Scattering amplitudes: Celestial and Carrollian

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Flat space holography

Celestial approach

1) In this approach, the dual theory is a co-dimension 2 QFT. For 4d asymptotically flat spacetimes the proposed dual is a 2d CFT that lives on the celestial sphere.

2) This 2d CFT computes 4d bulk scattering amplitude in terms of its correlation functions. The ward identities of this 2d CFT successfully capture the rich infrared behaviour of the scattering amplitudes.

3) However, it is not entirely evident how these results can be rediscovered by taking a suitable flat space limit of AdS-CFT.

Carrollian approach

1) In this parallel approach, the dual theory is a 3d field theory that lives on the null boundary of asymptotically flat spacetimes and enjoys BMS symmetries.

2) This approach has been very successful in lower dimensional cases. Particularly 2d Carrollian CFTs exactly reproduced various observables (eg. Entropy, stress tensor correlates etc) of 3d asymptotically flat spacetimes.

3) But this approach has not been as fruitful for 4d bulk theories because its relation with the bulk scattering amplitudes was not known.

References→ Arjun's talk

Carroll,Null and BMS

- The Carroll group is the ultra-relativistic contraction of the Poincare group, where the speed of light $c \to 0$. In this limit, the light cones close up; consequently, this limit indicates ultra locality.

- The associated kinematical structures give rise to Carrollian Manifolds. They are defined by a degenerate twice symmetric metric $h_{\mu\nu}$ and its kernel vector field τ^{μ}.

- These structures have many potential applications, including flat space holography, because they appear in the intrinsic geometries of null hypersurfaces (for example - null infinity ($\mathscr{I}$), black hole horizons)

* It is possible to study the conformal isometries of these structures. The conformal killing equations are given by

$$\mathscr{L}_\xi \tau^\mu = -\frac{\lambda}{N}\tau^\mu \quad \mathscr{L}_\xi h_{\mu\nu} = \lambda h_{\mu\nu}$$

The 'dynamical exponent' N captures the relative scaling between the space-like and time-like vielbeins.

* For $N = 2$, they scale homogeneously and the isometries become isomorphic to the BMS group.

* In the particular example of $\mathscr{I}$ (which is topologically $R \times S^2$) , these conformal killing vectors are given by

$$\xi = [\alpha(x^i) + \frac{u}{2}\nabla_i f^i(x^i)]\partial_u + f^i(x^i)\partial_i$$

Here, $\alpha(x^i)$ are arbitrary functions but $f^i(x^j)$ need to satisfy the conformal killing equations on S^2. These equations fix f^z and $f^{\bar{z}}$ to be holomorphic and anti-holomorphic functions only

* A particular useful representation of these vector fields is given by

$$L_n = -z^{n+1}\partial_z - \frac{1}{2}(n+1)z^n u\partial_u \quad \bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}} - \frac{1}{2}(n+1)\bar{z}^n u\partial_u \quad M_{r,s} = z^r\bar{z}^s\partial_u$$

* These generators close to form the BMS_4 algebra

$$[L_n, L_m] = (n-m)L_{n+m} \quad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m}$$

$$[L_n, M_{r,s}] = (\frac{n+1}{2} - r)M_{n+r,s} \quad [\bar{L}_n, M_{r,s}] = (\frac{n+1}{2} - s)M_{r,n+s}$$

$$[M_{r,s}, M_{p,q}] = 0$$

* $L_n, \bar{L}_n \rightarrow$ superrotations, $M_{r,s} \rightarrow$ supertranslations

Field theory on $\mathscr{I}$

- Conformal field theories on null infinity would naturally inherit these conformal isometries and be BMS invariant.

- We can assume highest weight representation defined by

$$[L_n, \Phi(0)] = 0, \quad [\bar{L}_n, \Phi(0)] = 0, \quad \forall n > 0 \quad [M_{r,s}, \Phi(0)] = 0 \quad \forall r, s > 0$$

and label the states by $[L_0, \Phi(0)] = h\Phi(0), \quad [\bar{L}_0, \Phi(0)] = \bar{h}\Phi(0)$

- The transformation rules of these fields at an arbitrary point are then given by

$$\delta_{L_n}\Phi(u, z, \bar{z}) = [z^{n+1}\partial_z + (n+1)z^n(h + \frac{1}{2}u\partial_u)]\Phi(u, z, \bar{z}) \quad \text{and}$$

$$\delta_{M_{r,s}}\Phi(u, z, \bar{z}) = z^r\bar{z}^s\partial_u\Phi(u, z, \bar{z})$$

also, a similar relation holds for the anti-holomorphic piece

* These boundary fields can be mapped with the bulk fields by using the **modified Mellin transformations**, which is

$$\Phi^{\epsilon}_{h,\bar{h}}(u,z,\bar{z}) = \int_0^{\infty} d\omega \omega^{\Delta-1} e^{-i\epsilon\omega u} a(\epsilon\omega, z, \bar{z}, \sigma)$$

Here $a(\omega, z, \bar{z}, \sigma)$ is a bulk field with momenta $p^{\mu} = \frac{p^0}{1+z\bar{z}}\left(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\right)$ and helicity σ.

* Scattering amplitudes in the bulk is typically given by

$$\mathcal{S}_n \sim < \prod_i^n a_i(\epsilon_i \omega_i, z_i, \bar{z}_i) >$$

* Thus the n-point function of Carrollian conformal primary fields computes the scattering amplitudes via

$$< \prod_i^n \Phi^{\epsilon^i}_{h_i,\bar{h}_i}(u_i, z_i, \bar{z}_i) > \sim \prod_i^n \int_0^{\infty} d\omega_i e^{-i\epsilon_i \omega_i u_i} \omega_i^{\Delta_i - 1} \mathcal{S}_n$$

* This modified Mellin transformation is used in Celestial holography to compute the graviton scattering amplitudes in GR. Because of the typical UV behaviour, Mellin integrals don't converge and this additional u dependence piece works as a regulator and gives finite results.

* This modified Mellin transformation also maps the bulk fields into 3D boundary fields as opposed to the Celestial holography. In Celestial holography the action of translations shifts the weights of the Celestial primary fields. For example

$$[H, \Phi^{\epsilon}_{h,\bar{h}}(z,\bar{z})] = -\epsilon(1+z\bar{z})\Phi^{\epsilon}_{h+\frac{1}{2},\bar{h}+\frac{1}{2}}(z,\bar{z})$$

* But the additional piece in the modified Mellin integral makes the boundary fields transform naturally under the bulk translations.

$$[H, \Phi^{\epsilon}_{h,\bar{h}}(u,z,\bar{z})] = (1+z\bar{z})(-i\partial_u)\Phi^{\epsilon}_{h,\bar{h}}(u,z,\bar{z})$$

Two point correlation function

* The global part of the conformal Carroll algebra is spanned by $L_{0,\pm 1}, \bar{L}_{0,\pm 1}$ and $M_{00}, M_{01}, M_{10}, M_{11}$. This subgroup is isomorphic to the bulk Poincare group in one higher dimension. Demanding invariances under these symmetries we can fix the two-point function for these theories entirely.

* Crucially Carroll boost invariance implies the two-point function should satisfy

$$(z\partial_u + z'\partial_{u'})G^2(u-u', z-z', \bar{z}-\bar{z}') = 0$$

This equation has two independent solutions that give rise to two independent sets of correlation functions.

1) CFT branch : This branch throws away u dependence in the correlation function and then the rest of the generators fixes it to be a 2d CFT correlator.

$$G^2(u-u', z-z', \bar{z}-\bar{z}') = \frac{\delta_{h,h'}\delta_{\bar{h},\bar{h}'}}{(z-z')^{2h}(\bar{z}-\bar{z}')^{2\bar{h}}}$$

2) Delta function branch: In another case, we keep u dependence in the correlation function, which restricts the correlation to having only a delta function in the space variables. This solution looks like

$$G^2(u-u', z-z', \bar{z}-\bar{z}') = \frac{\delta^2(z-z', \bar{z}-\bar{z}')}{(u-u')^{\Delta+\Delta'-2}}\delta_{\sigma+\sigma',0}$$

Here $\Delta = \dfrac{h+\bar{h}}{2}, \quad \sigma = \dfrac{h-\bar{h}}{2}$

- It is natural to have a delta function in the correlation function for the Carrollian theories as it reflects the ultra-local behaviour.

- Also, a noteworthy difference from the CFT branch is that it allows non-trivial correlation functions for an arbitrary value of Δ and Δ'.

Free scalar example:

$$S = \int d^2x^i e\tau^\mu\tau^\nu\partial_\mu\Phi\partial_\nu\Phi \quad e = \det(\tau^\mu, e_\mu)$$

- This example can be checked to be invariant under the conformal Carroll group.

- The non-trivial stress tensor components for this theory are

With
$$T^u_u = \frac{1}{2}(\partial_u\Phi)^2 \quad T^u_i = \frac{3}{4}\partial_u\Phi D_i\Phi - \frac{1}{4}\Phi\partial_u D_i\Phi$$

$$T^i_u = 0, \quad T^i_j = -\frac{1}{2}\delta^i_j T^u_u$$

This particular structure of stress tensor reflects the underlying Carrollian symmetries of the action.

- We can construct the charges directly using these stress tensor components. They are given by

$$Q[\alpha] = \int\sqrt{q}d^2x^i\alpha(x^i)\frac{1}{2}(\partial_u\Phi)^2$$

$$Q[f] = \int\sqrt{q}d^2x^i[\big(\frac{1}{2}(\partial_u\Phi)^2(\alpha(x^i) + \frac{u}{2}D_if^i(x^i))\big) + \big((\frac{3}{4}\partial_u\Phi D_i\Phi - \frac{1}{4}\Phi\partial_u D_i\Phi)f^i(x^i)\big)]$$

* It is possible to verify using these expressions of the charges that the scalar field $\Phi(u, z, \bar{z})$ would transform like a BMS-primary with weight $h = \frac{1}{4}$ and $\bar{h} = \frac{1}{4}$.

* We also compute the two-point correlation function for this theory explicitly by computing the Green's functions.

$$\partial_u^2 G(u-u', z^i - z^{'i}) = \delta^3(u-u', z^i - z^{'i})$$

In Fourier space this equation becomes

$$\tilde{G}(k_u, k_i) = -\frac{1}{k_u^2}$$

transforming it back to position space yields

$$G(u-u', z^i - z^{'i}) = \int \frac{dk_u}{k_u^2 + \mu^2} e^{-ik_u u} \int d^2\overrightarrow{k}\, e^{-ik_i(z^i - z^{'i})}$$

$$= \frac{i}{2}[\frac{1}{\mu} - (u-u')]\delta^2(z^i - z^{'i})$$

* Here we have used μ as a regulator. After throwing away the divergent piece, this answer agrees with the one derived from symmetry arguments.

Connections with scattering amplitudes

- This two-point function described previously can be obtained by modified Mellin transformation of free propagation amplitude.

- The free propagation amplitude of a massless particle is given by

$$< p_1, \sigma_1 | p_2, \sigma_2 > = (2\pi)^3 2E_{p_1} \delta^3(p_1 - p_2)\delta_{\sigma_1+\sigma_2,0}$$

Now if we use the following parametrisation $p^\mu = \omega(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$

then

$$< p_1, \sigma_1 | p_2, \sigma_2 > = 4\pi^3 \frac{\delta(\omega_1 - \omega_2)(\delta^2(z_1 - z_2))}{\omega_1}\delta_{\sigma_1+\sigma_2,0}$$

- Now modified Mellin transformation of this expression yields

$$\mathscr{M}(u_1, z_1, \bar{z}_1, u_2, z_2, \bar{z}_2) = 4\pi^3 \delta_{\sigma_1+\sigma_2,0} \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \omega_1^{\Delta_1 - 1} \omega_2^{\Delta_2 - 1} e^{-i\omega_1 u_1} e^{-i\omega_2 u_2} \frac{\delta(\omega_1 - \omega_2)(\delta^2(z_1 - z_2))}{\omega_1}$$

$$= 4\pi^3 \Gamma(\Delta_1 + \Delta_2 - 2) \frac{\delta^2(z_1 - z_2)}{(i(u_1 - u_2))^{\Delta_1 + \Delta_2 - 2}} \delta_{\sigma_1+\sigma_2,0}$$

* It is also possible to show that the three-point function corresponding to this branch vanishes. This is consistent with the fact the scattering amplitude of three massless particles vanishes due to momentum conservation.

* The central claim made based on this evidence is the following

The time-dependent correlation functions of primaries in Carrollian CFTs compute the bulk scattering amplitudes in one higher dimension.

$$\mathcal{M}(u_i, z_i, \bar{z}_i) = \prod_i^n \int d\omega_i \omega_i^{\Delta_i - 1} e^{-i\epsilon_i \omega_i u} S_n(u_i, z_i, \bar{z}_i) = \prod_i < \Phi_{h_i, \bar{h}_i}(u_i, z_i, \bar{z}_i) >$$

Future directions

* Higher point correlation functions → dynamical aspects of scattering

* Carroll ward identities and soft factorisation

* Interacting examples

Thank you