On the Relation Between Asymptotic Charges, the Failure of Peeling and Late-time Tails

Based on the papers The Case Against Smooth Null Infinity I-III (IV-V), a joint paper with the same title as the talk with Dejan Gajic (Leipzig University), and upcoming joint work with Hamed Masaood (Imperial College London)

Leonhard Kehrberger

Cambridge University

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It turns out that this is very closely related to the issue of modelling isolated systems in general relativity.

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2. The Question of Late-Time Asymptotics/Tails

3. The Question of Early-Time Asymptotics/Peeling/Smooth Null Infinity

4. Bringing everything together

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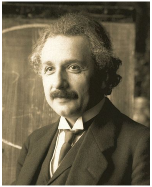
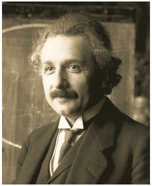
General Relativity



- Formulated by Einstein during 1912–1915
- Contemporary understanding of gravitational physics
- Many new predictions: gravitational waves, black holes, singularities, cosmology . . .



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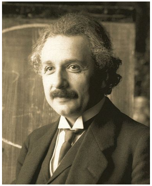
- Contemporary understanding of gravitational physics
- Many new predictions: gravitational waves, black holes, singularities, cosmology ...

- The objects of study are (3+1)-dimensional Lorentzian manifolds $(\mathcal{M}, g)$ with signature $\text{sign}(g) = (-, +, +, +)$ solving the Einstein equations ($\Lambda = 0$):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2T_{\mu\nu}, \tag{EE}$$

where $T_{\mu\nu}$ corresponds to matter fields.

General Relativity



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$$R_{\mu\nu} = 0. \qquad \text{(EE)}$$

The initial value problem in GR

- General relativity is a *dynamical* theory.
- Einstein equations are hyperbolic (in suitable gauge) and admit well-posed initial value formulation.
- Initial data are given by a 3d Riemannian manifold $(\Sigma, \bar{g})$ together with a symmetric 2-tensor k.

Theorem (Choquet-Bruhat, 1952, (1969 with Geroch), Sbierski 2013).
For suitably regular initial data $(\Sigma, \bar{g}, k)$ solving the constraint equations, there exists a unique maximal globally hyperbolic development $(\mathcal{M}, g)$ solving the Einstein equations (EE).

- Penrose diagrams are extremely practical tools for visualising the causal structure of a spacetime. Take e.g. the Minkowski spacetime $(\mathbb{R}^{3+1}, -dt^2 + dr^2 + r^2 d\Omega_{\mathbb{S}^2})$.

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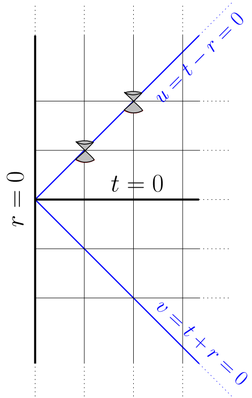

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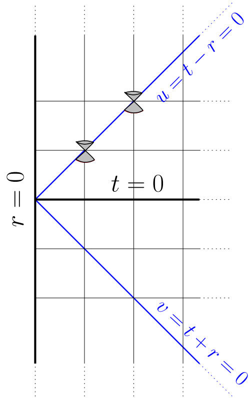
- In double null coordinates $u = t - r$, $v = t + r$, the metric reads $-4dudv + r^2 d\Omega_{\mathbb{S}^2}$.



Mapping the double null coordinates (u, v) to a set of bounded double null coordinates, (e.g. $U = \arctan u$, $V = \arctan v$) gives:

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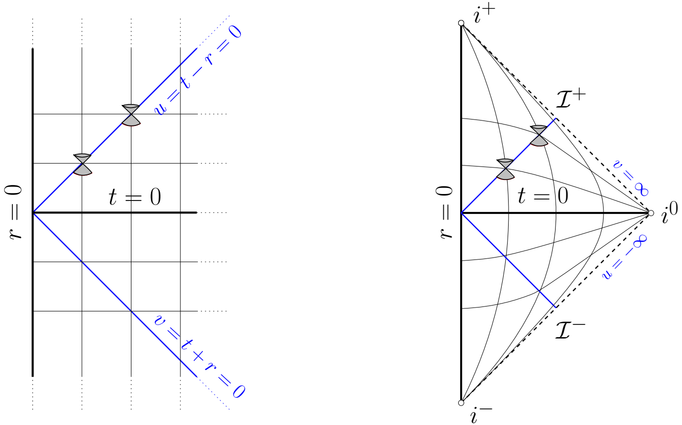

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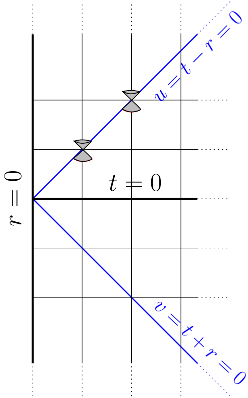

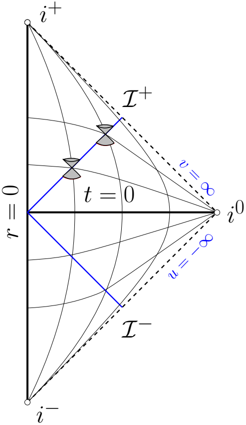

$\mathcal{I}^+$ corresponds to the set of limit points $\{v = \infty\}$, $\mathcal{I}^-$ corresponds to $\{u = -\infty\}$.

- For $M > 0$, define $(\mathcal{M}_M, g_M)$ with $\mathcal{M}_M = \mathbb{R}_t \times (2M, \infty)_r \times \mathbb{S}^2$ and

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_{\mathbb{S}^2}. \tag{1}$$

These are solutions to the Einstein vacuum equations and describe the exterior of a spherically symmetric black hole of mass M.

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- Define $r^* = r + 2M \log |r/2M - 1|$, and let $u = t - r^*$, $v = t + r^*$. Then

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THE SCHWARZSCHILD BLACK HOLE EXTERIOR

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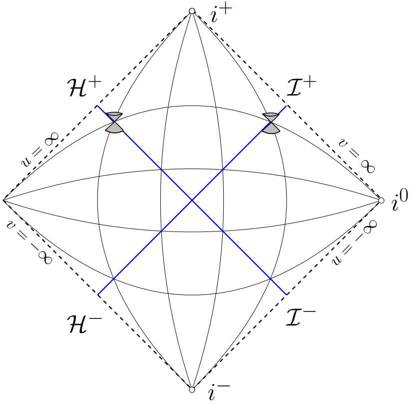

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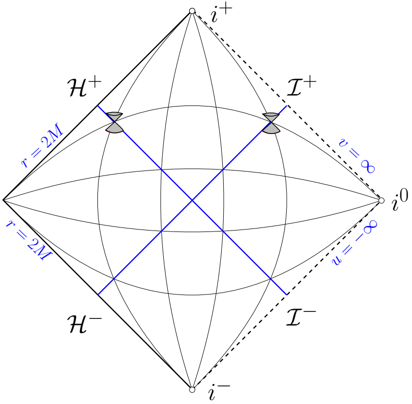
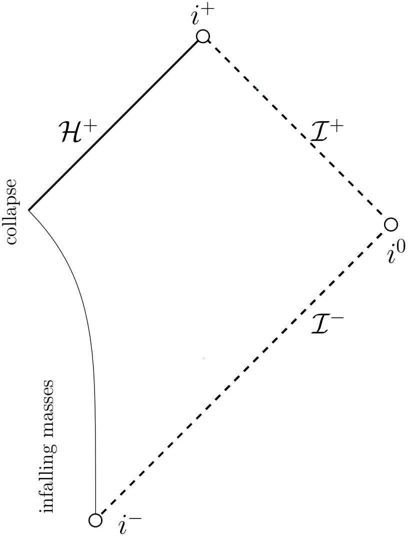


Four overarching questions



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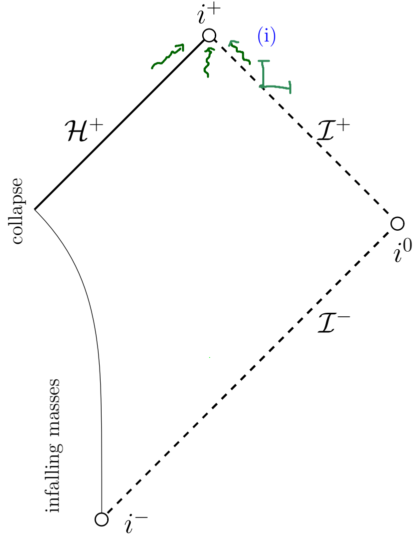

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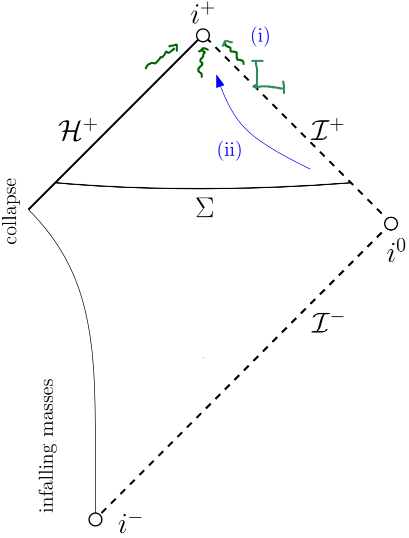

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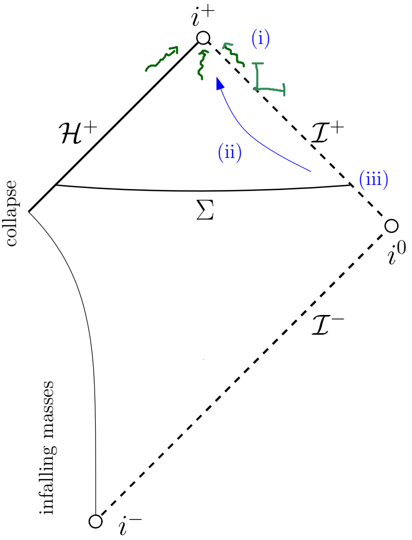

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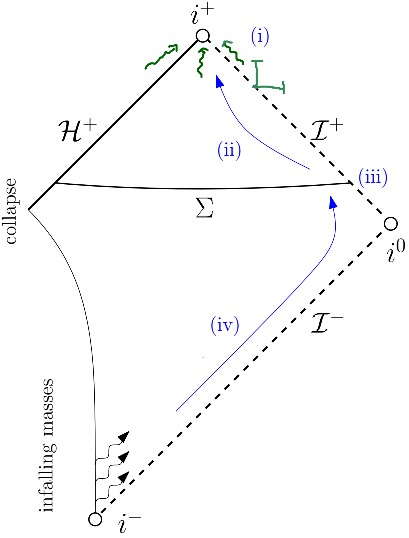

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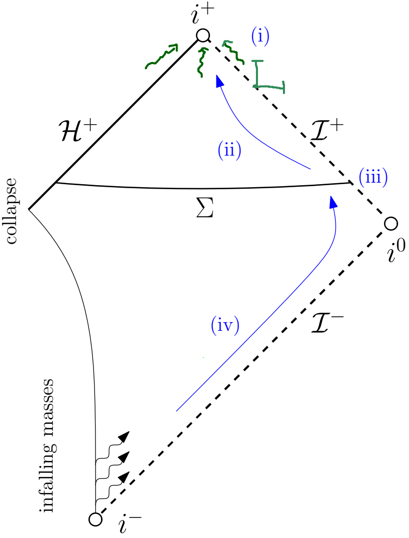

Aim of this talk is to show how all these questions are related and to provide answers to these questions within a simple model!

THE SETUP

Consider the linearised Einstein vacuum equations around the exterior of Schwarzschild:

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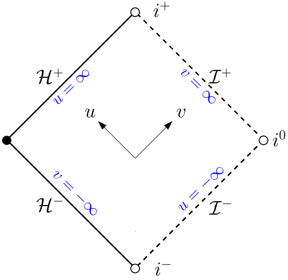

5.2.1 The complete list of unknowns

The equations will concern a set of quantities

$$\mathscr{S} = \left(\overset{(1)}{\hat{\not{g}}}\,,\ \overset{(1)}{\sqrt{\not{g}}}\,,\ \overset{(1)}{\Omega}\,,\ \overset{(1)}{b}\,,\ (\Omega \overset{(1)}{tr} \chi)\,,\ (\Omega \overset{(1)}{tr} \underline{\chi})\,,\ \overset{(1)}{\hat{\chi}}\,,\ \overset{(1)}{\hat{\underline{\chi}}}\,,\ \overset{(1)}{\underline{\eta}}\,,\ \overset{(1)}{\eta}\,,\ \overset{(1)}{\omega}\,,\ \overset{(1)}{\underline{\omega}}\,,\ \overset{(1)}{\alpha}\,,\ \overset{(1)}{\beta}\,,\ \overset{(1)}{\rho}\,,\ \overset{(1)}{\sigma}\,,\ \overset{(1)}{\underline{\beta}}\,,\ \overset{(1)}{\underline{\alpha}}\,,\ \overset{(1)}{K} \right) \tag{142}$$

of smooth (to be defined precisely below) functions, $S^2_{u,v}$-vectors and tensors defined on domains of the Schwarzschild manifold $(\mathcal{M}, g)$. Specifically, the quantities

5.2.2 Equations for the linearised metric components

The equations for the metric components read

$$\underline{D}\left(\frac{\sqrt{\overset{(1)}{\not g}}}{\sqrt{\not g}}\right) = (\Omega \overset{(1)}{tr} \underline{\chi}) \quad , \quad D\left(\frac{\sqrt{\overset{(1)}{\not g}}}{\sqrt{\not g}}\right) = (\Omega \overset{(1)}{tr}\chi) - \not{d}\mathrm{iv}\, \overset{(1)}{b}, \tag{144}$$

$$\sqrt{\not g}\, \underline{D}\left(\frac{\overset{(1)}{\hat{\not g}}_{AB}}{\sqrt{\not g}}\right) = 2\Omega\, \overset{(1)}{\hat{\underline{\chi}}}_{AB} \quad , \quad \sqrt{\not g}\, D\left(\frac{\overset{(1)}{\hat{\not g}}_{AB}}{\sqrt{\not g}}\right) = 2\Omega\, \overset{(1)}{\hat{\chi}}_{AB} + 2\left(\not{\mathcal{D}}_2^\star \overset{(1)}{b}\right)_{AB}, \tag{145}$$

$$\partial_u \overset{(1)}{b}{}^A = 2\Omega^2 \left(\overset{(1)}{\eta}{}^A - \overset{(1)}{\underline{\eta}}{}^A\right) . \tag{146}$$

5.2.3 Equations for the linearised Ricci coefficients

We start with the equations for the weighted linearised traces of the second fundamental forms:

$$D(\Omega \overset{(1)}{tr}\underline{\chi}) = \Omega^2 \left(2\slashed{div}\, \overset{(1)}{\underline{\eta}} + 2\overset{(1)}{\rho} + 4\rho\,\Omega^{-1}\overset{(1)}{\Omega} \right) - \frac{1}{2}\Omega tr\chi \left((\Omega \overset{(1)}{tr}\underline{\chi}) - (\Omega \overset{(1)}{tr}\chi) \right), \tag{147}$$

$$\underline{D}(\Omega \overset{(1)}{tr}\chi) = \Omega^2 \left(2\slashed{div}\, \overset{(1)}{\eta} + 2\overset{(1)}{\rho} + 4\rho\,\Omega^{-1}\overset{(1)}{\Omega} \right) - \frac{1}{2}\Omega tr\chi \left((\Omega \overset{(1)}{tr}\underline{\chi}) - (\Omega \overset{(1)}{tr}\chi) \right), \tag{148}$$

$$D(\Omega \overset{(1)}{tr}\chi) = -\left(\Omega tr\chi\right)\left(\Omega \overset{(1)}{tr}\chi\right) + 2\omega(\Omega \overset{(1)}{tr}\chi) + 2\left(\Omega tr\chi\right)\overset{(1)}{\omega}, \tag{149}$$

$$\underline{D}(\Omega \overset{(1)}{tr}\underline{\chi}) = -\left(\Omega tr\underline{\chi}\right)\left(\Omega \overset{(1)}{tr}\underline{\chi}\right) + 2\underline{\omega}(\Omega \overset{(1)}{tr}\underline{\chi}) + 2\left(\Omega tr\underline{\chi}\right)\overset{(1)}{\underline{\omega}}. \tag{150}$$

For the traceless parts we have

$$\begin{aligned} \slashed{\nabla}_3 \left(\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}\right) + \Omega^{-1}\left(tr\underline{\chi}\right)\overset{(1)}{\hat{\underline{\chi}}} &= -\Omega^{-1}\overset{(1)}{\underline{\alpha}}, \\ \slashed{\nabla}_4 \left(\Omega^{-1}\overset{(1)}{\hat{\chi}}\right) + \Omega^{-1}\left(tr\chi\right)\overset{(1)}{\hat{\chi}} &= -\Omega^{-1}\overset{(1)}{\alpha}, \end{aligned} \tag{151}$$

$$\slashed{\nabla}_3 \left(\Omega\overset{(1)}{\hat{\chi}}\right) + \frac{1}{2}\left(\Omega tr\underline{\chi}\right)\overset{(1)}{\hat{\chi}} + \frac{1}{2}\left(\Omega tr\chi\right)\overset{(1)}{\hat{\underline{\chi}}} = -2\Omega\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\eta}, \tag{152}$$

$$\slashed{\nabla}_4 \left(\Omega\overset{(1)}{\hat{\underline{\chi}}}\right) + \frac{1}{2}\left(\Omega tr\chi\right)\overset{(1)}{\hat{\underline{\chi}}} + \frac{1}{2}\left(\Omega tr\underline{\chi}\right)\overset{(1)}{\hat{\chi}} = -2\Omega\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\underline{\eta}}. \tag{153}$$

THE EQUATIONS OF LINEARISED GRAVITY

For $\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}$ the equations read

$$\nabla_3 \overset{(1)}{\underline{\eta}} = \frac{1}{2}\left(tr\underline{\chi}\right)\left(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}\right) + \overset{(1)}{\underline{\beta}} \quad , \quad \nabla_4 \overset{(1)}{\eta} = -\frac{1}{2}\left(tr\chi\right)\left(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}\right) - \overset{(1)}{\beta}. \tag{154}$$

The equations for the linearised lapse and its derivatives are given by

$$D\overset{(1)}{\underline{\omega}} = -\Omega\left(\overset{(1)}{\rho} + 2\rho\Omega^{-1}\overset{(1)}{\Omega}\right) , \tag{155}$$

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$$\overset{(1)}{\omega} = D\left(\Omega^{-1}\overset{(1)}{\Omega}\right) \ , \ \overset{(1)}{\underline{\omega}} = \underline{D}\left(\Omega^{-1}\overset{(1)}{\Omega}\right) \ , \ \overset{(1)}{\eta}_A + \overset{(1)}{\underline{\eta}}_A = 2\nabla_A\left(\Omega^{-1}\overset{(1)}{\Omega}\right). \tag{157}$$

Finally we have the linearised Codazzi equations

$$\begin{aligned} d\!\!/iv\overset{(1)}{\hat{\underline{\chi}}} &= -\frac{1}{2}\left(tr\underline{\chi}\right)\overset{(1)}{\eta} + \overset{(1)}{\underline{\beta}} + \frac{1}{2\Omega}\nabla\!\!\!/\left(\Omega\overset{(1)}{tr}\underline{\chi}\right), \\ d\!\!/iv\overset{(1)}{\hat{\chi}} &= -\frac{1}{2}\left(tr\chi\right)\overset{(1)}{\underline{\eta}} - \overset{(1)}{\beta} + \frac{1}{2\Omega}\nabla\!\!\!/\left(\Omega\overset{(1)}{tr}\chi\right), \end{aligned} \tag{158}$$

and

$$c\!\!/url\overset{(1)}{\eta} = \overset{(1)}{\sigma} \quad , \quad c\!\!/url\overset{(1)}{\underline{\eta}} = -\overset{(1)}{\sigma}, \tag{159}$$

as well as the linearised Gauss equation

$$\overset{(1)}{K} = -\overset{(1)}{\rho} - \frac{1}{4}\frac{tr\chi}{\Omega}\left(\left(\Omega\overset{(1)}{tr}\underline{\chi}\right) - \left(\Omega\overset{(1)}{tr}\chi\right)\right) + \frac{1}{2}\Omega^{-1}\overset{(1)}{\Omega}\left(tr\chi tr\underline{\chi}\right) . \tag{160}$$

5.2.4 Equations for linearised curvature components

We complete the system of linearised gravity with the linearised Bianchi equations:

$$\nabla_3 \overset{(1)}{\alpha} + \frac{1}{2} tr\underline{\chi}\,\overset{(1)}{\alpha} + 2\underline{\hat{\omega}}\,\overset{(1)}{\alpha} = -2\mathcal{D}_2^\star \overset{(1)}{\beta} - 3\rho\,\overset{(1)}{\hat{\chi}}\,, \tag{161}$$

$$\nabla_4 \overset{(1)}{\beta} + 2(tr\chi)\overset{(1)}{\beta} - \hat{\omega}\overset{(1)}{\beta} = d\!\!/iv\,\overset{(1)}{\alpha}\,, \tag{162}$$

$$\nabla_3 \overset{(1)}{\beta} + (tr\underline{\chi})\overset{(1)}{\beta} + \underline{\hat{\omega}}\overset{(1)}{\beta} = \mathcal{D}_1^\star\left(-\overset{(1)}{\rho}, \overset{(1)}{\sigma}\right) + 3\rho\,\overset{(1)}{\eta}\,, \tag{163}$$

$$\nabla_4 \overset{(1)}{\rho} + \frac{3}{2}(tr\chi)\overset{(1)}{\rho} = d\!\!/iv\,\overset{(1)}{\beta} - \frac{3}{2}\frac{\rho}{\Omega}\overset{(1)}{(\Omega tr\chi)}\,, \tag{164}$$

$$\nabla_3 \overset{(1)}{\rho} + \frac{3}{2}(tr\underline{\chi})\overset{(1)}{\rho} = -d\!\!/iv\,\overset{(1)}{\underline{\beta}} - \frac{3}{2}\frac{\rho}{\Omega}\overset{(1)}{(\Omega tr\underline{\chi})}\,, \tag{165}$$

$$\nabla_4 \overset{(1)}{\sigma} + \frac{3}{2}(tr\chi)\overset{(1)}{\sigma} = -cu\!\!/rl\,\overset{(1)}{\beta}\,, \tag{166}$$

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$$\nabla_3 \overset{(1)}{\underline{\beta}} + 2(tr\underline{\chi})\overset{(1)}{\underline{\beta}} - \underline{\hat{\omega}}\overset{(1)}{\underline{\beta}} = -d\!\!/iv\,\overset{(1)}{\underline{\alpha}}\,, \tag{169}$$

$$\nabla_4 \overset{(1)}{\underline{\alpha}} + \frac{1}{2}(tr\chi)\overset{(1)}{\underline{\alpha}} + 2\hat{\omega}\overset{(1)}{\underline{\alpha}} = 2\mathcal{D}_2^\star \overset{(1)}{\underline{\beta}} - 3\rho\,\overset{(1)}{\underline{\hat{\chi}}}\,. \tag{170}$$

THE SETUP

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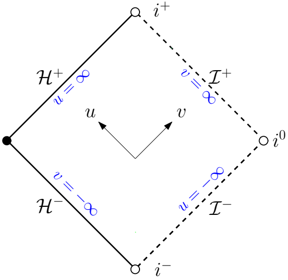

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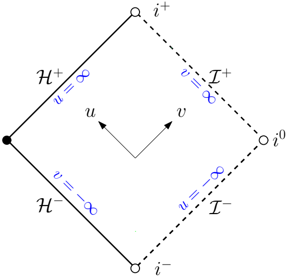

- Miraculously, the two extremal components of the Weyl curvature tensor Ψ^0, Ψ^4, then satisfy decoupled wave equations, from which one can moreover control* the rest of the system:

$$\mathcal{T}^{[s]}_{g_M} \Psi^{|s|\pm s} = 0, \quad s = \pm 2 \qquad \text{(Teukolsky)}$$

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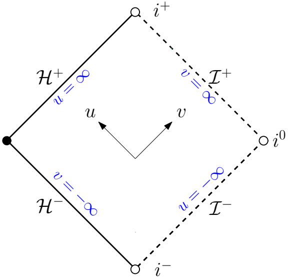

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- To ease presentation, we will occasionally focus on the simpler wave equation

$$\Box_{g_M}\phi (= \nabla^\mu \nabla_\mu \phi) = 0 \qquad \text{(Wave)}$$

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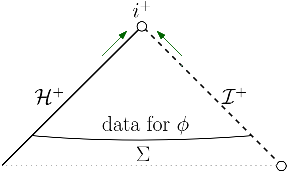
- Given data for ϕ on some hyperboloidal initial hypersurface Σ, what is the asymptotic behaviour of ϕ near i^+?



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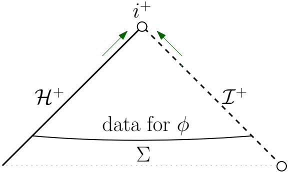

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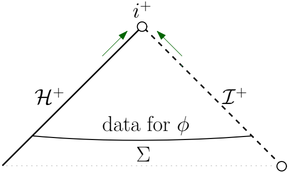

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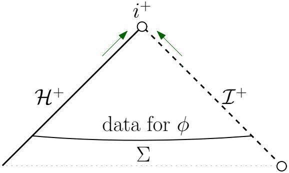

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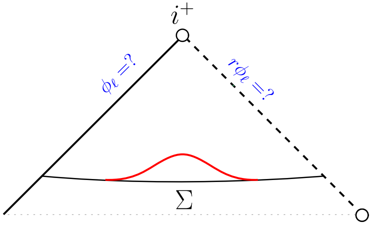
Of course, the asymptotics one obtains will depend on the exact assumptions one makes on data. But what assumptions to make on data?

Case (i): Initial data for ϕ are of compact support



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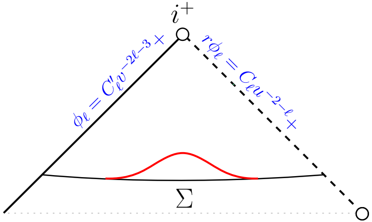

- Project ϕ onto spherical harmonics $Y_{\ell m}$, suppress m-index
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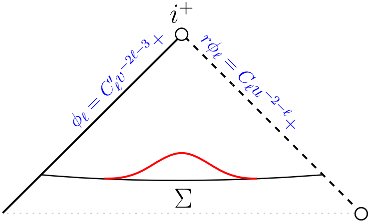

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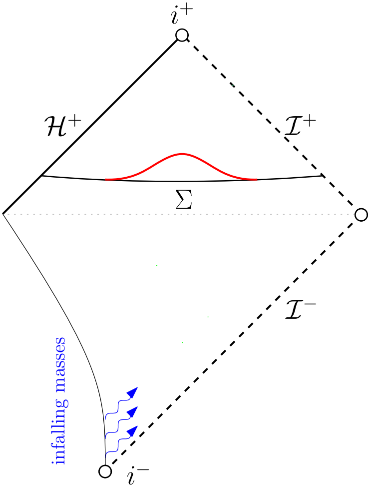

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Aside: Modelling Isolated Systems in GR

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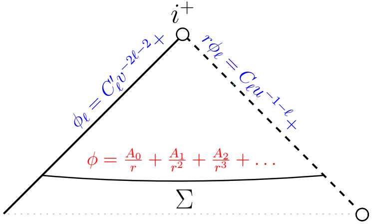
In particular, the following decay behaviour of the Weyl tensor is implied:

$$\Psi^j = \mathcal{O}(r^{-5+j}) \text{ towards } \mathcal{I}^+, \Psi^{4-j} = \mathcal{O}(r^{-5+j}) \text{ towards } \mathcal{I}^-$$

Case (ii): Conformally regular/ Peeling initial data

In the spirit of Penrose, let us now assume data that are conformally regular and satisfy peeling, i.e. data that have an expansion in powers of $1/r$.



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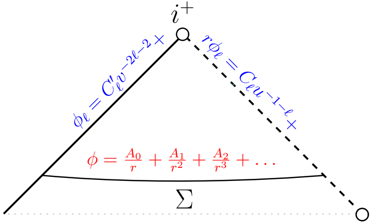

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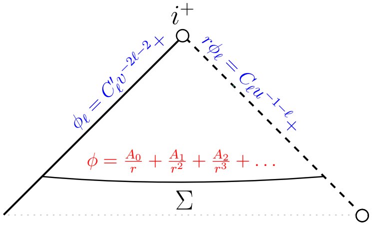

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- Decay rates one power slower than in case of localised data
- Constants C_ℓ, C'_ℓ are linear combinations of $M^\ell A_1, M^{\ell-1} A_2 \dots, A_{\ell+1}$
- Faster decay for higher ℓ-modes related to existence of certain conserved charges. In Minkowski ($M = 0$):

$$\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)^\ell(r\phi_\ell)) = 0 \tag{3}$$

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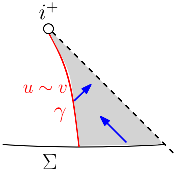

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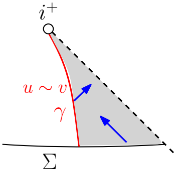

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- For higher ℓ-modes, can now perform a similar argument, but with $r\phi_0$ replaced by $(r^2\partial_v)^\ell(r\phi_\ell)$. (Recall $\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)^\ell(r\phi_\ell)) = 0$ in Minkowski.)
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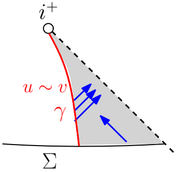

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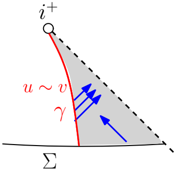

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Note: The actual "conserved quantity" is not $(r^2\partial_v)^\ell(r\phi_\ell)$, but

$$\Phi_\ell := \sum_{i=0}^{\ell} x_i^{(\ell)} \cdot M^i \cdot \left(\frac{r^2\partial_v}{1-\frac{2M}{r}}\right)^{\ell-i} (r\phi_\ell). \tag{7}$$

To be precise, $I_\ell^{\mathrm{NP}}[\phi] := \lim_{v\to\infty} r^2\partial_v\Phi_\ell$ is conserved along $\mathcal{I}^+$.

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- For instance, for compactly supported data, one would get

$$r\Psi^4_{\ell=2}|_{\mathcal{I}^+} \sim r\phi_{\ell=4}|_{\mathcal{I}^+} \sim u^{-\ell-2} = u^{-6}.$$

  For conformally smooth data, one would get

$$r\Psi^4_{\ell=2}|_{\mathcal{I}^+} \sim r\phi_{\ell=4}|_{\mathcal{I}^+} \sim u^{-\ell-1} = u^{-5}$$

  This has recently been proved by [Ma–Zhang].

Case (iii): Conformally irregular initial data

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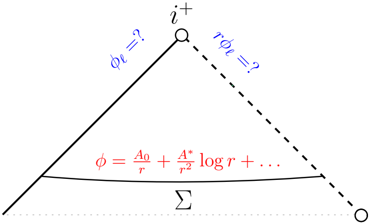
What happens if we assume data that are not conformally regular?



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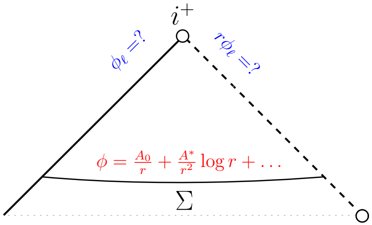

Let's revisit the previous proof!

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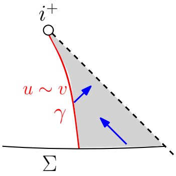

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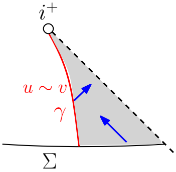
- **If $M \neq 0$,** no longer have global conservation law. Instead:

$$v^2\log^{-1}v\cdot\partial_u\partial_v(r\phi_0) = -\left(1-\frac{2M}{r}\right)\frac{2M\cdot r\phi_0}{r^3}\cdot v^2\log^{-1}v \to 0 \tag{13}$$

- This implies the conservation of the modified $\ell = 0$-Newman–Penrose charge:

$$\lim_{v\to\infty} v^2\log^{-1}v\partial_v(r\phi_0) =: I_0^{\mathrm{NP},\log}[\phi] \equiv -A_1 \tag{14}$$



- Can moreover extend this conservation law a bit away from $\mathcal{I}^+$: $\partial_v(r\phi_0) \sim I_0^{\mathrm{NP},\log}[\phi]v^{-2}\log v$ in depicted region.
- Finally, integrate this from γ:

$$r\phi_0 - r\phi_0|_\gamma \sim I_0^{\mathrm{NP},\log}[\phi]\left(\frac{\log u}{u} - \frac{\log v}{v}\right)$$

$$v\to\infty: \quad \implies r\phi_0|_{\mathcal{I}^+} \sim \frac{I_0^{\mathrm{NP},\log}[\phi]\log u}{u}$$

SKETCH OF THE PROOF II

- For higher ℓ-modes, can now perform a similar argument, but with $r\phi_0$ replaced by $(r^2\partial_v)^\ell(r\phi_\ell)$. (Recall $\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)^\ell(r\phi_\ell)) = 0$ in Minkowski.)
- The main observation is that if the data are conformally regular $(\phi = \frac{A_0}{r} + \frac{A_1}{r^2} + \frac{A_2}{r^3} + \dots)$, then

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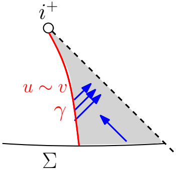
for any $\ell > 0$, even though extra r-weights are introduced!



- Can again extend this a bit away from $\mathcal{I}^+$: $\partial_v(r^2\partial_v)^\ell(r\phi_\ell) \sim v^{-2}$ in depicted region.
- Finally, integrate this $\ell + 1$ times from γ, each time picking up a $1/u$-factor:

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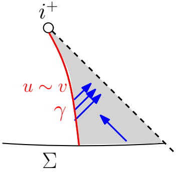

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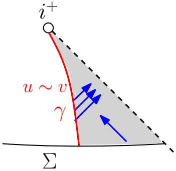

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The above is a simplification. The actual conserved quantity is

$$I_\ell^{\mathrm{NP},r^{-\ell}}[\phi]:=\lim_{v\to\infty} r^{-\ell}\cdot r^2\partial_v\Phi_\ell(u,v) \tag{17}$$

Case (iii): Conformally irregular initial data

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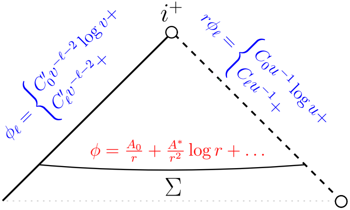

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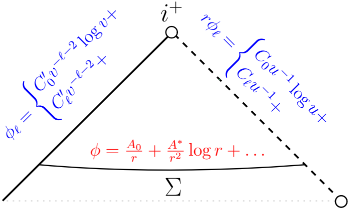

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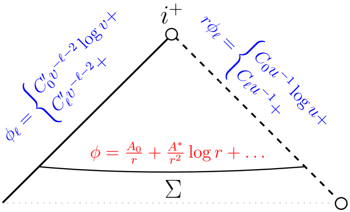

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$\Longrightarrow$ If your solution is conformally irregular, then the cause of this irregularity is precisely what you would measure in the late-time tails!

SUMMARY

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  - For instance, it is expected that in the non-linear setting, the non-stationary terms will dominate for higher ℓ-modes (or higher spin fields) if the data are compactly supported. [Bizoń–Chmaj–Rostworowski, upcoming work by Luk–Oh]
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- We will now try and understand *dynamically* what the behaviour towards $\mathcal{I}^+$ should be!

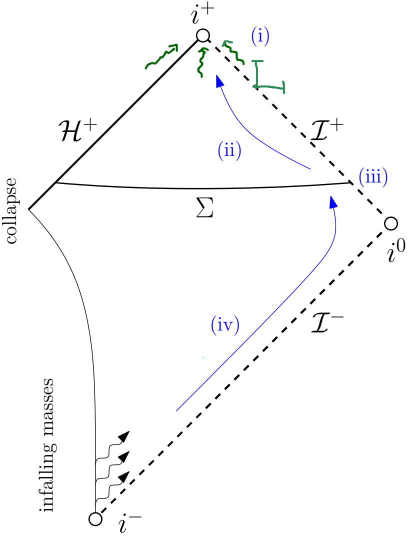
Table of Contents

(i) In gravitational collapse, what is the (measurable?) asymptotic behaviour of gravitational radiation at late times?

(ii) How is this asymptotic behaviour **along** $\mathcal{I}^+$ related to asymptotic behaviour **towards** $\mathcal{I}^+$?

(iii) What is the asymptotic behaviour of gravitational radiation towards $\mathcal{I}^+$? To what degree is peeling satisfied? Is $\mathcal{I}^+$ smooth in the sense of Penrose?

(iv) How is the asymptotic behaviour towards $\mathcal{I}^+$ related to the structure of gravitational radiation in the infinite past?



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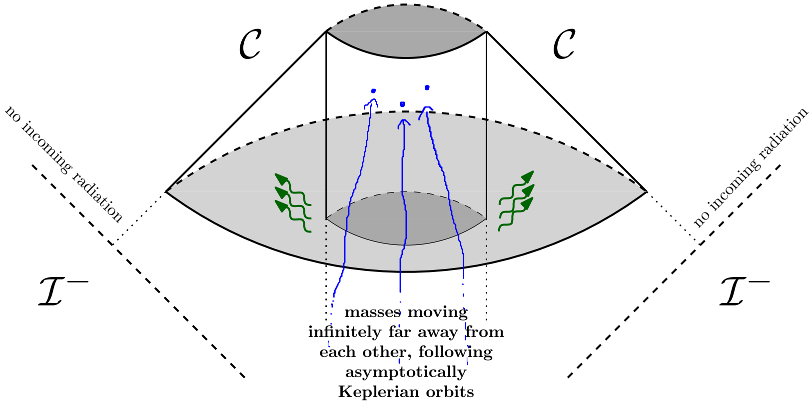

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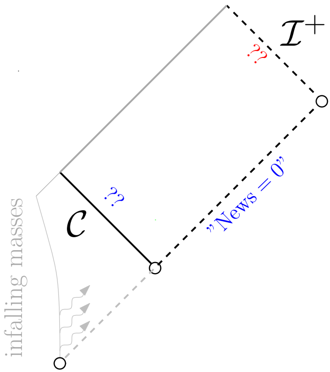

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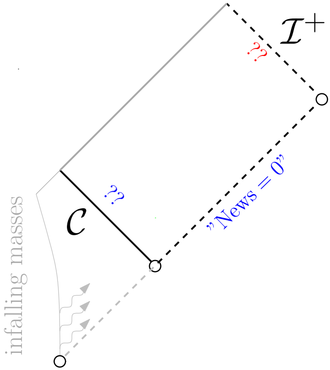

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This will give rise to a scattering problem for the linearised Einstein vacuum equations!

SKETCH OF THE POST-NEWTONIAN PREDICTION

- [MTW, Thorne '80: Multipole expansions of gravitational radiation] Decompose into multipoles

$$h_{jk}^{\mathrm{TT}} = \sum_{\ell \geq 2} \sum_{m=-\ell}^{\ell} [r^{-1\,(\ell)} I^{\ell,m}(t-r) T_{jk}^{\ell,m} + \dots], \tag{18}$$

where the ${}^{(\ell)}I^{\ell,m}$ are the (ℓ)-th derivatives of the mass multipole moments, which are general functions of retarded time $u = t - r$.

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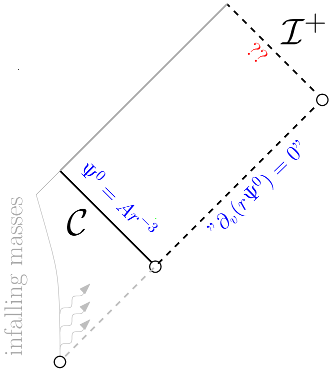

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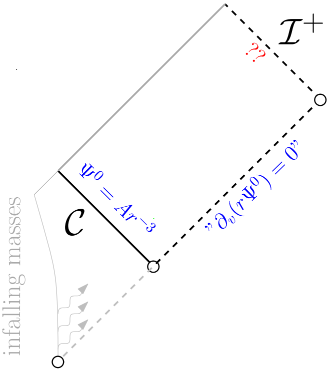

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Very roughly, can now uniquely solve this scattering problem (joint work with H. Masaood)!

The approximate conservation law for the Teukolsky equation

The asymptotic analysis of the solutions arising from this scattering problem again makes crucial use of certain approximate conservation laws for the Teukolsky equations. Each fixed angular mode Ψ_ℓ^0 satisfies:

$$\partial_u \left(\left(\frac{1 - \frac{2M}{r}}{r^2} \right)^\ell \partial_v \left(\frac{r^2 \partial_v}{1 - \frac{2M}{r}} \right)^\ell (r^5 \Psi_\ell^0) + \dots \right) = M C_\ell r^{-2\ell-3} \left(\frac{r^2 \partial_v}{1 - \frac{2M}{r}} \right)^\ell (r^5 \Psi_\ell^0) + \dots, \quad (19)$$

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$$\partial_u (r^{-4} \partial_v (r^5 \Psi^0)) = M r^{-7} \cdot r^5 \Psi^0 .$$

ANALYSIS OF THE CORRESPONDING SOLUTION

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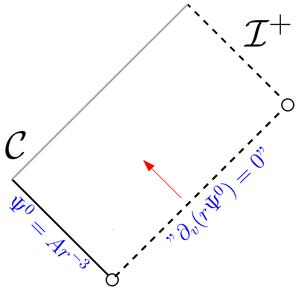

- Insert this into (20) and integrate from $u = -\infty$:

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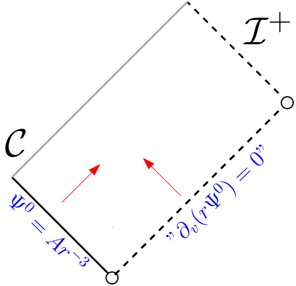

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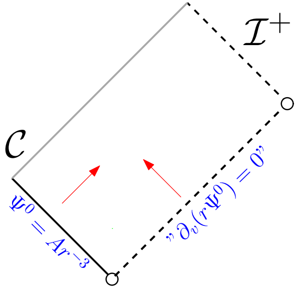

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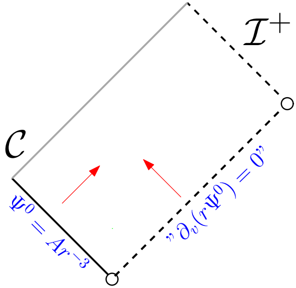

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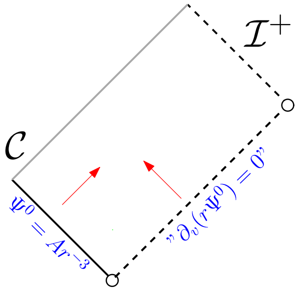

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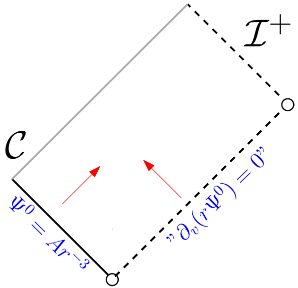

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In particular, we now have $r^5\Psi^0 = Au^2 + \dots$. Finally, inserting this back into (20) gives

$$r^{-4}\partial_v(r^5\Psi^0) = \int_{-\infty}^{u} \frac{MAu^2}{r^7} + \dots = \frac{MA}{4r^4} + \dots \implies r^5\Psi^0 = Au^2 + \frac{MAr}{4} + \dots \tag{21}$$

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- Finally, we remark that the limit $\lim_{r\to\infty, u=const} r^4\Psi^0$ is conserved along $\mathcal{I}^+$, and entirely determines the leading order late-time asymptotics.

Table of Contents

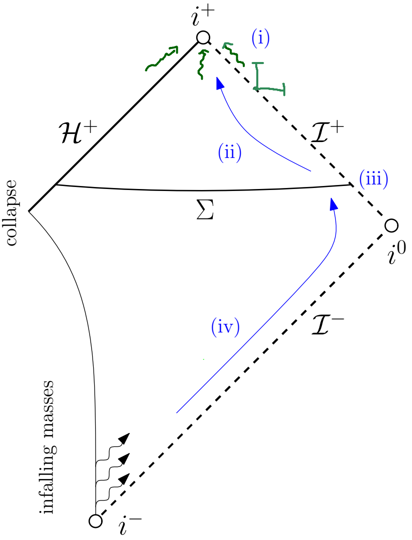
Four overarching questions

(i) In gravitational collapse, what is the (measurable?) asymptotic behaviour of gravitational radiation at late times?

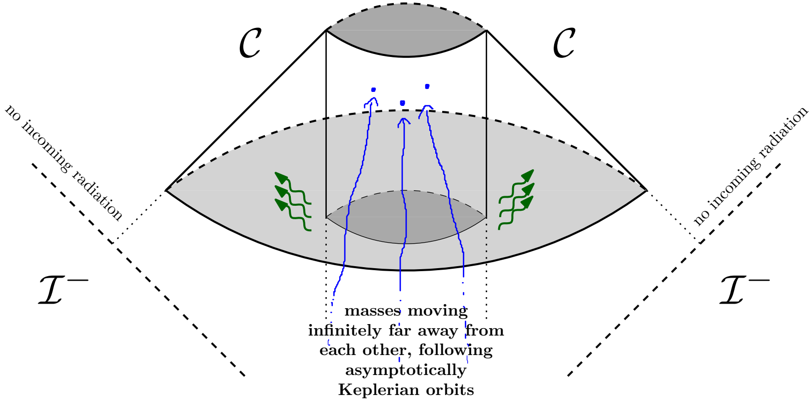
(ii) How is this asymptotic behaviour **along** $\mathcal{I}^+$ related to asymptotic behaviour **towards** $\mathcal{I}^+$?

(iii) What is the asymptotic behaviour of gravitational radiation towards $\mathcal{I}^+$? To what degree is peeling satisfied? Is $\mathcal{I}^+$ smooth in the sense of Penrose?

(iv) How is the asymptotic behaviour towards $\mathcal{I}^+$ related to the structure of gravitational radiation in the infinite past?



The schematic picture



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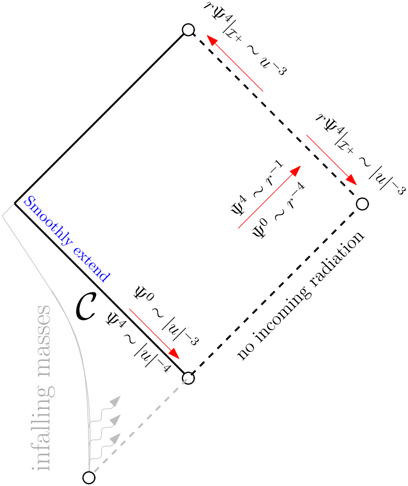

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Situation for gravitational perturbations

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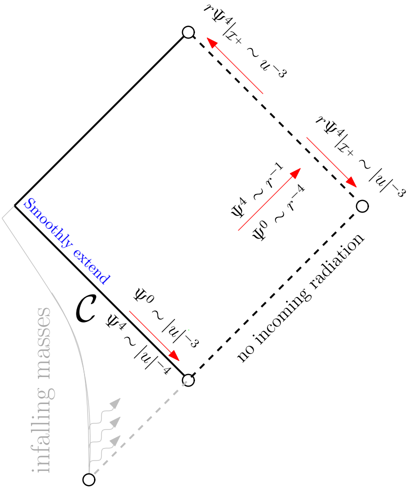

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- This failure of smoothness/peeling translates into something measurable at late times: $r\Psi^4|_{\mathcal{I}^+} \sim MAu^{-3} + \dots$
- To be contrasted with Price's law for compactly supported Cauchy data: $r\Psi^4|_{\mathcal{I}^+} = Cu^{-6} + \dots$

What we haven't talked about and what is to come

- We focussed only on gauge-invariant quantities Ψ^0 and Ψ^4. Rest of the system? Scattering construction? Questions of gauge? Can you Bondi normalise the solutions?

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Thank you so much for your attention :)