Hamiltonian Mechanics with external sources

Adrien Fiorucci

Technische Universität Wien — Institut für Theoretische Physik

Second Carroll Workshop @ UMONS

15th September 2022

Radiative phase spaces in General Relativity

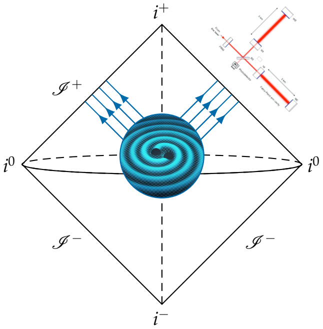
Gravitational waves (GW) : spacetime oscillations propagating at c.
Typical sources : periodic motions of massive bodies like binaries.

- Some historical facts :
  - **1918** – *Einstein* : linearized perturbations around flat space.
  - **1974** – *Hulse, Taylor* : angular momentum loss by GW emission in binary pulsar `PSR B1913+16`.
  - **2015** – *LIGO Collaboration* : direct detection of GW produced by black hole merger `GW150914`.

  $\rightarrow$ era of *gravitational astronomy* + ultimate tests for *G.R.*

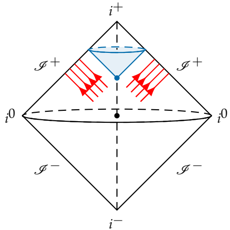
- Phenomenology $\in$ ***flux-balance laws*** (loss of energy/angular momentum) at *null infinity* $\mathscr{I}$ for the charges related to ***asymptotic symmetries***.

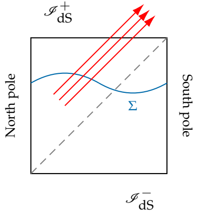
- Consequences :
  - Radiative solutions $\rightarrow$ ***leaky boundary conditions*** at $\mathscr{I}$;
  - Boundary $\mathscr{I}$ permeable $\Rightarrow$ non-equilibrium physics
    $\rightarrow$ ***non-conserved*** / ***non-integrable*** charges ;
  - Holography : GW $\sim$ ***sources*** for the dual quantum theory.

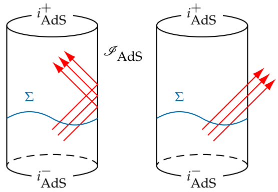


Leaky boundary conditions for all Λ's

- **$\Lambda = 0$** : asymptotically Minkowskian b.c. for $\Lambda = 0$ are *leaky* ! [Bondi-van der Burg-Metzner-Sachs 1962...]
  - ∞-dimensional asymptotic group: $\boxed{\text{BMS}_4 = SO(3,1) \ltimes \text{Supertranslations}} \geq$ Poincaré ;
  - Non-conservation of BMS charges ∋ "*Bondi mass-loss*".
- **$\Lambda > 0$** : essential to consider leaky b.c., otherwise, constrains on the Cauchy problem !
- **$\Lambda < 0$** :
  - Conservative b.c. are widely discussed (suitable for holography) ;
  - However, leaky b.c. are appealing ! [Fiorucci-Ruzziconi 2021]
    - → natural for GW physics, needed in recent discussions on black hole evaporation ("*islands*")...
    - → // to flat case ⇒ promising route towards $4d/3d$ flat holography with flat limit process $\Lambda \to 0$.







Well-known facts about asymptotically flat gravity

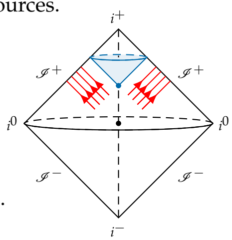
See Romain Ruzziconi's talk for details and notations

- Model for observation of GW emission from astrophysically remote localized sources.
  - Asymptotic flatness $\Rightarrow$ ***boundary structure*** at conformal boundary $\mathscr{I}$.

    Foliation $\boldsymbol{T} = \partial_u$ on $\mathscr{I}$ & metric $\mathring{q}_{AB}$ on $S_\infty \perp \boldsymbol{T}$

  - Induces a ***Carrollian structure*** on $\mathscr{I}$. [Geroch 1977]
  - *Dual theory:* QFT with conformal Carroll (BMS) symmetries $\xi = (f, Y^A)$

$$f_{[1,2]} = Y_1^A D_A f_2 + \frac{1}{2} f_1 D_A Y_2^A - (1 \leftrightarrow 2), \quad Y^A_{[1,2]} = Y_1^B D_B Y_2^A - (1 \leftrightarrow 2).$$



- In presence of radiation,
  - BMS charges are ***not conserved*** in time *e.g.* energy flux at $\mathscr{I}$ (*Bondi mass loss*) [Trautman 1958, Bondi-Metzner-Sachs 1962].
  - BMS charges are ***non-integrable*** [Barnich-Troessaert 2011].

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{M} = -\frac{1}{2}\oint_{S_\infty} \mathrm{d}^2x\, N^{AB} N_{AB} \leq 0, \quad \int_{\mathscr{I}} i_{\delta_T}\boldsymbol{\omega} = \delta\mathcal{M} + \frac{1}{2}\oint_{S_\infty} \mathrm{d}^2x\, N^{AB}\delta C_{AB}.$$

  - Action ***not stationary*** on-shell: $\delta S = \frac{1}{2}\int_{\mathscr{I}} \mathrm{d}^3x\, N^{AB}\delta C_{AB} \neq 0$ [Compère-Fiorucci-Ruzziconi 2019].

All reverse back to normal iff Bondi news $N_{AB} \equiv 0$ (no radiation). [units: $16\pi G = 1$]

Radiative spacetimes with cosmological constant

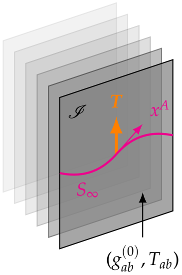
The Λ-BMS program [Compère-Fiorucci-Ruzziconi 2019]

- Using the gauge freedom at $\mathscr{I}$ (conformal boundary), isolate radiative DoF.
  - → Boundary structure: *boundary foliation* T and *fixed volume form* on $S_\infty \perp T$.
  - → Dynamical boundary metric $g^{(0)}_{AB}(t, x^C) \Rightarrow$ non-Dirichlet b.c.!
  - → Asymptotic symmetries form ***Λ-BMS algebroid***.

$$f_{[1,2]} = Y_1^A D_A f_2 + \frac{1}{2} f_1 D_A Y_2^A - \delta_{\xi_1} f_2 - (1 \leftrightarrow 2),$$

$$Y^A_{[1,2]} = Y_1^B D_B Y_2^A + \frac{3}{\Lambda} f_1 g^{AB}_{(0)} D_B f_2 - \delta_{\xi_1} Y_2^A - (1 \leftrightarrow 2).$$

✅ Reduces to BMS with Diff(S^2) super-Lorentz when $\Lambda \to 0$.



- With radiation in the bulk:
  - Action principle [de Haro-Solodukhin-Skenderis 2001] ***not stationary***!
  - Gravitational charges ***not conserved*** in time.

$$\delta S = -\frac{1}{2} \int_{\mathscr{I}} \mathrm{d}^3x\, T^{AB} \delta g^{(0)}_{AB} \neq 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}^{(\Lambda)} \propto \oint_{S_\infty} \mathrm{d}^2x\, T^{AB} \mathcal{L}_T g^{(0)}_{AB} \neq 0$$

  - → Non-global hyperbolicity if $\Lambda < 0 \Leftrightarrow$ dual CFT couples to an external system!
  - → Symplectic couple $(T^{AB}_{TF}, g^{(0)}_{AB})$ generalizes the *Bondi news* (N^{AB}, C_{AB}) in AF spacetimes.
  - → Flat limit $\Lambda \to 0$ gives AF radiative phase space.

New challenge: holography with sources

- In presence of radiation (*i.e.* leaky b.c.), the holographic theory $\forall\Lambda$ has *akward properties*:
  1. Noether charges are ***not conserved*** in time.
     - → Flux driven by external DoF obey no EOM ("*source*").
     - → Expect modifications in quantum Ward identities (see **Romain's talk**).
  2. Charges are ***non-integrable*** on the phase space.
     - → Definition of Hamiltonian requires prescription.
- Natural to consider ***field theories with external sources*** has holographic duals!
  - → **Example 1.** [Donnay-Fiorucci-Herfray-Ruzziconi 2022]

    $4d$ asymptotically flat gravity ⟷ *Flat space holography* ⟷ $3d$ sourced conformal Carrollian field theory

    Source $\sim$ Bondi news N_{AB}
  - → **Example 2.** [Fiorucci-Ruzziconi 2021] (also [Bakas-Skenderis 2014])

    $4d$ asymptotically AdS gravity with GW ⟷ *Non-equilibrium AdS/CFT* ⟷ $3d$ sourced conformal field theory

    Source $\sim g^{(0)}_{AB}$

Objectives of the talk

- Challenges to describe holographic theories with GW in the bulk:
  - → *"How to restore Noether symmetries in presence of external sources?"*
  - → *"How to understand non-conservation laws from generalized Noether principle?"*
  - → *"How to define Hamiltonian for each restored global symmetry?"*
  - → *"How to close a charge algebra under (possibly extended) Poisson bracket?"*
  - → *"Does this algebra represent the symmetry algebra?"*

✅ Easier to address these questions in *classical mechanics*. All the features are already present!

Classical mechanics with sources	—inspiration→	Field theories with sources	←holographic duality→	Gauge field theories with leaks
Today		Global symmetries		Asymptotic symmetries

In the future (Field theories with sources and Gauge field theories with leaks)

In this talk: work to be published, in collaboration with **Glenn Barnich** & **Romain Ruzziconi**. Building on [Troessaert 2015] (see also [Wieland 2020, Freidel 2021...]).

1 **Sourced Hamiltonian systems**.

- 1.1 Main definitions and examples.
- 1.2 Variational principle with sources.
- 1.3 Definition of generalized symmetries.
- 1.4 Non-conservation / non-integrability of the charges.

  Example. particle in $\mathbb{R}^d$ with an external force.

2 **Charge algebra for sourced systems**.

- 2.1 Review: the integrable case.
- 2.2 Non-integrable case: modified Poisson bracket.
  - 2.2.1 Cotangent Lie algebroid & Koszul bracket.
  - 2.2.2 Split maps on differential forms.
  - 2.2.3 Definition of the modified bracket & properties.
- 2.3 Representation theorem with sources.

  Example. particle in $\mathbb{R}^d$ with an external force.

Sourced Hamiltonian systems

- We consider a mechanical system Σ with Hamiltonian $\mathcal{H}$.

  We say that Σ is "sourced" if some external factor ("source") break Noether symmetries.

  A few examples:

  - Particle in $\mathbb{R}^d$ with an external force: $\mathcal{H} = \frac{1}{2m} p^i p_i - q^i f_i \Rightarrow m\ddot{q}^i = f^i$.

Hypotheses on f_i	0	Time-indep.	Central	Both	General
Noether sym.	Galileo	∂_t $(\dot{E} = 0)$	∂_φ $(\dot{J} = 0)$	$\partial_t, \partial_\varphi$ $(\dot{E} = 0 = \dot{J})$	$\varnothing$

  - Damping: $\mathcal{H} = \frac{p^2}{2m} e^{-2\gamma t} + V(q) e^{2\gamma t} \Rightarrow \ddot{q} + 2m\gamma\dot{q} + V'(q) = 0$. $\rightarrow$ motions in viscous media.
  - Forced harmonic oscillator: $\mathcal{H} = \omega a^* a - j(t) a^* - j^*(t) a \Rightarrow \dot{a} = -i(\omega a - j)$. $\rightarrow$ basics for QFT.
  - Systems in contact with thermal bath (*ex.* your coffie cup), *etc.*

- **Fundamental properties**:

  - Sources are ***allowed to transform*** in order to
    (i) "restore" symmetries & *(ii)* have symmetries of EOM.
  - The statement "*the source is on/off*" is ***physical*** (source not removable by frame choice).

Variational principle and presymplectic structure

- $(\mathscr{P}, \boldsymbol{\omega})$: phase space of Σ. We assume $\dim(\mathscr{P}) = 2k + \ell$, $k, \ell \in \mathbb{N}$.

  Notations: $\delta \overset{\text{not}}{=}$ exterior derivative on $\mathscr{P}$ ("variation"). Local coordinates are

$\{z^A\}_{A=1}^{2k}$	***Dynamical fields***	ω_{AB} is non-degenerate	$\delta z^A \neq 0$	Obey Hamiltonian EOM
$\{\sigma^m\}_{m=1}^{\ell}$	***External sources***	Kernel directions $\boldsymbol{\omega}$	$\delta \sigma^m \neq 0$	No associated EOM

- Sourced Hamiltonian action:

$$S[z|\sigma] = \int_{t_1}^{t_2} \mathrm{d}t\, L(z|\sigma) = \int_{t_1}^{t_2} \mathrm{d}t \left[\theta_A(z)\dot{z}^A - \mathcal{H}(z|\sigma)\right]$$

  where $\boldsymbol{\theta}^{(H)} = \theta_A \delta z^A$ is the *symplectic potential* and $\boldsymbol{\omega}^{(H)} = \delta \boldsymbol{\theta}^{(H)}$ defines the *symplectic form*.

- After variation,

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \mathrm{d}t \left[\left(\omega_{AB}\dot{z}^B - \frac{\partial \mathcal{H}}{\partial z^A}\right)\delta z^A + \frac{\mathrm{d}}{\mathrm{d}t}\left(\theta_A \delta z^A\right) - \frac{\partial \mathcal{H}}{\partial \sigma^m}\delta\sigma^m\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left[\Big(\text{Hamiltonian EOM}\Big)\delta z^A + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\text{boundary term}\Big) + \text{external flux } \boldsymbol{\Theta}^{(H)}\right].\end{aligned}$$

Variational principle and presymplectic structure

- For a given external source, the Hamiltonian EOM read as

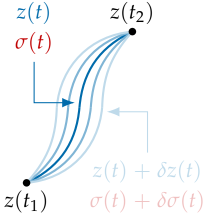
$$\omega_{AB}\dot{z}^B - \frac{\partial \mathcal{H}}{\partial z^A} = 0 \quad \Leftrightarrow \quad i_{X_t}\boldsymbol{\omega}^{(H)} + \delta\mathcal{H}\Big|_{\sigma} = 0$$

where the vector field $X_t \equiv \dot{z}^A \frac{\partial}{\partial z^A} + \dot{\sigma}^m \frac{\partial}{\partial \sigma^m}$ generates *time evolution* on $\mathscr{P}$.

- Evaluating δS on-shell for the standard time b.c.

$$\theta_A \delta z^A\Big|_{t_1,t_2} = 0 \quad \Rightarrow \quad \boxed{\delta S = -\int_{t_1}^{t_2} \mathrm{d}t\, \frac{\partial \mathcal{H}}{\partial \sigma^m}\delta\sigma^m = \int_{t_1}^{t_2} \mathrm{d}t\, \boldsymbol{\Theta}^{(H)}.}$$

$\rightarrow$ Stationary action in the absence of sources *or* if sources are fixed.
$\rightarrow$ With sources, action generally *not* stationary on solutions!



- ***Generalized definition of "well-defined" variational principle***

The variational principle is "well-defined" if the action is stationary on solutions of the Hamiltonian EOM, *up to terms involving variations of the sources* collectively called *external flux*.

Example. *Point particle with an external force.* [Troessaert 2015]

- Consider a particle of inertial mass m in $\mathbb{R}^d = \text{Span}\{e_i\}$ subject to a total external force $f(t) = f^i(t)e_i$. In Darboux coordinates (q^i, p_j), the action integral is

$$S[q^i, p_j|f_i] = \int_{t_1}^{t_2} \mathrm{d}t \left[p_i \dot{q}^i - \mathcal{H}(q^i, p_j|f_i) \right], \qquad \mathcal{H}(q^i, p_j|f_i) = \frac{1}{2m} p_i p^i - q^i f_i.$$

- A variation gives

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left[\left(f_i - \dot{p}^i \right) \delta q^i + \left(\dot{q}^i - \tfrac{1}{m} p^i \right) \delta p_i + \frac{\mathrm{d}}{\mathrm{d}t} (p_i \delta q^i) + q^i \delta f_i \right].$$

  - Equations of motion: $\dot{q}^i = \frac{1}{m} p^i$, $\dot{p}_i = f_i \Rightarrow m\ddot{q}^i = f^i$ (*Newton's second law*);
  - Boundary term $\boldsymbol{\theta}^{(H)} = p_i \delta q^i$ vanishes for "standard" b.c. *i.e.* $\delta q^i(t_1) = 0 = \delta q^i(t_2)$.

    $\to$ *Remark.* The symplectic form is $\boldsymbol{\omega}^{(H)} = \delta \boldsymbol{\theta}^{(H)} = \delta p_i \wedge \delta q^i$ as usual.
  - The *external flux* $\boldsymbol{\Theta}^{(H)} = q^i \delta f_i$ prevents stationarity of the action on-shell.

    $\to$ A variation of the source maps a solution on another!

- $X = X^A_{(z)} \frac{\partial}{\partial z^A} + X^m_{(\sigma)} \frac{\partial}{\partial \sigma^m} \in \mathfrak{X}(\mathscr{P})$ generates a *generalized variational symmetry* of the theory if

  1. The Lagrangian transforms as $\boxed{\delta_X L(z|\sigma) = \frac{\mathrm{d}}{\mathrm{d}t} B_X(z|\sigma) + V_X(z|\sigma);}$
  2. $V_X(z|\sigma)$ and $X^m_{(\sigma)}(z|\sigma)$ vanish in the absence of sources, *i.e.* when $\sigma^m = 0$, $m = 1, \ldots, \ell$;
  3. X is a symmetry of the sourced equations of motion, *i.e.* $\delta_X(\text{EOM}) = 0$ on-shell.

Remarks:

- Enhancement of the hypotheses of [Troessaert 2015] where $X^m_{(\sigma)}$, V_X were independent on z.
- Hypothesis 3 fixes the dependence in z^A of V_X!

$$\delta_X \frac{\delta L}{\delta z^A} = \frac{\delta}{\delta z^A} \delta_X L - \frac{\partial X^B_{(z)}}{\partial z^A} \frac{\delta L}{\delta z^B} - \frac{\partial X^m_{(\sigma)}}{\partial z^A} \frac{\delta L}{\delta \sigma^m} \quad \Rightarrow \quad \frac{\partial V_X}{\partial z^A} = (\mathcal{L}_X \omega_{AC})\, \omega^{CB} \frac{\delta L}{\delta z^B} + \frac{\partial X^m_{(\sigma)}}{\partial z^A} \frac{\delta L}{\delta \sigma^m}.$$

- ***Important properties:***

  - If X_1, X_2 generate two generalized symmetries $\Rightarrow [\delta_{X_1}, \delta_{X_2}] F \equiv \delta_{[X_1, X_2]} F, \forall F \in \mathcal{F}(\mathscr{P})$.
  - Any Noether symmetry of the unsourced system is promoted to a generalized symmetry.

Example. *Point particle with an external force.* [Troessaert 2015]

- When $f_i = 0$, the dynamics is invariant under *Galilean transformations* $\delta_X q^i = X^i(q; a, b, v, \varpi)$

$$\delta_X q^i \equiv a^i + b\frac{p^i}{m} + tv^i - \varpi^i{}_j q^j \quad \text{where} \quad \begin{cases} a^i : \text{ spatial translations, } b : \text{ time translation,} \\ v^i : \text{ change of inertial frame, } \varpi^i{}_j : \text{ spatial rotations.} \end{cases}$$

- When $f_i \neq 0$, demanding that X is a symmetry of the EOM implies

$$\begin{cases} \dot{q}^i = \frac{p^i}{m} & \Rightarrow \quad \delta_X p_i = b f_i + m v_i + p_j \varpi^j{}_i, \\ \dot{p}_i = f_i & \Rightarrow \quad \delta_X f_i = b\dot{f}_i + f_j \varpi^j{}_i. \end{cases}$$

→ The presence of $f_i \neq 0$ *breaks* the Galilean invariance (*e.g.* $q^i f_i \in \mathcal{H}$ spoils translation invariance).

- X is a *generalized Noether symmetry* because

$$\delta_X L_H = \frac{\mathrm{d}}{\mathrm{d}t} B_X(q,p|f) + V_X(f) \text{ with } \begin{cases} B_X(q,p|f) \equiv b\left(\frac{1}{m}p^i p_i - \mathcal{H}\right) + v^i p_i, \\ V_X(f) \equiv (a^i + tv^i) f_i. \quad \leftarrow \textit{not surprising!} \end{cases}.$$

→ $\delta_X f_i = 0 = V_X(f)$ when $f_i \equiv 0$. Here V_X does not depend on fields (not true in general!).

- Using $\delta_X L_H = \frac{\mathrm{d}}{\mathrm{d}t} B_X + V_X$ and developing the l.h.s. $\delta_X(\theta_A \dot{z}^A - \mathcal{H})$ as

$$X^B_{(z)} \frac{\partial \theta_A}{\partial z^B} \dot{z}^A + \theta_A \dot{X}^A_{(z)} - X^A_{(z)} \frac{\partial \mathcal{H}}{\partial z^A} - X^m_{(\sigma)} \frac{\partial \mathcal{H}}{\partial \sigma^m} = \left(\omega_{AB} \dot{z}^B - \frac{\partial \mathcal{H}}{\partial z^A}\right) X^A_{(z)} + \frac{\mathrm{d}}{\mathrm{d}t}\left(i_X \boldsymbol{\theta}^{(H)}\right) + i_X \boldsymbol{\Theta}^{(H)},$$

one obtains $\frac{\mathrm{d}}{\mathrm{d}t}(B_X - i_X \boldsymbol{\theta}^{(H)}) = (\omega_{AB} \dot{z}^B - \frac{\partial \mathcal{H}}{\partial z^A}) X^A_{(z)} + i_X \boldsymbol{\Theta}^{(H)} - V_X(z|\sigma)$.

Theorem. The *Noether charge* $H_X \equiv B_X - i_X \boldsymbol{\theta}^{(H)}$ associated with a generalized symmetry X satisfies the following on-shell *non-conservation law*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} H_X(z|\sigma) = i_X \boldsymbol{\Theta}^{(H)} - V_X(z|\sigma).}$$

→ Conservation iff $\sigma^m \equiv 0$. Otherwise, H_X is *not* conserved on-shell!

Example. *Point particle with an external force.*

- Galilean charge found to be $H_X = m v_i q^i - (a^i + t v^i - \omega^i{}_j q^j) p_i - b\,\mathcal{H}$.
- A straightforward computation gives $\frac{\mathrm{d}}{\mathrm{d}t} H_X = \underbrace{q^i (b \dot{f}_i + \omega_i{}^j f_j)}_{i_X \boldsymbol{\Theta}^{(H)}} - \underbrace{f_i (a^i + t v^i)}_{V_X}$.

Momentum map

How to deduce the charge from the symplectic structure?

- In absence of sources, considering the conservation law off-shell gives

$$\frac{\mathrm{d}}{\mathrm{d}t}H_X = \frac{\partial H_X}{\partial t} + \frac{\partial H_X}{\partial z^A}\dot{z}^A = X^A \omega_{AB}\dot{z}^B - X^A \frac{\partial \mathcal{H}}{\partial z^A} \quad \Rightarrow \quad \boxed{i_X \boldsymbol{\omega}^{(H)} = \delta H_X, \quad \mathcal{L}_X \mathcal{H} + \frac{\partial H_X}{\partial t} = 0.}$$

  → X is Noether ⇔ X is Hamiltonian, *i.e.* $\exists H_X \in \mathcal{F}(\mathscr{P}) : i_X \boldsymbol{\omega}^{(H)} = \delta H_X$ (“*the charge is integrable*”).
  → This defines the *momentum map* $X \mapsto H_X = \int_{\mathscr{C}} i_X \boldsymbol{\omega}^{(H)}$ (up to a constant).
  → *Corollary*: X is a symplectomorphism, *i.e.* $\mathcal{L}_X \boldsymbol{\omega}^{(H)} = i_X \delta \boldsymbol{\omega}^{(H)} + \delta i_X \boldsymbol{\omega}^{(H)} = 0$.

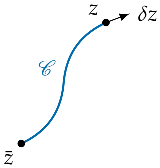
- In presence of sources, the non-conservation law implies

$$\frac{\mathrm{d}}{\mathrm{d}t}H_X = \frac{\partial H_X}{\partial t} + \frac{\partial H_X}{\partial z^A}\dot{z}^A + \frac{\partial H_X}{\partial \sigma^m}\dot{\sigma}^m = X^A \omega_{AB}\dot{z}^B - X^A \frac{\partial \mathcal{H}}{\partial z^A} + i_X \boldsymbol{\Theta}^{(H)} - V_X.$$

Theorem. The vector X generates a generalized symmetry iff it obeys the following requirements:

$$\boxed{\mathcal{L}_X \boldsymbol{\omega}^{(H)}\Big|_\sigma = 0 \Leftrightarrow i_X \boldsymbol{\omega}^{(H)} = \delta H_X\Big|_\sigma, \quad \mathcal{L}_X \mathcal{H} + \frac{\partial H_X}{\partial t} = -\frac{\partial H_X}{\partial \sigma^m}\dot{\sigma}^m - V_X.}$$

Non-integrability



- In presence of sources, momentum map $X \mapsto \delta H_X$ only for fixed sources $\delta\sigma^m = 0$.

$$\boxed{i_X \boldsymbol{\omega}^{(H)} = \delta H_X + \Xi_X.} \quad \text{where } H_X \in \mathcal{F}(\mathscr{P}),\ \Xi_X \in \Omega^1(\mathscr{P}).$$

  → Dynamical sources are responsible for the ***non-integrability*** of the charges!
  → Momentum map $X \mapsto \int_{\mathscr{C}} i_X \boldsymbol{\omega}^{(H)}$ depends on the path $\mathscr{C}$ on $\mathscr{P}$!
  → *Consequence*: X is *no longer* a symplectomorphism!

$$\mathcal{L}_X \boldsymbol{\theta}^{(H)} = i_X \boldsymbol{\omega}^{(H)} + \delta i_X \boldsymbol{\theta}^{(H)} = \delta H_X + \Xi_X + \delta i_X \boldsymbol{\theta}^{(H)} = \delta B_X + \Xi_X \quad \Rightarrow \quad \mathcal{L}_X \boldsymbol{\omega}^{(H)} = \delta \Xi_X \neq 0.$$

- Using $i_X \boldsymbol{\omega}^{(H)} = \delta H_X \Big|_\sigma$, we have

$$\delta H_X = \frac{\partial H_X}{\partial z^A}\Big|_\sigma \delta z^A + \frac{\partial H_X}{\partial \sigma^m}\Big|_z \delta\sigma^m \quad \Rightarrow \quad \boxed{\Xi_X \equiv -\delta H_X\Big|_z = -\frac{\partial H_X}{\partial \sigma^m}\Big|_z \delta\sigma^m.}$$

  → Prescription to fix the ambiguity between integrable and non-integrable parts in the charge!

Non-integrability

Example. *Point particle with an external force.*

- Galilean charge is $H_X = m v_i q^i - (a^i + t v^i - \varpi^i{}_j q^j) p_i - b\,\mathcal{H}$. A direct computation yields

$$i_X \omega = \delta H_X - b\, q^i \delta f_i \quad \Rightarrow \quad \boxed{\Xi_X(q,p|\delta f) = -b\, q^i \delta f_i = -\frac{\partial H_X}{\partial f_i} \delta f_i.}$$

→ Non-integrability if and only if f_i is allowed to vary!
→ Sources only from $\mathcal{H}$ in $H_X \Rightarrow$ only time translation has non-integrable charge:

$$\delta\!\!\!/ H_b = -b \left(\delta\mathcal{H} + q^i \delta f_i \right).$$

If $f_i \equiv 0$, we recover (minus) the *total mechanical energy* of the particle for $b = 1$.

- Non-conservation of the energy controlled as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p^i p_i}{2m} + q^i f_i \right) = q^i \dot{f}_i = -i_{X_t} \mathbf{\Theta}^{(H)} = -i_{X_t} \mathbf{\Xi}_X.$$

→ Energy conserved if $\delta_{X_t} f_i = 0$, integrable if $\delta f_i = 0$ (stronger condition).

Key equations (summary)

$X_t \equiv \dot{z}^A \partial_A + \dot{\sigma}^m \partial_m.$

$$\frac{\mathrm{d}}{\mathrm{d}t} H_X \overset{(1)}{=} i_X \boldsymbol{\Theta}^{(H)} - V_X, \quad i_X \boldsymbol{\omega}^{(H)} \overset{(2)}{=} \delta H_X \Big|_\sigma, \quad \mathcal{L}_X \mathcal{H} + \frac{\partial H_X}{\partial t} \overset{(3)}{=} i_{X_t} \boldsymbol{\Xi}_X - V_X.$$

1 *Non-integrability entangled with non-conservation.*

$$\frac{\mathrm{d}}{\mathrm{d}t} H_X = \frac{\partial H_X}{\partial t} + i_{X_t} \delta H_X \overset{(2)}{=} \frac{\partial H_X}{\partial t} + i_{X_t} i_X \boldsymbol{\omega} - i_{X_t} \boldsymbol{\Xi}_X \overset{(3)}{=} i_X \boldsymbol{\Theta}^{(H)} - V_X \equiv (1).$$

→ Usually, $i_{X_t} i_X \boldsymbol{\omega}$ is the Poisson bracket ⇒ (non-)conservation controlled by charge algebra!

2 *The Hamiltonian flow X_t is always a generalized symmetry.*

(*i*) The EOM read as $i_{X_t} \boldsymbol{\omega}^{(H)} + \delta \mathcal{H}|_\sigma = 0 \Rightarrow H_{X_t} = -\mathcal{H}$.

(*ii*) On-shell, $\mathcal{L}_{X_t}(\mathcal{H}) = \frac{\partial \mathcal{H}}{\partial z^A} \dot{z}^A + \frac{\partial \mathcal{H}}{\partial \sigma^m} \dot{\sigma}^m = \frac{\partial \mathcal{H}}{\partial \sigma^m} \dot{\sigma}^m = -i_{X_t} \boldsymbol{\Theta}^{(H)}$.

Hence $\boldsymbol{\Xi}_{X_t} \equiv -\boldsymbol{\Theta}^{(H)}$ and $V_{X_t} \equiv 0$ from (3).

→ $\boldsymbol{\Theta}^{(H)}$ controls non-conservation of energy ("*external flux*");

→ $\boldsymbol{\Theta}^{(H)}$ induces non-integrability of energy.

Charge algebra for sourced systems

What we did so far...

- Modification of the variational principle in presence of sources.
- Restoration of broken Noether symmetries ⇒ definition of *generalized symmetries* + non-conservation and non-integrability of the charges.

What we do in the second half of the talk...

- Define closing charge algebras for generalized symmetries.
  - *Main obstruction*: non-integrability.
    - → We have to live with it in presence of sources!
  - *Solution*: define a modified Poisson bracket taking this into account!
    - → First (empirical) attempt in [Barnich-Troessaert 2011] (context of BMS symmetries).
    - → Further refinements in the context of the *covariant phase space formalism* by [Troessaert 2015, Wieland 2020, Freidel-Oliveri-Pranzetti-Speziale 2021, Freidel 2021, Fiorucci 2021...].
    - → Our constructive definition reproduces [Freidel 2021] for Einstein gravity.

Useful structures

- *Assumptions:* $(\mathscr{P}, \boldsymbol{\omega})$ of dimension $2k+\ell$. $\boldsymbol{\omega}$ is of rank $2k$ with a ℓ-dim. kernel.
  $\rightarrow$ ω_{AB} is invertible, and $\omega^{AB}\omega_{BC} = \delta_A{}^C$.
- *Poisson bivector:* $\pi = \pi^{AB}\partial_A \wedge \partial_B$ on $\mathscr{P}$ where $\pi^{AB} = -\omega^{AB}$.
  $\rightarrow$ Coordinate-free definition of ***Poisson bracket***:

$$\{f,g\} \equiv \pi(\delta f, \delta g) = \pi^{AB}\partial_A f \partial_B g \text{ for } f,g \in \mathscr{F}(\mathscr{P}).$$

- *Symplectic "musical isomorphism"*:

$$\pi^\sharp : T^\star\mathscr{P} \to T\mathscr{P} \text{ such that } i_{\pi^\sharp \boldsymbol{\beta}}\boldsymbol{\alpha} = \pi(\boldsymbol{\alpha}, \boldsymbol{\beta}), \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in T^\star\mathscr{P},$$
$$\omega^\flat : T\mathscr{P} \to T^\star\mathscr{P} : X \mapsto i_X\boldsymbol{\omega} \quad \Rightarrow \quad \omega^\flat = (\pi^\sharp)^{-1}.$$

- For two *Noether symmetries* X_1, X_2, *i.e.* ***Hamiltonian vectors*** such that $\omega^\flat X_a = \delta H_{X_a}$ $(a = 1,2)$,

$$\{H_{X_1}, H_{X_2}\} \equiv \pi(\delta H_{X_1}, \delta H_{X_2}) = \boldsymbol{\omega}(X_1, X_2) = i_{X_2} i_{X_1}\boldsymbol{\omega}.$$

Well-known fact: $[X_1, X_2]$ is also Hamiltonian because

$$i_{[X_1,X_2]}\boldsymbol{\omega} = [\mathcal{L}_{X_1}, i_{X_2}]\boldsymbol{\omega} = \mathcal{L}_{X_1} i_{X_2}\boldsymbol{\omega} - i_{X_2}\cancel{\mathcal{L}_{X_1}\boldsymbol{\omega}} = i_{X_1}\delta\cancel{i_{X_2}\boldsymbol{\omega}} + \delta\left(i_{X_1} i_{X_2}\boldsymbol{\omega}\right) \equiv \delta H_{[X_1,X_2]}.$$

- Coupling both pieces of information leads to

Representation theorem. The Hamiltonian functions H_X represent the vector algebra under the Poisson bracket up to central extensions, *i.e.* for all pairs X_1, X_2 of Hamiltonian vector fields,

$$\boxed{\{H_{X_1}, H_{X_2}\} = -H_{[X_1,X_2]} + K_{X_1,X_2}.}$$

The constant $K_{X_1,X_2} \in \mathbb{R}$ is a Lie algebra 2-cocycle⋆ over $T\mathscr{P}$.

- *More generally:* for integrable forms $\boldsymbol{\alpha} = \delta f_\alpha$, $\boldsymbol{\beta} = \delta f_\beta$, π induces the Lie bracket $[\boldsymbol{\alpha}, \boldsymbol{\beta}] \equiv \delta\{f_\alpha, f_\beta\}$.
  - $\pi^\sharp$ is a *Lie algebra isomorphism* because

$$\pi^\sharp([\boldsymbol{\alpha},\boldsymbol{\beta}]) = \pi^\sharp\delta\left(\pi(\boldsymbol{\alpha},\boldsymbol{\beta})\right) = \pi^\sharp\delta i_{\pi^\sharp\boldsymbol{\beta}}\boldsymbol{\alpha} = \pi^\sharp\mathcal{L}_{\pi^\sharp\boldsymbol{\beta}}\boldsymbol{\alpha} = \mathcal{L}_{\pi^\sharp\boldsymbol{\beta}}\pi^\sharp\boldsymbol{\alpha} = -[\pi^\sharp\boldsymbol{\alpha}, \pi^\sharp\boldsymbol{\beta}].$$

    → This forms an embryonal ***Lie algebroid structure*** over $\mathscr{P}$ where $\pi^\sharp$ plays the rôle of anchor.
    → **Good property!** Links symmetry algebra with local charge algebra if $\boldsymbol{\alpha} = \omega^\flat X_1$, $\boldsymbol{\beta} = \omega^\flat X_2$.
  - For this bracket, $f_{[\alpha,\beta]} = \{f_\alpha, f_\beta\} + K_{\alpha,\beta}$ is *immediate* by construction ⇒ *representation theorem*!

⋆ $K_{X_1,X_2} = -K_{X_2,X_1}$, $K_{[X_1,X_2],X_3} + \text{cyclic}(1,2,3) = 0$.

Cotangent Lie algebroid & Koszul bracket

- A natural generalization brings the following theorem:

Theorem. The ***cotangent Lie algebroid*** of the Poisson manifold $(\mathscr{M},\pi)$, defined as the Lie algebroid structure on $T^\star\mathscr{P}$ with $\pi^\sharp$ as anchor map and the ***Koszul bracket*** [Koszul 1985]

$$[\boldsymbol{\alpha},\boldsymbol{\beta}] = \mathcal{L}_{\pi^\sharp\boldsymbol{\beta}}\boldsymbol{\alpha} - \mathcal{L}_{\pi^\sharp\boldsymbol{\alpha}}\boldsymbol{\beta} - \delta\left(\pi(\boldsymbol{\alpha},\boldsymbol{\beta})\right)$$

as Lie bracket on sections $\Gamma(T^\star\mathscr{P})$, is the unique one with the property that the anchor maps δg to X_g such that $X_g[f] = \{f,g\}$ and $[\delta f,\delta g] = \delta\{f,g\}$.

- ☑ Lie algebroid structure to map vector algebra on one-form algebra;

$$\pi^\sharp([\omega^\flat X_1,\omega^\flat X_2]) = -[X_1,X_2] \Rightarrow \omega^\flat[X_1,X_2] = -[\omega^\flat X_1,\omega^\flat X_2] \Rightarrow \text{Charge algebra.}$$

- ☑ Reduces to $\delta\{f_\alpha,f_\beta\}$ when $\boldsymbol{\alpha},\boldsymbol{\beta}$ integrable.

- For generalized symmetry X, $\omega^\flat X = \delta H_X + \Xi_X[\delta\sigma]$.
  - → Representation property fails because X is *no longer* Hamiltonian!
  - → **Goal:** extract a *Poisson bracket* from the Koszul bracket to extend the ***representation theorem*** for non-integrable charges!

Split maps of one-forms

- **Intermediate task:** formalize the notion of split between integrable and non-integrable pieces.
- For each one-form $\boldsymbol{\alpha}$, it is always possible to write $\boldsymbol{\alpha} = \delta f_{\boldsymbol{\alpha}} + \Xi_{\boldsymbol{\alpha}}$. The writing is ambiguous and we have the following equivalence relation

$$(f_{\boldsymbol{\alpha}}, \Xi_{\boldsymbol{\alpha}}) \sim_\rho (f'_{\boldsymbol{\alpha}}, \Xi'_{\boldsymbol{\alpha}}) \Leftrightarrow \exists N_{\boldsymbol{\alpha}} \in \mathscr{F}(\mathscr{P}) : \begin{cases} f'_{\boldsymbol{\alpha}} = f_{\boldsymbol{\alpha}} - N_{\boldsymbol{\alpha}}, \\ \Xi'_{\boldsymbol{\alpha}} = \Xi_{\boldsymbol{\alpha}} + \delta N_{\boldsymbol{\alpha}}. \end{cases}$$

The map $S : T^\star\mathscr{P} \to (\mathscr{F}(\mathscr{P}) \times T^\star\mathscr{P}) \big/ \rho : \boldsymbol{\alpha} \mapsto [(f_{\boldsymbol{\alpha}}, \Xi_{\boldsymbol{\alpha}})]$ is an *isomorphism*.

- Providing a *prescription* to isolate a particular integrable part in $\boldsymbol{\alpha}$ gives rise to the ***split map***

$$\boxed{s : T^\star\mathscr{P} \to \mathscr{F}(\mathscr{P}) \times T^\star\mathscr{P} : \boldsymbol{\alpha} \mapsto (f_{\boldsymbol{\alpha}}, \Xi_{\boldsymbol{\alpha}}) .}$$

  - Unlike S, a split map is *not* necessarily linear (*i.e.* $N_{\boldsymbol{\alpha}}$ depends linearly on $\boldsymbol{\alpha}$).
  - A linear split map $\bar{s}$ such that $\bar{s}(\delta g) = (g + K, \mathbf{0})$ is called *fundamental*.
    $\to$ **Proposition.** $\bar{s}(i_X \omega^{(H)}) = (H_X, \Xi_X)$ defines a fundamental split map!

Effect of split maps on the Koszul bracket

- Let us pick a linear split map $s(\boldsymbol{\alpha}) = (f_\alpha, \Xi_\alpha)$, $s(\boldsymbol{\beta}) = (f_\beta, \Xi_\beta)$. We want to determine the couple

$$\Big(f_{[\alpha,\beta]}, \Xi_{[\alpha,\beta]}\Big) \text{ such that } [\boldsymbol{\alpha},\boldsymbol{\beta}] = \delta f_{[\alpha,\beta]} + \Xi_{[\alpha,\beta]} \text{ and } s([\boldsymbol{\alpha},\boldsymbol{\beta}]) \equiv \Big(f_{[\alpha,\beta]}, \Xi_{[\alpha,\beta]}\Big)\,.$$

- Let us perform a simple computation:

$$[\boldsymbol{\alpha},\boldsymbol{\beta}] = \delta\Big(\pi(\delta f_\alpha, \delta f_\beta) - \pi(\Xi_\alpha, \Xi_\beta)\Big) + \mathcal{L}_{\pi^\sharp\beta}\Xi_\alpha - \mathcal{L}_{\pi^\sharp\alpha}\Xi_\beta,$$

which reveals the ***Barnich-Troessaert bracket*** [Barnich-Troessaert 2011]

$$\boxed{\{f_\alpha, f_\beta\}^s_{BT} = i_{\pi^\sharp\beta}\delta f_\alpha + i_{\pi^\sharp\alpha}\Xi_\beta = \pi(\delta f_\alpha, \delta f_\beta) - \pi(\Xi_\alpha, \Xi_\beta).}$$

✅ Manifestly antisymmetric under $\boldsymbol{\alpha} \leftrightarrow \boldsymbol{\beta}$ and reduces to the Poisson bracket $\pi(\delta f_\alpha, \delta f_\beta)$ if (i) one argument is *exact* & (ii) $s = \bar{s}$ is a *fundamental* split map.

Barnich-Troessaert bracket

$$\boxed{\begin{aligned}[\boldsymbol{\alpha},\boldsymbol{\beta}] &= \delta\{f_{\boldsymbol{\alpha}},f_{\boldsymbol{\beta}}\}^s_{BT} + \mathcal{L}_{\pi^\sharp\boldsymbol{\beta}}\Xi_{\boldsymbol{\alpha}} - \mathcal{L}_{\pi^\sharp\boldsymbol{\alpha}}\Xi_{\boldsymbol{\beta}},\\ \{f_{\boldsymbol{\alpha}},f_{\boldsymbol{\beta}}\}^s_{BT} &= i_{\pi^\sharp\boldsymbol{\beta}}\delta f_{\boldsymbol{\alpha}} + i_{\pi^\sharp\boldsymbol{\alpha}}\Xi_{\boldsymbol{\beta}} = \pi(\delta f_{\boldsymbol{\alpha}},\delta f_{\boldsymbol{\beta}}) - \pi(\Xi_{\boldsymbol{\alpha}},\Xi_{\boldsymbol{\beta}}).\end{aligned}}$$

- First derived for BMS charge algebra in 4d asymptotically flat spacetimes. [Barnich-Troessaert 2011]
- In this context, the BMS charge algebra closes up to a *field-dependent* 2-cocycle.
  - → Raised some debate in the "community": physical interpretation, quantization,...
  - → BT bracket is Poisson only if supplementary relations hold for the cocycle.
  - → General proof exists (with general form of the 2-cocycle)! [Fiorucci 2021]

$$\{f_{\boldsymbol{\alpha}},f_{\boldsymbol{\beta}}\}^s_{BT} = f_{[\boldsymbol{\alpha},\boldsymbol{\beta}]} + \underbrace{\int_{\mathscr{C}}\left(\mathcal{L}_{\pi^\sharp\boldsymbol{\alpha}}\Xi_{\boldsymbol{\beta}} - \mathcal{L}_{\pi^\sharp\boldsymbol{\beta}}\Xi_{\boldsymbol{\alpha}} + \Xi_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\right)}_{\text{field-dependent 2-cocycle}} + \text{central charge.}$$

- Unsatisfactory for our construction, because if we set $f_{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = p_1\circ s([\boldsymbol{\alpha},\boldsymbol{\beta}]) \stackrel{!}{=} \{f_{\boldsymbol{\alpha}},f_{\boldsymbol{\beta}}\}^s_{BT}$, there is *no reason* that in general $\Xi_{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = p_2\circ s([\boldsymbol{\alpha},\boldsymbol{\beta}]) \equiv \mathcal{L}_{\pi^\sharp\boldsymbol{\beta}}\Xi_{\boldsymbol{\alpha}} - \mathcal{L}_{\pi^\sharp\boldsymbol{\alpha}}\Xi_{\boldsymbol{\beta}}$ for the *same* split map s.

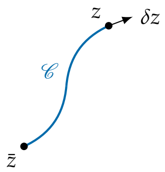
Modified Poisson bracket

- Indeed, recall that given a split map s, the objects $f_\eta = p_1 \circ s(\boldsymbol{\eta})$ and $\Xi_\eta = p_2 \circ s(\boldsymbol{\eta})$ are intended as *unambiguous functions* of $\boldsymbol{\eta} \in T^\star \mathscr{P}$. In particular, from $s([\boldsymbol{\alpha}, \boldsymbol{\beta}])$, $f_{[\alpha,\beta]}$ and $\Xi_{[\alpha,\beta]}$ have been *uniquely determined* as functions of $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$.
- We thus require

$$[\boldsymbol{\alpha}, \boldsymbol{\beta}] = \underbrace{\delta\left(\{f_\alpha, f_\beta\}^s_{BT} - N_{[\alpha,\beta]}\right)}_{\stackrel{!}{=} f_{[\alpha,\beta]}} + \underbrace{\mathcal{L}_{\pi^\sharp \beta}\Xi_\alpha - \mathcal{L}_{\pi^\sharp \alpha}\Xi_\beta + \delta N_{[\alpha,\beta]}}_{\stackrel{!}{=} \Xi_{[\alpha,\beta]}} .$$

The second condition is solved as

$$\boxed{N_{[\alpha,\beta]} = \int_{\mathscr{C}} \left(\mathcal{L}_{\pi^\sharp \alpha}\Xi_\beta - \mathcal{L}_{\pi^\sharp \beta}\Xi_\alpha + \Xi_{[\alpha,\beta]}\right) + K_{\alpha,\beta}}$$



where $\mathscr{C}$ denotes a path in $\mathscr{P}$ and $K_{\alpha,\beta} \in \mathbb{R}$ is an arbitrary integration constant.
$\rightarrow$ **Proposition.** The expression of $N_{[\alpha,\beta]}$ does *not* depend on $\mathscr{C}$.

- This motivates the definition of the ***modified Poisson bracket***

$$\boxed{\{f_\alpha, f_\beta\}^s_\star = \{f_\alpha, f_\beta\}^s_{BT} - \int_{\mathscr{C}} \left(\mathcal{L}_{\pi^\sharp \alpha}\Xi_\beta - \mathcal{L}_{\pi^\sharp \beta}\Xi_\alpha + \Xi_{[\alpha,\beta]}\right)}$$

Modified Poisson bracket

$$\boxed{\{f_\alpha, f_\beta\}^s_\star = \{f_\alpha, f_\beta\} - \pi(\Xi_\alpha, \Xi_\beta) - \int_{\mathscr{C}} \left(\mathcal{L}_{\pi^\sharp \alpha} \Xi_\beta - \mathcal{L}_{\pi^\sharp \beta} \Xi_\alpha + \Xi_{[\alpha,\beta]} \right)}$$

- Some crucial properties:
  - **Proposition 1.** $\{\cdot,\cdot\}^s_\star$ defines a ***Poisson bracket*** on $\mathscr{F}(\mathscr{P})$.
  - **Proposition 2.** The definition of the bracket is ***independent*** on the path $\mathscr{C}$.
  - **Proposition 3.** $\{\cdot,\cdot\}^s_\star$ reduces to $\{\cdot,\cdot\}$ if the arguments are exact *and* s is a fundamental split map.
- For a given split map s, we have $s([\alpha,\beta]) = (f_{[\alpha,\beta]}, \Xi_{[\alpha,\beta]}) = (\{f_\alpha, f_\beta\}^s_\star - K_{\alpha,\beta}, \Xi_{[\alpha,\beta]})$ by definition, where $K_{\alpha,\beta} \in \mathbb{R}$ is a Lie algebra 2-cocycle over $T^\star\mathscr{P}$.

Lemma. The set of scalar functions $\{f_\alpha = p_1 \circ s(\alpha) \,|\, \alpha \in T^\star\mathscr{P}\}$ provided by the split map s forms a Poisson algebra under the modified bracket $\{\cdot,\cdot\}^s_\star$. This algebra represents the algebra of differential forms under the Koszul bracket up to central extensions.

$$\boxed{\{f_\alpha, f_\beta\}^s_\star = f_{[\alpha,\beta]} + K_{\alpha,\beta}.}$$

$\rightarrow$ *Property:* This algebra is *not* modified under changes of split maps: $f_\alpha \to f_\alpha - N_\alpha$, $\Xi_\alpha \to \Xi_\alpha + \delta N_\alpha$.

Generalized representation theorem

- For $X \in \mathcal{X}(\mathscr{F})$ a generalized symmetry, we have the *local charge* $\omega^\flat X = \delta H_X + \Xi_X$ where Ξ_X is uniquely fixed by the *fundamental split map* s $\Rightarrow$ Generalization of the "*momentum map*":

$$\boxed{\zeta : T\mathscr{P} \to (\mathscr{F}(\mathscr{P}), T^\star \mathscr{P}) : X \mapsto \zeta(X) \equiv s \circ \omega^\flat X \equiv (H_X, \Xi_X).}$$

- Using the Lie algebra isomorphism $\omega^\flat$ and the dictionary

$$\boldsymbol{\alpha} \to \omega^\flat X_1, \; \boldsymbol{\beta} \to \omega^\flat X_2 \text{ and } [\boldsymbol{\alpha}, \boldsymbol{\beta}] = [\omega^\flat X_1, \omega^\flat X_2] = -\omega^\flat([X_1, X_2]),$$

the definition of the modified Poisson bracket is adapted as

$$\boxed{\{H_{X_1}, H_{X_2}\}^\zeta_\star \equiv \overbrace{\underbrace{\{H_{X_1}, H_{X_2}\} - i_{X_2}\Xi_{X_1}}_{\equiv \delta_{X_2} H_{X_1}} + i_{X_1}\Xi_{X_2}}^{\text{Barnich-Troessaert bracket}} - \underbrace{\int_{\mathscr{C}} \left(\mathcal{L}_{X_1}\Xi_{X_2} - \mathcal{L}_{X_2}\Xi_{X_1} - \Xi_{[X_1,X_2]} \right)}_{\text{field-dependent 2-cocycle}}.}$$

- The representation property translates into

$$(H_{[X_1,X_2]}, \Xi_{[X_1,X_2]}) = \zeta([X_1, X_2]) = -\zeta([\omega^\flat X_1, \omega^\flat X_2]) \overset{\text{(lemma)}}{\equiv} (-\{H_{X_1}, H_{X_2}\}^\zeta_\star + K_{X_1,X_2}, \Xi_{[X_1,X_2]}).$$

Generalized representation theorem

Generalized representation theorem. The algebra of integrable parts $\{H_X = p_1 \circ \zeta(X) | X \in T\mathscr{P}\}$ of the infinitesimal charges $\omega^\flat X$, equipped with the modified bracket $\{\cdot,\cdot\}^\zeta_\star$, represents the vector algebra up to central extensions, *i.e.*

$$\boxed{\{H_{X_1}, H_{X_2}\}^\zeta_\star = -H_{[X_1,X_2]} + K_{X_1,X_2},}$$

where K_{X_1,X_2} is a *Lie algebra 2-cocycle* over $T\mathscr{P}$.

- Generalized symmetries form a sub-algebra of $\mathfrak{diff}(\mathscr{P}) \Rightarrow$ their charges close a sub-algebra in $\mathscr{F}(\mathscr{P})$. In that case, ζ is a *fundamental split map* $\Rightarrow$ standard representation theorem when sources are absent.
- **Crucial property:** *the charge algebra controls the non-conservation laws.* Using $\frac{\partial H_X}{\partial t} = H_{[X,X_t]} = -\{H_X, H_{X_t}\}^\zeta_\star - K_{X,X_t}$ and recalling that $\boldsymbol{\Xi}_{X_t} = -\boldsymbol{\Theta}$:

$$\frac{\mathrm{d}H_X}{\mathrm{d}t} = \frac{\partial H_X}{\partial t} + i_{X_t}\delta H_X = -i_X \boldsymbol{\Xi}_{X_t} + \int_{\mathscr{C}} \underbrace{\left(\mathcal{L}_X \boldsymbol{\Xi}_{X_t} + \mathcal{L}_{X_t}\boldsymbol{\Xi}_X - \boldsymbol{\Xi}_{[X,X_t]}\right)}_{\propto -\delta V_X} - K_{X,X_t} = i_X \boldsymbol{\Theta} - V_X\,!$$

Generalized representation theorem

Example. *Point particle with an external force.*

- Galilean symmetries form an sub-algebra $\mathfrak{G} \subset \mathfrak{diff}(\mathscr{P})$: $\delta_{[X_1,X_2]} q^i \equiv \hat{a}^i + \hat{b}\frac{p^i}{m} + t\hat{v}^i - \hat{\omega}^i_j q^j$, with $\hat{a}^i = (b_2 v^i_1 - b_1 v^i_2) + (\omega^i_{1j} a^j_2 - \omega^i_{2j} a^j_1)$, $\hat{b} = 0$, $\hat{v}^i = \omega^i_{1j} v^j_2 - \omega^i_{2j} v^j_1$ and $\hat{\omega}^i_j = \omega^i_{1k}\omega^k_{2j} - \omega^i_{2k}\omega^k_{1j}$.
- Fundamental split map: $\bar{\zeta}(i_X \omega^{(H)}) = (H_X, \Xi_X) = \big(m v_i q^i - (a^i + t v^i - \omega^i{}_j q^j) p_i - b\,\mathcal{H}, -b\, q^i \delta f_i\big)$.
  - Computing the Barnich-Troessaert bracket gives

$$\{H_{X_1}, H_{X_2}\}^{\bar{\zeta}}_{BT} \equiv \delta_{X_2} H_{X_1} + i_{X_1}\Xi_{X_2} = -H_{[X_1,X_2]} - m(a^i_1 v^2_i - a^i_2 v^1_i) + f_i \left[b_1 a^i_2 - b_2 a^i_1 + t(b_1 v^i_2 - b_2 v^i_1) \right].$$

    $\rightarrow$ Charge algebra closes *up to a source-dependent 2-cocycle!*

  - Since, $\hat{b} = 0$, $\Xi_{[X_1,X_2]} = 0$, and $\mathcal{L}_{X_1}\Xi_{X_2} - \mathcal{L}_{X_2}\Xi_{X_1} = \delta\left[(b_1 a^i_2 - b_2 a^i_1) f_i + t(b_1 v^i_2 - b_2 v^i_1) f_i\right]$.

    $\rightarrow$ The modified Poisson bracket allows to delete the cocycle:

$$\boxed{\{H_{X_1}, H_{X_2}\}^{\bar{\zeta}}_{\star} = -H_{[X_1,X_2]} - m(a^i_1 v^2_i - a^i_2 v^1_i).} \qquad [\ \textbf{Bargmann algebra}\ ]$$

$\Rightarrow$ *The charge algebra represents the Galilean algebra up to a central extension!* ☺

Conclusions

- **Summary.** In this talk, we discussed a *formalism* to study *sourced Hamiltonian systems in mechanics*.
  - ✎ Apparent violation of the traditional variational principle.
  - ✎ Possible to re-install the Noether symmetries,
    - $\rightarrow$ at the price to have *non-conserved* & *non-integrable* charges.
  - ✎ Poisson bracket has to be modified to define meaningful charge algebra.
- Not the only alternative to treat "dissipation" in mechanics (see *e.g.* [Kaufman 1983, Figotin-Schenker 2007, Galley 2012], and many works in engineering physics!) *but* in our construction,
  - ☑ Economy of new concepts, robust derivation $\Rightarrow$ enough generality;
  - ☑ Very close to what is known in symplectic geometry;
  - ☑ Easily exploitable in field theory and QFT (*cf.* **Romain's talk**).
- **Future perspectives?** Publish the paper soon 😄
  - ? Discuss the *Lagrangian* version in mechanics + full extension to field theories.
  - ? Application to covariant phase space formalism for *leaky gauge theories*.
  - ? Hope to be useful in *holography with sources* and in many (practical?) applications.

Thank you for your attention!

Work supported by the
Austrian Science Fund

FWF Der Wissenschaftsfonds.

INSTITUTE FOR THEORETICAL PHYSICS