The many faces of Carroll/Galilei duality

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In the UK (at least) we only just entered a new *carolian* era.

Carroll

Galilei

“Through the looking-glass”

or

“Through a glass darkly”?

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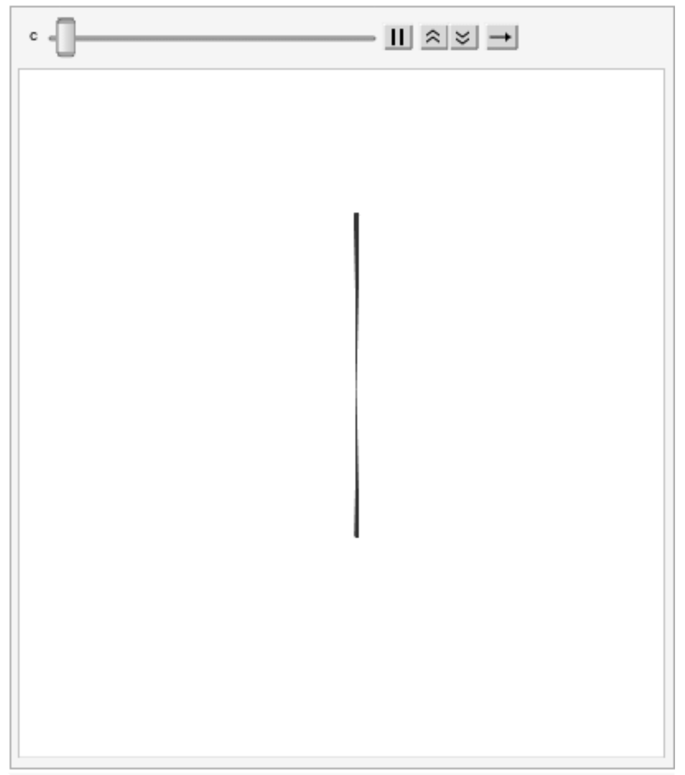
Naive duality

Naive Carroll/Galilei duality

$0 \leftarrow c$ $\quad\quad$ $\frac{1}{c} \to 0$

Carroll $\longleftarrow$ **Lorentz** $\longrightarrow$ **Galilei**

[Lévy-Leblond 1965]



$c \leftrightarrow \frac{1}{c}$ duality

(superficially reminiscent of Kramers-Wannier duality)

[Kramers+Wannier 1941]

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Geometric duality

Galilean geometry

[Weyl 1919]
[Künzle 1972]

(N, τ, γ)

$\tau \in \Omega^1(N)$ nowhere-vanishing **clock one-form**

$\gamma \in \Gamma(\odot^2 TN)$ corank-1, positive-semidefinite **spatial cometric** $\gamma(\tau, -) = 0$

Generally of 3 kinds:

$d\tau = 0$	**torsionless**
$d\tau \neq 0, d\tau \wedge \tau = 0$	**twistless-torsional**
$d\tau \wedge \tau \neq 0$	**torsional**

[Christensen+Hartong+Obers+Rollier 2013]

Carrollian geometry

[Henneaux 1979]
[Duval+Gibbons+Horvathy+Zhang 2014]

(N, κ, h)

$\kappa \in \mathscr{X}(N)$ nowhere-vanishing **carrollian vector field**

$h \in \Gamma(\odot^2 T^* N)$ corank-1, positive-semidefinite **spatial metric** $h(\kappa, -) = 0$

Now 4 kinds:

$\mathscr{L}_\kappa h = 0$	**totally geodesic**
$\mathscr{L}_\kappa h = f h$	**totally umbilic**
$\operatorname{tr}(\mathscr{L}_\kappa h) = 0$	**minimal**
none of the above	**generic**

[JMF 2020]

Geometric Carroll/Galilei duality

A galilean structure is a section of $T^*N \oplus \odot^2 TN$

vector bundle duality

A carrollian structure is a section of $TN \oplus \odot^2 T^*N$

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Categorical duality

Bargmann geometry

[Duval+Burdet+Künzle+Perrin 1985]
[Papadopoulos 2018]
[JMF 2020]

(M, g, ξ)

(M, g) lorentzian manifold

$\xi \in \mathscr{X}(M)$ nowhere-vanishing null vector field

Assume that

ξ is Killing ($\mathscr{L}_\xi g = 0$) and

it integrates to an action of Γ which is free and proper

Then $\pi : M \to M/\Gamma =: N$ is a principal-Γ bundle

Null reduction

[Duval+Burdet+Künzle+Perrin 1985]
[Julia+Nicolai 1995]

(M, g, ξ)

$$\begin{array}{c} M \\ \big\downarrow \pi \\ N \end{array}$$

$$\xi^\flat = g(\xi, -) = \pi^* \tau \qquad \tau \in \Omega^1(N)$$

$$g^{-1}(\pi^*\alpha, \pi^*\beta) = \pi^*(\gamma(\alpha, \beta)) \qquad \gamma \in \Gamma(\odot^2 TN)$$

(N, τ, γ) is a galilean manifold

Null hypersurfaces

(M, g, ξ)

ξ defines a distribution $\xi^{\perp} \subset TM$

If $d\xi^{\flat} \wedge \xi^{\flat} = 0$ so that $\xi^{\perp}$ is integrable, M is foliated by **null hypersurfaces**

$i : N \to M$ admitting a carrollian structure given by ξ and $h = i^{*}g$

[Hartong 2015]

Categorical Carroll/Galilei duality

[Duval+Gibbons+Horvathy+Zhang 2014]

	embedding		fibration	
Carroll	$\longrightarrow$	**Bargmann**	$\longrightarrow$	**Galilei**
	mono		epi	
	subobject		quotient	

In category theory, monos/epis and subobject/quotient are dual under

$$\mathscr{C} \leftrightarrow \mathscr{C}^{\mathrm{op}}$$

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Geometric duality revisited

Local frames

M an n-dimensional manifold

$\{U_\alpha\}$ an open cover with $TU_\alpha \cong U_\alpha \times \mathbb{R}^n$

$\left(e_1^{(\alpha)}, \ldots, e_n^{(\alpha)}\right)$ a local frame on U_α

On nonempty $U_\alpha \cap U_\beta$

$$e_i^{(\beta)} = e_j^{(\alpha)} (g_{\alpha\beta})^j{}_i$$

for some $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$

G-structures

Let $G \subset \mathrm{GL}(n, \mathbb{R})$. A G-structure on an n-dimensional manifold M is a principal G-subbundle of the frame bundle.

In other words, as in the previous slide, but the transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

Galilean and carrollian structures are examples of G-structures.

Carroll and Galilei G-structures

The Galilei G-structure has

$$G_{\text{gal}} = \left\{ \begin{pmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{v} & A \end{pmatrix} \middle| \boldsymbol{v} \in \mathbb{R}^{n-1}, \quad A \in O(n-1) \right\} \subset \mathrm{GL}(n, \mathbb{R})$$

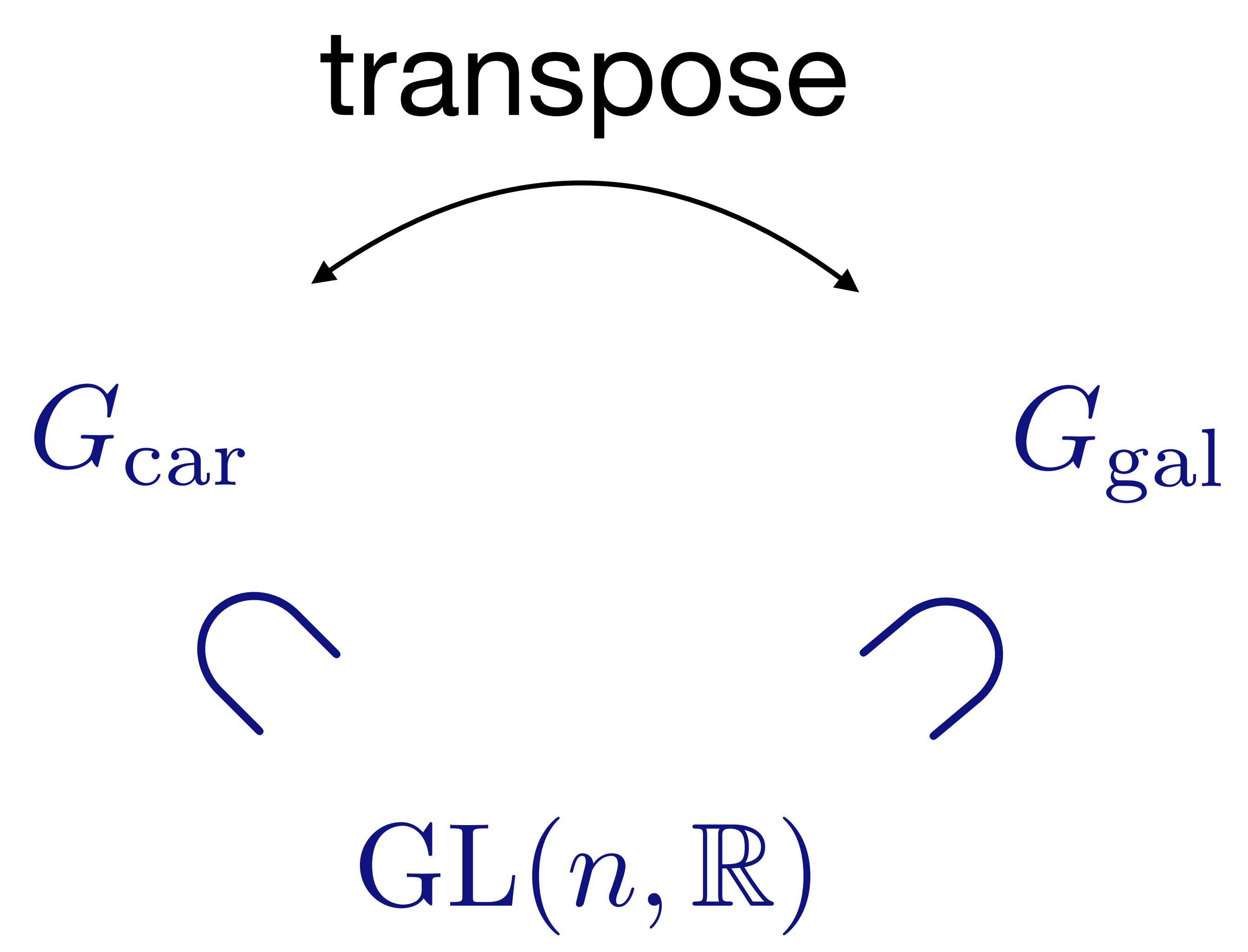
The Carroll G-structure has

$$G_{\text{car}} = \left\{ \begin{pmatrix} 1 & \boldsymbol{v}^T \\ \mathbf{0} & A \end{pmatrix} \middle| \boldsymbol{v} \in \mathbb{R}^{n-1}, A \in O(n-1) \right\} \subset \mathrm{GL}(n, \mathbb{R})$$

$G_{\text{gal}} \cong G_{\text{car}}$ abstractly, but they are **not** conjugate in $\mathrm{GL}(n, \mathbb{R})$

Indeed, $G_{\text{car}} = \left(G_{\text{gal}}\right)^T$ and two such subgroups of $\mathrm{GL}(n, \mathbb{R})$ are conjugate only if they are abelian, since conjugation preserves but transposition reverses the order of multiplication.

Carroll/Galilei duality manifests itself via transposition



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Lie algebraic duality

Which Lie groups admit a bi-invariant galilean/carrollian structure?

Lie groups with bi-invariant metrics

Let G be a connected Lie group and g a bi-invariant metric. The metric restricts at the identity to an inner product $\langle -, - \rangle$ on the Lie algebra $\mathfrak{g}$ which is ad-invariant:

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle \qquad \forall X, Y, Z \in \mathfrak{g}$$

Such Lie algebras are called **metric Lie algebras** and they can be characterised as the class of Lie algebras generated by the simple and the one-dimensional Lie algebras under the operations of orthogonal direct sum and **double extension**.

[Medina+Revoy 1985]
[JMF+Stanciu 1995]

One-dimensional double extension

Let $(\mathfrak{g}, \langle -, - \rangle)$ be a metric Lie algebra and let D be a skew-symmetric derivation:

$$D[X,Y] = [DX,Y] + [X,DY]$$

$$\langle DX,Y \rangle = - \langle X, DY \rangle \qquad \forall X,Y \in \mathfrak{g}$$

This defines a central extension $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}Z$ with brackets

$$[X + \alpha Z, Y + \beta Z]_{\hat{\mathfrak{g}}} = [X,Y]_{\mathfrak{g}} + \langle DX, Y \rangle Z$$

And also a double extension $\hat{\hat{\mathfrak{g}}} = \mathfrak{g} \oplus \mathbb{R}Z \oplus \mathbb{R}D$ with brackets

$$[X + \alpha Z, Y + \beta Z]_{\hat{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} + \langle DX, Y \rangle \, Z$$

$$[D, X] = DX$$

The double extension has an invariant inner product given by

$$\langle X, Y \rangle_{\hat{\hat{\mathfrak{g}}}} = \langle X, Y \rangle_{\mathfrak{g}} \qquad \langle D, Z \rangle_{\hat{\hat{\mathfrak{g}}}} = 1$$

making $\hat{\hat{\mathfrak{g}}}$ into a metric Lie algebra of signature $(p+1, q+1)$ where $\mathfrak{g}$ has signature (p, q)

Galilean Lie algebras

Let us say that a Lie algebra $\mathfrak{g}$ is **galilean** if its corresponding simply-connected Lie group admits a bi-invariant galilean structure.

Equivalently, $\mathfrak{g}$ admits ad-invariant $\tau \in \mathfrak{g}^*$ (assumed nonzero) and $\gamma \in \odot^2 \mathfrak{g}$, with γ of corank-1 whose radical is spanned by τ. In particular, τ annihilates brackets:

$$\tau([X,Y]) = 0 \qquad \forall X, Y \in \mathfrak{g}$$

Let $\mathfrak{g}_0 = \ker \tau$. It is an ideal of $\mathfrak{g}$ and hence we have a short exact sequence

$$0 \longrightarrow \mathfrak{g}_0 \longrightarrow \mathfrak{g} \xrightarrow{\tau} \mathbb{R} \longrightarrow 0$$

The sequence splits. Choose any $D \in \mathfrak{g}$ with $\tau(D) \neq 0$. It follows that γ induces an inner product γ_0 on $\mathfrak{g}_0^*$ and hence its inverse gives an inner product on $\mathfrak{g}_0$, which is invariant under the action of $\mathfrak{g}$. This says that $\mathfrak{g}_0$ is a metric Lie algebra and $[D, -\,]$ is a skew-symmetric derivation.

This is the same data which defines a one-dimensional double extension $\hat{\hat{\mathfrak{g}}}$ and we have the following short exact sequence

$$0 \longrightarrow \mathbb{R}Z \longrightarrow \hat{\hat{\mathfrak{g}}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

In other words, the one-dimensional double extension of $\mathfrak{g}_0$ is a central extension of the galilean Lie algebra $\mathfrak{g}$.

Carrollian Lie algebras

Let us say that a Lie algebra $\mathfrak{g}$ is **carrollian** if its corresponding simply-connected Lie group admits a bi-invariant carrollian structure.

Equivalently, $\mathfrak{g}$ admits a nonzero central element $Z \in \mathfrak{g}$ and a symmetric bilinear form $h \in \odot^2 \mathfrak{g}^*$, with h of corank-1 whose radical is spanned by Z.

Let $\mathfrak{g}_0 := \mathfrak{g}/\mathbb{R}Z$ on which h induces an inner product, so that it is a metric Lie algebra. We have a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R}Z \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_0 \longrightarrow 0$$

showing that $\mathfrak{g}$ is a one-dimensional central extension of $\mathfrak{g}_0$.

A central extension of a metric Lie algebra $\mathfrak{g}_0$ defines a skew-symmetric derivation D of $\mathfrak{g}_0$ by

$$[X,Y] = [X,Y]_0 + \langle DX, Y\rangle Z$$

So a carrollian Lie algebra is determined by a metric Lie algebra and a skew-symmetric derivation, which again is the data defining a one-dimensional double extension $\hat{\hat{\mathfrak{g}}}$. In fact we now get a short exact sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow \hat{\hat{\mathfrak{g}}} \longrightarrow \mathbb{R}D \longrightarrow 0$$

In other words, the one-dimensional double extension of $\mathfrak{g}_0$ is an "extension by a skew-symmetric derivation" of the carrollian Lie algebra $\mathfrak{g}$.

Summary

The data consisting of a metric Lie algebra $\mathfrak{g}_0$ and a skew-symmetric derivation D allows us to construct three Lie algebras:

- a metric (Bargmann) Lie algebra $\mathfrak{g}_{\text{bar}}$
- a galilean Lie algebra $\mathfrak{g}_{\text{gal}}$, which is a quotient of $\mathfrak{g}_{\text{bar}}$
- a carrollian Lie algebra $\mathfrak{g}_{\text{car}}$, which is an ideal of $\mathfrak{g}_{\text{bar}}$

This is summarised by the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{R}Z & = & \mathbb{R}Z & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathfrak{g}_{\mathrm{car}} & \longrightarrow & \mathfrak{g}_{\mathrm{bar}} & \longrightarrow & \mathbb{R}D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathfrak{g}_{0} & \longrightarrow & \mathfrak{g}_{\mathrm{gal}} & \longrightarrow & \mathbb{R}D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

We say that $\mathfrak{g}_{\mathrm{car}}$ and $\mathfrak{g}_{\mathrm{gal}}$ are **dual**, and the same holds for their simply-connected Lie groups.

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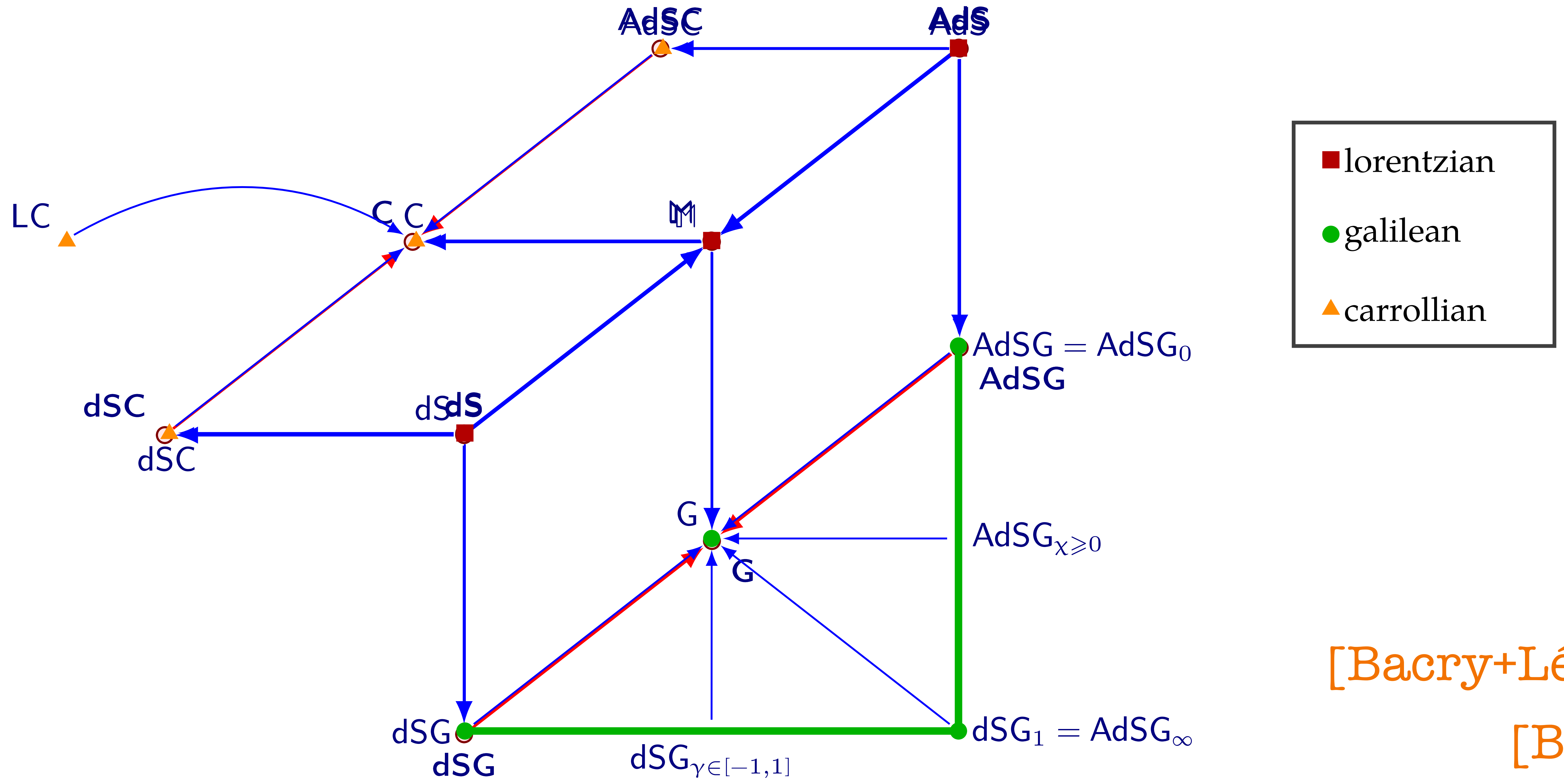
A glitch?

Spatially isotropic homogeneous spacetimes



[Bacry+Lévy-Leblond 1968]

[Bacry+Nuyts 1986]

[JMF+Prohazka 2018]

Why this (seeming) asymmetry between galilean and carrollian spacetimes?

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Algebraic duality
(Sanity restored?)

The Bargmann algebra

[Bargmann 1954]

The Bargmann algebra is the **universal central extension** of the Galilei algebra.

Let $\mathfrak{g}$ denote the **Galilei algebra** with basis (L_{ab}, B_a, P_a, H) and nonzero brackets

$$
\begin{aligned}
[L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{ad}L_{bc} \\
[L_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b \\
[L_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b \\
[B_a, H] &= P_a
\end{aligned}
$$

The Bargmann central extension $\widehat{\mathfrak{g}}$ has an additional generator M with

$$[B_a, P_b] = \delta_{ab}M$$

This can be summarised as an exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R}M \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

which shows that $\widehat{\mathfrak{g}}$ is indeed a one-dimensional (central) extension of $\mathfrak{g}$

It turns out that the Bargmann algebra also admits a **carrollian** description.

A carrollian description of the Bargmann algebra

Let $\mathfrak{c}$ denote the **Carroll algebra** spanned by (L_{ab}, B_a, P_a, H) with nonzero brackets

$$
\begin{aligned}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc} \\
[L_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \\
[B_a, P_b] &= \delta_{ab} H
\end{aligned}
$$

Let $\delta : \mathfrak{c} \to \mathfrak{c}$ be the derivation defined by

$$
\delta(L_{ab}) = \delta(P_a) = \delta(H) = 0 \qquad \delta(B_a) = -P_a
$$

Let $\widehat{\mathfrak{c}} = \mathfrak{c} \oplus \mathbb{R}D$ denote the extension of $\mathfrak{c}$ by the derivation $\delta = [D, -\,]$ with additional bracket

$$[B_a, D] = P_a$$

This Lie algebra is an extension of the one-dimensional Lie algebra $\mathbb{R}D$ by the Carroll algebra $\mathfrak{c}$

$$0 \longrightarrow \mathfrak{c} \longrightarrow \widehat{\mathfrak{c}} \longrightarrow \mathbb{R}D \longrightarrow 0$$

Fun fact: $\widehat{\mathfrak{c}} \cong \widehat{\mathfrak{g}}$

$\widehat{\mathfrak{c}}$	L_{ab}	B_a	P_a	D	H
$\widehat{\mathfrak{g}}$	L_{ab}	B_a	P_a	H	M

An algebraic Carroll/Galilei duality

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

reversing arrows

and $\mathfrak{c} \leftrightarrow \mathfrak{g}$

$$0 \longrightarrow \mathfrak{c} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathbb{R} \longrightarrow 0$$

Something similar occurs with the torsional galilean spacetimes, whose transitive Lie algebras admit a carrollian description.

How does this help?

Null reduction to the rescue

The Bargmann algebra is the subalgebra of the Poincaré algebra (in one dimension higher) which centralises a null translation.

And Galilei spacetime is the null reduction of Minkowski spacetime along that null translation.

Question: Can we do the same with the torsional galilean spacetimes?

Answer: Yes! **(with some small print)**

Spatially isotropic homogeneous galilean spacetimes

Let $\mathfrak{g}_{\alpha,\beta}$ be the Lie algebra spanned by (L_{ab}, B_a, P_a, H) with nonzero brackets

$$
\begin{aligned}
[L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{ad}L_{bc} \\
[L_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b \\
[L_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b \\
[B_a, H] &= P_a \\
[H, P_a] &= \alpha B_a + \beta P_a
\end{aligned}
$$

The spatially isotropic homogeneous galilean spacetime $\mathscr{M}_{\alpha,\beta}$ is described

by the Klein pair $(\mathfrak{g}_{\alpha,\beta}, \mathfrak{h})$ where $\mathfrak{h}$ is the subalgebra spanned by (L_{ab}, B_a).

Let $\widehat{\mathfrak{g}}_{\alpha,\beta} = \mathfrak{g}_{\alpha,\beta} \oplus \mathbb{R}M$ denote the one-dimensional extension with additional brackets

$$[B_a, P_b] = \delta_{ab} M \qquad [H, M] = \beta M$$

$\widehat{\mathfrak{g}}_{\alpha,\beta}$ is a deformation of the centrally extended static kinematical Lie algebra.

[JMF 2018]

The extension is **central** if and only if $\beta = 0$. If that is the case, we can rescale α to be one of $\{0, \pm 1\}$, corresponding to the Bargmann and the centrally extended **Newton—Hooke** algebras.

$\widehat{\mathfrak{g}}_{\alpha,\beta}$ appear as the Lie algebra of conserved charges associated to free particle motion on $\mathscr{M}_{\alpha,\beta}$, in the same way that the Bargmann algebra is the Lie algebra of conserved charges associated to free particle motion on Galilei spacetime.

[JMF+Görmez+Van den Bleeken 2022]

The homogeneous space $\widehat{\mathscr{M}}_{\alpha,\beta}$ with Klein pair $\left(\hat{\mathfrak{g}}_{\alpha,\beta}, \hat{\mathfrak{h}}\right)$ where $\hat{\mathfrak{h}} \cong \mathfrak{h}$ is the subalgebra spanned by $\left(L_{ab}, B_a\right)$ is **lorentzian** and the Killing vector field corresponding to M is null, and the corresponding null reduction is the homogeneous spacetime $\mathscr{M}_{\alpha,\beta}$. The induced galilean structure is only homothetic to the invariant one.

[JMF+Grassie+Prohazka 2022]

Null geodesics in these lorentzian manifolds describe the Eisenhart lifts of geodesic motion on $\mathscr{M}_{\alpha,\beta}$ relative to the canonical invariant connection.

[JMF+Görmez+Van den Bleeken 2022]

Carrollian description of $\widehat{\mathfrak{g}}_{\alpha,\beta}$

As in the case of the Bargmann algebra, $\widehat{\mathfrak{g}}_{\alpha,\beta}$ is an extension of the Carroll algebra by a derivation $\delta = [D, -\,]$ defined by

$$\delta(L_{ab}) = 0 \qquad \delta(B_a) = -P_a \qquad \delta(P_a) = \alpha B_a + \beta P_a \qquad \delta(H) = \beta H$$

so that

$$0 \longrightarrow \mathfrak{c} \longrightarrow \widehat{\mathfrak{g}}_{\alpha,\beta} \longrightarrow \mathbb{R}D \longrightarrow 0$$

Algebraic Carroll/Galilei duality

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{\mathfrak{g}}_{\alpha,\beta} \longrightarrow \mathfrak{g}_{\alpha,\beta} \longrightarrow 0$$

reversing arrows

and $\mathfrak{c} \leftrightarrow \mathfrak{g}_{\alpha,\beta}$

$$0 \longrightarrow \mathfrak{c} \longrightarrow \widehat{\mathfrak{g}}_{\alpha,\beta} \longrightarrow \mathbb{R} \longrightarrow 0$$

Always the Carroll algebra, but the derivation **changes.**

Does this mean that all homogeneous galilean spacetimes are dual to the Carroll spacetime?

Symmetries

All galilean spacetimes are locally isomorphic and their symmetry algebras are isomorphic to the **Coriolis algebra**:

$$0 \longrightarrow C^\infty(\mathbb{R}, \mathfrak{iso}(n-1)) \longrightarrow \mathfrak{Coriolis} \longrightarrow \mathbb{R} \longrightarrow 0$$

[Duval 1993]

[JMF+Grassie+Prohazka 2019]

whereas the symmetry algebra of the Carroll spacetime is

$$0 \longrightarrow C^\infty(\mathbb{E}^{n-1}) \longrightarrow \mathfrak{Carroll} \longrightarrow \mathfrak{iso}(n-1) \longrightarrow 0$$

[Duval+Gibbons+Horvathy 2014]

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Conclusions and questions

Conclusions and open questions

- I hope to have convinced you that the Carroll/Galilei duality is present at a ***kinematical*** level
- **Two open questions:**
  - What is the galilean dual of the lightcone?
  - Is there a ***dynamical*** manifestation of this duality?