mathfrak{h},\mathfrak{p}]=\mathfrak{p}$	$[\mathfrak{p},\mathfrak{p}]=\mathfrak{h}$ (Curvature)	$[\mathfrak{p},\mathfrak{p}]=\mathfrak{p}$ (Torsion)
Leibniz	$[\mathbf{D},\mathbf{K}]=C$	$[\mathbf{K},H]=\mathbf{P}$, $[C,H]=M$, $[\mathbf{D},\mathbf{P}]=M$		
	$[\mathbf{D},\mathbf{K}]=C$	$[\mathbf{K},H]=\mathbf{P}$, $[C,H]=M$, $[\mathbf{D},\mathbf{P}]=M$		$[H,\mathbf{P}]=\mathbf{P}$, $[H,M]=M$
$\alpha_+\geq 0$	$[\mathbf{D},\mathbf{K}]=C$	$[\mathbf{K},H]=\mathbf{P}$, $[C,H]=M$, $[\mathbf{D},\mathbf{P}]=M$	$[H,\mathbf{P}]=\mathbf{K}$, $[H,M]=C$	$[H,\mathbf{P}]=\alpha_+\mathbf{P}$, $[H,M]=\alpha_+M$
$\alpha_-\geq 0$	$[\mathbf{D},\mathbf{K}]=C$	$[\mathbf{K},H]=\mathbf{P}$, ,$[C,H]=M$, $[\mathbf{D},\mathbf{P}]=M$	$[H,\mathbf{P}]=-\mathbf{K}$, $[H,M]=-C$	$[H,\mathbf{P}]=\alpha_-\mathbf{P}$, $[H,M]=\alpha_-M$

thus allowing to solve the homogeneous lifting problem for Leibnizian geometry.

Lifting homogeneous Galilean manifolds

K.M. Part II

Each of these Leibnizian algebras can be realised as isometry algebra

$$\mathfrak{g} := \left\{ X \in \Gamma\left(T\mathscr{M}\right) \mid \mathcal{L}_X \xi = 0, \mathcal{L}_X \psi = 0, \mathcal{L}_X \gamma = 0 \text{ and } \mathcal{L}_X \nabla = 0 \right\}$$

of the flat Leibniz structure:

$$\xi = \partial_u \quad , \quad \psi = dt \quad \text{and where} \quad \gamma = \delta_{ij}\, dx^i \otimes dx^j \text{ acts on } \mathrm{Ker}\ \psi$$

endowed with a compatible connection:

	Γ^u_{ut}	Γ^u_{tt}	Γ^i_{tt}	Γ^i_{jt}
$(0,0)$	0	0	0	0
$(1,0)$	-1	0	0	$-\delta^i_j$
$(\alpha_+,1)$	$-\alpha_+$	u	x^i	$-\alpha_+\delta^i_j$
$(\alpha_-,1)$	$-\alpha_-$	$-u$	$-x^i$	$-\alpha_-\delta^i_j$

These connections all satisfy $\mathcal{L}_\xi \nabla \sim \xi$ and $T(\xi, \cdot) \sim \xi$ and hence are projectable.

Homogeneous (projective) Leibnizian manifolds

The quotient $\mathscr{M}/\mathbb{R}$ is isomorphic to the flat Galilei structure:

$$\psi = dt \quad \text{and where} \quad \gamma = \delta_{ij}\, dx^i \otimes dx^j \text{ acts on } \mathrm{Ker}\ \psi$$

endowed with the compatible connection:

	Γ^i_{tt}	Γ^i_{jt}
$(0,0)$	0	0
$(1,0)$	0	$-\delta^i_j$
$(\alpha_+,1)$	x^i	$-\alpha_+\delta^i_j$
$(\alpha_-,1)$	$-x^i$	$-\alpha_-\delta^i_j$

Homogeneous Galilean manifolds

Mirroring ambient algebras

K.M. Part II

- Any kinematical algebra in $(d+2)$ spacetime dimensions induces an ambient kinematical algebra in d spatial dimensions via:

$$H \mapsto H \quad , \quad P_0 \mapsto M \quad , \quad \mathbf{P} \mapsto \mathbf{P} \quad , \quad K_0 \mapsto C \quad , \quad \mathbf{K} \mapsto \mathbf{K} \quad , \quad J_{0i} \mapsto \mathbf{D} \quad , \quad \mathbf{J} \mapsto \mathbf{J}.$$

Example: Galilei in ambient form

$[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$	$[\mathfrak{h},\mathfrak{p}] = \mathfrak{p}$
$[\mathbf{D}, C] \sim -\mathbf{K}$, $[\mathbf{D}, \mathbf{K}] \sim C$, $[\mathbf{D}, \mathbf{D}] \sim -\mathbf{J}$	$[\mathbf{K}, H] \sim \mathbf{P}$, $[C, H] \sim M$, $[\mathbf{D}, M] \sim -\mathbf{P}$, $[\mathbf{D}, \mathbf{P}] \sim M$

- Performing the exchange $\mathbf{D} \leftrightarrow \mathbf{P}$ yields the mirror algebra:

Example: Mirror image of Galilei in ambient form (with $M \mapsto -M$)

$[\mathfrak{h},\mathfrak{p}] = \mathfrak{p}$	$[\mathfrak{p},\mathfrak{p}] = \mathfrak{h}$ (Curvature)	$[\mathfrak{h},\mathfrak{p}] = \mathfrak{h}$ (Non-reductivity)
$[\mathbf{D}, \mathbf{P}] \sim M$, $[C, H] \sim -M$	$[M, \mathbf{P}] \sim -\mathbf{D}$, $[\mathbf{P}, \mathbf{P}] \sim -\mathbf{J}$	$[\mathbf{K}, H] \sim \mathbf{D}$, $[C, \mathbf{P}] \sim \mathbf{K}$, $[\mathbf{K}, \mathbf{P}] \sim -C$

- The obtained pair $(\mathfrak{g}, \mathfrak{h})$ is Leibnizian and effective, albeit non-reductive.
- It provides a non-reductive embedding of dS Carroll into a Leibnizian pair.

Homogeneous problem

(Leibniz version) K.M. Part II

Homogeneous lifting problem into Leibniz:

- Galilei
- (A)dS Galilei
- Torsional Galilei

Homogeneous embedding problem into Leibniz:

- Carroll
- (A)dS Carroll

Chapter 1

Alice was beginning
to get very tired of sitting

Conclusion

Main point

- Ambient is fun: many novel algebraic and geometric structures to explore
- Leibniz as a « maximal » alternative to Bargmann:

Ambient manifold	**Ambient problem**	**Lifting** (Galilean)	**Embedding** (Carrollian)
Bargmann	Generic	Full	Full
	Torsionfree	Partial	Partial
	Homogeneous	Full	Partial
Leibniz	Generic	Full	Full
	Torsionfree	Full	Full
	Homogeneous	Full	Full

Perspectives

- Classify ambient kinematical and Aristotelian effective Klein pairs
- Explore the corresponding Cartan geometries and their embedding power
- BMS like extension of Leibnizian algebras

Ambient Aristotelian geometries

Starting from a Leibnizian manifold $(\mathscr{M}, \xi, \psi, \gamma, \nabla)$ one can define various ambient Aristotelian geometries by supplementing it with an additional (compatible) canonical structure:

1. G**-Ari manifolds** endowed with a canonical Ehresmann connection A

$$A(\xi) = 1 \text{ and } \nabla A = 0$$

2. C**-Ari manifolds** endowed with a canonical field of observers N

$$\psi(N) = 1 \text{ and } \nabla N = 0.$$

3. **Bargmannian manifolds** endowed with a canonical Lagrangian metric g

$$g(\xi) = \psi \quad , \quad g|_{\text{Ker } \psi} = \gamma \text{ and } \nabla g = 0.$$

The intersection between any two of the above structures is called a **Lifshitzian manifold**.