An open sigma model for celestial gravity

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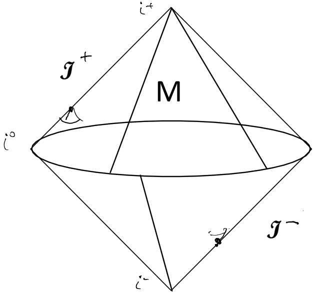
2nd Caroll workshop, Mons, Sept. 2022

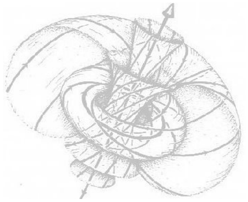
Start from: Math.DG/0504582, Duke Math (2007), w/ C. LeBrun
and: Adamo, M. & Sharma 2103.16984,
to construct global SD metrics & full amplitudes from $\mathscr{I}$.
Work in progress with C. LeBrun
+ some extra on amplitudes & Strominger's $Lw_{1+\infty}$.

Holography from null infinity, and amplitudes

- Celestial Holography seeks to find boundary theory that constructs 4d gravity from $\mathscr{I}$.
- Newman '70's: tries to rebuild space-time from 'cuts' of $\mathscr{I}$.
- Yields instead '$\mathcal{H}$-space' a complex self-dual space-time.



- Penrose: $\rightsquigarrow$ asymptotic Twistor space $P\mathscr{T} \sim \mathbb{CP}^3$, the *nonlinear graviton*.
- Embodies integrability of SD sector.
- Chiral sigma models in twistor space give full 4d gravity S-matrix expanding around self-dual sector.



Flat holography: the split signature story from $\mathscr{I}$

Caroll geometry for split signature

Now $\mathscr{I} = \mathbb{R} \times S^1 \times S^1$ with real coords $(u, \lambda, \tilde{\lambda})$, $\lambda = \lambda_1/\lambda_0$.

$$ds^2 = \frac{1}{R^2}\left(du dR - d\lambda d\tilde{\lambda} + R\sigma d\tilde{\lambda}^2 + R\tilde{\sigma} d\lambda^2 + \dots\right) ,$$

where $R = 1/r$, and $\mathscr{I} = \{R = 0\}$.

- The $\sigma, \tilde{\sigma}$ are now *real* asymptotic *shears* that encode gravitational data.
- σ encodes SD sector and $\tilde{\sigma}$ the ASD sector.
- Split signature $\rightsquigarrow$ real 'twistors' $=$ totally null SD 2-planes.
- Twistors intersect $\mathscr{I}$ in null geodesics in $\lambda =$ const. planes:

$$u = Z(\lambda, \tilde{\lambda}), \qquad \frac{\partial^2 Z}{\partial \tilde{\lambda}^2} = \sigma(Z, \lambda, \tilde{\lambda}) .$$

- We will show how twistor construction encodes $(\sigma, \tilde{\sigma})$ into twistor data $h(U), \tilde{h}(\tilde{U})$ encoding $Lw_{1+\infty}$ action.

SD sector arises by solving open disk chiral sigma model, and gives formulae for perturbations about SD sector.

Conformal self-duality in 4d, split signature

Recall on 4d manifold (M^4, g),

$$\Omega^2_M = \begin{pmatrix} \Omega^{2+} \\ \oplus \\ \Omega^{2-} \end{pmatrix}, \qquad \text{Riem} = \begin{pmatrix} \text{Weyl}^+ + S\delta & \text{Ricci}_0 \\ \text{Ricci}_0 & \text{Weyl}^- + S\delta \end{pmatrix}.$$

This talk: focus on Ricci $= 0 =$ Weyl$^-$, so Ω^{2-} is flat.

Conformal group $= SO(3,3)$ acts on global models:

- Conformally flat models: $S^2 \times S^2$ or $S^2 \times S^2/\mathbb{Z}_2$:

$$ds^2 = \Omega^2(ds^2_{S^2_{\mathbf{x}}} - ds^2_{S^2_{\mathbf{y}}}),$$

  Coordinates $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $|\mathbf{x}| = |\mathbf{y}| = 1$.
- $\mathbb{Z}_2$ acts by $(\mathbf{x}, \mathbf{y}) \to (-\mathbf{x}, -\mathbf{y})$.
- For $\Lambda = 0 : \Omega = \frac{1}{x_3 - y_3}$, and $\mathscr{I} = \mathbb{R} \times S^1 \times S^1$.
- (For $\Lambda \neq 0$: $\Omega = 1/y_3$, and $\mathscr{I} = S^2 \times S^1$.)

α and β-surfaces and the Zollfrei condition

The split signature conformally flat metric

$$ds^2 = \Omega^2 (ds^2_{S^2_{\mathbf{x}}} - ds^2_{S^2_{\mathbf{y}}}) ,$$

admits a 3-parameter family of β-planes denoted by $\mathbb{PT}_{\mathbb{R}}$:

- respectively totally null ASD S^2s given by

$$\mathbf{x} = A\mathbf{y} , \qquad A \in SO(3) = \mathbb{RP}^3 .$$

- Weyl$^- = 0 \Rightarrow \beta$-planes survive as β-surfaces.
- β-surfaces are projectively flat.
- If compact, β-surfaces are necessarily S^2 or $\mathbb{RP}^2$.
- Null geodesics are projectively $\mathbb{RP}^1$s or double cover.

Following Guillemin we define:

Definition

An indefinite space (M^d, g) is (strongly) Zollfrei if all null geodesics are embedded S^1s (of same projective length).

Conformally self-dual case

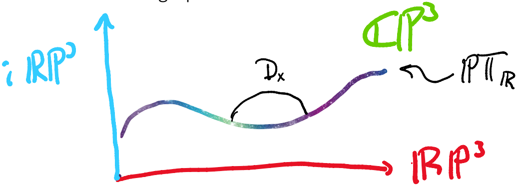
Theorem (LeBrun & M. [Duke Math J. 2007, math.dg/0504582.])

Let $(M^4, [g])$ be Zollfrei with SD Weyl-curvature. Then either

- $M = S^2 \times S^2/\mathbb{Z}_2$ *with the standard conformally flat conformal structure, or*
- $M = S^2 \times S^2$ *and there is a* $1:1$*-correspondence between*
  1. *SD conformal structures on* $S^2 \times S^2$ *near flat model &*
  2. *Deformations of the standard embedding of* $\mathbb{RP}^3 \subset \mathbb{CP}^3$ *modulo reparametrizations of* $\mathbb{RP}^3$ *and* $PGL(4, \mathbb{C})$ *on* $\mathbb{CP}^3$.

The deformed embedded $\mathbb{RP}^3$ is space of β planes $\mathbb{PT}_\mathbb{R}$ and $\mathbb{CP}^3$ is a complex twistor space.

Data encoded in graph



Reconstruction of M from twistor space $\mathbb{PT}_{\mathbb{R}}$

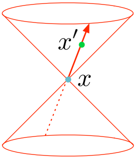
Each $x \in M \leftrightarrow$ holomorphic disc $D_x \subset \mathbb{CP}^3$ with $\partial D_x \subset \mathbb{PT}_{\mathbb{R}}$.

- D_x generates the degree-1 class in $H_2(\mathbb{CP}^3, \mathbb{PT}_{\mathbb{R}}, \mathbb{Z}) = \mathbb{Z}$.
- Reconstruct M from $\mathbb{PT}_{\mathbb{R}}$ space of all such disks:

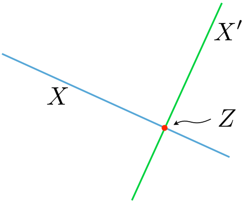
$$M = \{\text{Moduli of degree-1 hol. disks: } D_x \subset \mathbb{CP}^3, \partial D_x \subset \mathbb{PT}_{\mathbb{R}}\}$$

- Gives compact 4d moduli space
- M admits a conformal structure for which $\partial D_x \cap \partial D_{x'} = Z$ means that x, x' sit on same β-plane:

Space-time



Twistor Space



Restriction to Einstein vacuum case

Which $\mathbb{PT}_\mathbb{R} \subset \mathbb{CP}^3$ give SD Einstein $g \in [g]$ on $S^2 \times S^2$?

- Let Z^A, $A = 1, \dots, 4$ be homogenous coordinates for $\mathbb{CP}^3$.
- Introduce real skew ε^{ABCD} and

$$I_{AB} = I_{[AB]}, \quad I^{AB} = \frac{1}{2}\varepsilon^{ABCD} I_{CD}, \quad \text{with} \quad I^{AB} I_{AC} = 0\,.$$

- To define contact and Poisson structures on $\mathbb{CP}^3$

$$\theta = I_{AB} Z^A dZ^B \in \Omega^1(2)\,, \qquad \{f, g\} := I^{AB} \frac{\partial f}{\partial Z^A} \frac{\partial g}{\partial Z^B}$$

  of homogeneity degree 2 and -2 respectively & rank 2.

Theorem

A vacuum $g \in [g]$ exists when $\theta|_{\mathbb{PT}_\mathbb{R}}$ & $\{\,,\}_{\mathbb{PT}_\mathbb{R}}$ are real.

Generating functions for Einstein embeddings

Explicitly in homogeneous coordinates:

- Let $Z^A = U^A + iV^A, U^A, V^A \in \mathbb{R}^4$.
- Let $h(U)$ be an arbtrary function of homogeneity degree 2,

$$U \cdot \frac{\partial h}{\partial U} = 2h.$$

Proposition

All 'small' Einstein vacuum twistor data $\leftrightarrow h(U)$ by setting

$$\mathbb{T}_{\mathbb{R}} = \left\{ Z^A = U^A + iI^{AB} \frac{\partial h}{\partial U^B} \right\}$$

projectivising gives $\mathbb{PT}_{\mathbb{R}}$.
The corresponding SD $(2,2)$ vacuum metrics are Zollfrei on $S^2 \times S^2$ with null $\mathscr{I}$ modelled by $x_3 = y_3$.

The Poisson bracket underpins Strominger's $Lw_{1+\infty}$ structure, [Adamo, M., Sharma, 2110.06066.]. Here $Lw_{1+\infty}$ acts canonically on

$$\{\text{SD gravity phase space}\} = Lw^{\mathbb{C}}_{1+\infty}/Lw_{1+\infty} \ni h(U)$$

Holography: SD vacuum spaces from $\mathscr{I}$

Twistor space can be constructed from σ at $\mathscr{I}$:

- At fixed λ_α, real twistor coords $\mu^{\dot\alpha}$ parametrize null geodesics $u = Z(\tilde\lambda)$ in $\mathscr{I}$ where

$$\partial^2_{\tilde\lambda} Z = \sigma(Z, \tilde\lambda, \lambda).$$

  Defines projective structure on each $\lambda = \text{const.}$.
- Flat $\sigma = 0$ case has $u = \mu^{\dot\alpha}\tilde\lambda_{\dot\alpha}$.
- In general $\exists$ nonlinear correspondence [Lebrun & M, JDiffGeom. '02]:

$$\{\sigma \neq 0\} \stackrel{1:1}{\longleftrightarrow} \{h(U)\},$$

  gives $\mathscr{I} \leftrightarrow \mathbb{PT}_\mathbb{R} \subset \mathbb{PT}$ at each fixed λ.
- Transform between $\mathscr{I}$-data $(\sigma, \tilde\sigma)$ and twistor data $(h(U), \tilde h(U))$ is nonlinear analogue of radon transform

$$\sigma(u, \tilde\lambda, \lambda) = \partial_u^2 \int_{-\infty}^{\infty} dt\, h(\mu^{\dot\alpha} + t\tilde\lambda^{\dot\alpha}, \lambda_\alpha).$$

  in α-planes at $\mathscr{I}$ (cf light-ray transform).

Examples:

- Let $Z^A = (\lambda_\alpha, \mu^\alpha)$, $\alpha, \beta = 0, 1$; set $\varepsilon_{\alpha\beta} = \varepsilon_{[\alpha\beta]}$ and
$$\theta = \lambda_\alpha d\lambda_\beta \varepsilon^{\alpha\beta}\,, \qquad \{f, g\} = \varepsilon^{\alpha\beta} \frac{\partial f}{\partial \mu^\alpha} \frac{\partial g}{\partial \mu^\beta}\,,$$
- λ_α real on $\mathbb{PT}_\mathbb{R}$; if $\mu^{\dot\alpha} = u^\alpha + iv^\alpha$, take $h = h(u^\alpha\lambda_\alpha, \lambda_\alpha)$ so
$$v^\alpha = \lambda^\alpha \dot h\,.$$
- Use λ_α as homogeneous coordinates on the hol. disks, expressed as graphs by
$$\mu^\alpha = x^{\alpha\beta}\lambda_\beta + (t + g(x, \lambda))\lambda^\alpha, \qquad x^{\alpha\beta} = x^{(\alpha\beta)}\,.$$
where
$$g(x^{\alpha\beta}, \lambda) = \oint \frac{\lambda_0 - i\lambda_1}{\lambda'_0 - i\lambda'_1} \frac{1}{\langle\lambda\lambda'\rangle} \dot h((x^{\alpha\beta}\lambda'_\alpha\lambda'_\beta, \lambda'_\alpha) D\lambda'$$
- Gives split signature version of Gibbons-Hawking metrics
$$ds^2 = V d\mathbf{x}\cdot d\mathbf{x} + V^{-1}(dt + \omega)^2\,, \qquad dV =^* d\omega\,, \qquad V = \oint \ddot h D\lambda\,.$$
But now V satisfies $2 + 1$ wave equation!

Open chiral twistor sigma models

Hol. disks in $\mathbb{PT}$ with boundary on $\mathbb{PT}_{\mathbb{R}}$ are given in homogeneous coordinates by

$$Z^A(\sigma) : D \to \mathbb{T}\,, \qquad Z^A|_{\sigma=\bar\sigma} \in \mathbb{T}_{\mathbb{R}}\,.$$

representing D by upper-half-plane $D = \{\sigma \in \mathbb{C}, \Im\sigma \geq 0\}$.

- For k points $\sigma_i \in \mathbb{R}$, and $Z_i^A \in \mathbb{T}_{\mathbb{R}}$, $\exists!$ deg $k-1$ disk thru Z_i :

$$Z^A(\sigma) = \sum_{i=1}^{k} \frac{Z_i^A}{\sigma - \sigma_i} + M(\sigma)\,, \qquad M(\sigma) \text{ holomorphic on } D.$$

- For $Z = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}_{\mathbb{R}}$ implies λ_α real.
- Therefore $M^A = (0, m^{\dot\alpha})$, but $m^{\dot\alpha} \neq 0$ unless $h = 0$.
- Action for holomorphy and boundary conditions:

$$S_D[Z(\sigma), Z_i] = \int_D [m\,\bar\partial m] d\sigma + \oint_{\partial D} h(Z) d\sigma$$

using *spinor-helicity* notation $[\mu\,\nu] := \mu_{\dot\alpha}\nu^{\dot\alpha}$, $\langle 1\,2\rangle := \kappa_{1\alpha}\kappa_2^{\alpha}$.

Sigma model and gravity S-matrix on SD background

Amplitudes are functionals $\mathcal{M}[h,\tilde{h}_i]$ of gravitational data:

- $h \in C^\infty(\mathbb{PT}_\mathbb{R},\mathcal{O}(2))$for fully nonlinear SD part,
- $\tilde{h}_i \in C^\infty(\mathbb{PT}_\mathbb{R},\mathcal{O}(-6))$, $i=1,\ldots,k$, ASD perturbations.
- For eigenstates of momentum $k_{i\alpha\dot\alpha}=\kappa_{i\alpha}\tilde\kappa_{i\dot\alpha}$ take:

$$h_i=\int\frac{dt}{t^3}\delta^2(t\lambda_\alpha-\kappa_{i\alpha})\mathrm{e}^{it[\mu,\tilde\kappa_i]},\quad \tilde{h}_i=\int\frac{dt}{t^{-5}}\delta^2(t\lambda_\alpha-\kappa_{i\alpha})\mathrm{e}^{it[\mu,\tilde\kappa_i]}$$

Proposition (Adapted from [Adamo, M. & Sharma, 2103.16984] to split signature.)

The amplitude for k ASD perturbations on SD background h is

$$\mathcal{M}(h,\tilde{h}_i)=\int_{(S^1\times\mathbb{PT}_\mathbb{R})^k} S^{os}_D[h,Z_i,\sigma_i]\,\mathrm{det}\,'\tilde{\mathbb{H}}\prod_{i=1}^k\tilde{h}_i(Z_i)D^3Z_i d\sigma_i\,.$$

Here $S^{os}_D[h,Z_i,\sigma_i]$ *is the on-shell Sigma model action and*

$$\tilde{\mathbb{H}}_{ij}(Z_i)=\begin{cases}\frac{\langle\lambda_i\lambda_j\rangle}{\sigma_i-\sigma_j} & i\neq j\\ -\sum_l\frac{\langle\lambda_i\lambda_l\rangle}{\sigma_i-\sigma_j}\,, & i=j\,.\end{cases}$$

Ideas in proof

- Expand $h = h_{k+1} + \ldots + h_n$ to 1st order in momentum e-states h_i to give flat background perturbative amplitude.
- On shell action expands as tree correlator

$$S_D^{os}[h_{k+1} + \ldots + h_n, Z_i, \sigma_i] = \langle V_{h_{k+1}} \ldots V_{h_n} \rangle_{tree} + O(h_i^2)\,.$$

- Here the 'vertex operators' are $V_{h_i} = \int_{\partial D} h_i(\sigma_i) d\sigma_i\,.$
- Propagators for S_D give Poisson bracket $\{\,,\}$

$$\langle h_i h_j \rangle_{tree} = \frac{[\partial_\mu h_i\, \partial_\mu h_j]}{\sigma_i - \sigma_j} = \frac{[i\,j]}{\sigma_i - \sigma_j} h_i h_j\,, \qquad i \neq j\,.$$

- Matrix-tree theorem then gives

$$\langle h_{k+1} \ldots h_n \rangle_{\mathrm{tree}} = \det{}' \mathbb{H} \prod_{i=k+1}^{n} h_i\,, \qquad \mathbb{H}_{ij} = \frac{[ij]}{\sigma_i - \sigma_j}\,, \qquad i \neq j \text{ etc.}$$

$$\rightsquigarrow \qquad \mathcal{M}(h_i, \tilde{h}_i) = \int_{(S^1)^n \times (\mathbb{RP}^3)^k} \det{}' \mathbb{H} \det{}' \tilde{\mathbb{H}} \prod_{j=k+1}^{n} h_j d\sigma_j \prod_{i=1}^{k} \tilde{h}_i(Z_i) D^3 Z_i d\sigma_i\,.$$

This is now equivalent to the Cachazo-Skinner formula.

Relation to Einstein-Hilbert action at $k = 2$

[Adamo, M, Sharma, 2103.1239

At $k = 2$, $\det{}'\tilde{\mathbb{H}}$ and Mobius symmetry trivialises σ integrals so

$$\mathcal{M}[h, \tilde{h}_1, \tilde{h}_2] = \int d^2\mu_1 d^2\mu_2 \, \mathrm{e}^{i[\mu_1\, 1] + i[\mu_2\, 2]} S_D^{os}[h, Z_1, Z_2]$$

- Writing $x^{\alpha\dot{\alpha}} = (\mu_1^{\dot{\alpha}}, \mu_2^{\dot{\alpha}})$ this a space-time integral

$$\mathcal{M}[h, \tilde{h}_1, \tilde{h}_2] = \int d^4x \, \mathrm{e}^{ik_1\cdot x + ik_2 \cdot x} S_D^{os}[h, \mu_1, \mu_2]$$

Proposition

Let $\Omega(x) := S_D^{os}[h, \mu_1, \mu_2]$. Then Ω is the Plebanskis first potential (Kahler scalar) for the SD background metric

$$ds^2 = \frac{\partial^2 \Omega}{\partial \mu_1^{\dot{\alpha}} \partial \mu_2^{\dot{\beta}}} d\mu_1^{\dot{\alpha}} d\mu_2^{\dot{\beta}} \, .$$

The second variation of the Einstein-Hilbert action

$$\delta^2 S_{\mathrm{EH}}[h, \tilde{h}_1, \tilde{h}_2] = \int d^4x \mathrm{e}^{i(k_1+k_2)\cdot x} \, \Omega(x) = \mathcal{M}[h, \tilde{h}_1, \tilde{h}_2]$$

(Follows from Plebanski gravity action.)

Conclusions and open problems

- We have rigidity of conformally-flat SD split signature vacuum metrics with $\mathscr{I} = S^1 \times S^1 \times \mathbb{R}/\mathbb{Z}_2$.
- Have construction for split signature SD vacuum metrics on $S^2 \times S^2$ with $\mathscr{I} \simeq S^1 \times S^1 \times \mathbb{R}$ depending on smooth sections h of $\mathcal{O}(2)$ over $\mathbb{RP}^3$ defining deformed real slice.
- Similar results follow for $\Lambda \neq 0$ where $h \leftrightarrow 2+1$ signature conformal structure of $\mathscr{I} = S^2 \times S^1$.
- Reconstruction via open holomorphic discs leads to chiral open sigma model that computes gravity amplitudes.
- MHV formula gives theory underlying tree formalism of Bern et. al. from 1998.
- Framework gives full expression of $Lw_{1+\infty}$ symmetries. **Slogan:** SD gravity phase space $= Lw^{\mathbb{C}}_{1+\infty}/Lw_{1+\infty}$
- Split signature twistors avoid 'lightray transform' or Čech-Dolbeult manifesting $Lw_{1+\infty}$ directly.