Higher-spin extensions of BMS algebra

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joint work with Blagoje Oblak, arXiv:2209.02253

Broad motivations

Despite the wide array of no-go theorems against interacting massless theories in Minkowski spacetime (of dimension 4 and higher),

- the cardinal importance of flat spacetime for physical applications
- and the old issue of string theory symmetries in the tensionless limit

can be taken as broad motivation for studying higher-spin symmetries in flat spacetime.

Higher-spin motivations

Two tantalising questions:

1. What might be an analogue of the singleton in flat spacetime?
2. What might be higher-spin symmetry algebra in flat spacetime?

Higher-spin motivations

Two tantalising questions:

1. What might be an analogue of the singleton in flat spacetime?
2. What might be higher-spin symmetry algebra in flat spacetime?

Some candidates:

- Higher-spin symmetry algebras in flat spacetime are known in dimension 3 (Afshar-Bagchi-Fareghbal-Grumiller-Rosseel, 2013; Gonzalez-Matulich-Pino-Troncoso, 2013; ...)
- Flat limit of AdS higher-spin algebra (Campoleoni-Pekar, 2021) implicitly defines a Carrollian limit of the singleton on $\mathscr{I}$ (XB-Campoleoni-Pekar, to appear)
- Wick-rotated singleton
- Sachs representation

Kinematical tools

Two main tools available:

1. BMS representation theory
   - BMS_4: Seminal works
     - Sachs (1962)
     - Series of papers by McCarthy (1972-1975)
   - BMS_3: Barnich & Oblak (2014-2015)
   - $BMS_{>4}$: ?
2. BMS intrinsic geometry
   - Seminal works (Penrose, Geroch, Ashtekar, . . .)
   - Modern view as conformal Carroll (Duval-Gibbons-Horvathy, 2014)

Goals

Two main goals:

1. Discuss two BMS analogues of Rac (= scalar singleton)
   1. Wick-rotated Rac
      + looks natural and familiar
      − seems *not* unitarisable
      − is *not* faithful representation of BMS (nor Poincaré) only of Lorentz
   2. Sachs representation
      − qualitatively $\neq$ Rac (less degenerate)
      + faithful *and* unitary representation of BMS
      + corresponds to irrep of Poincaré (massless scalar, radiation solutions)

Goals

Two main goals:

1. Discuss two BMS analogues of Rac (= scalar singleton)
2. Construct the corresponding higher-spin extension(s) of (extended and generalised) BMS algebra(s)
    - Contains the higher-spin extension of Poincaré algebra (XB, 2010)
    - Make contact with BMS Killing tensors obtained from the asymptotic symmetries of free massless higher-spin fields (Campoleoni-Francia-Heissenberg, 2017-2020)

    Linear structure → Algebra structure

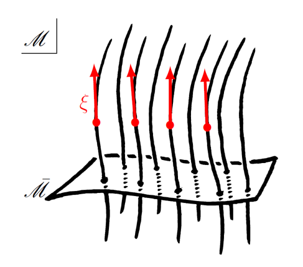
Outline

Principal bundle geometry

Fundamental vector field

Fundamental vector field: (essentially) equivalent data

- Nowhere vanishing vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on a manifold $\mathscr{M}$
- Congruence of parametrised curves from $\mathbb{R}$ to $\mathscr{M}$
- Principal $\mathbb{R}$-bundle $\mathscr{M}$ with fundamental vector field ξ



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- The curves are the integral lines of the fundamental vector field; they are also the orbits of the $\mathbb{R}$-action on $\mathscr{M}$.
- The space $\bar{\mathscr{M}}$ of such orbits is the base manifold of the principal bundle

$$\bar{\mathscr{M}} = \mathscr{M} / \mathbb{R}$$

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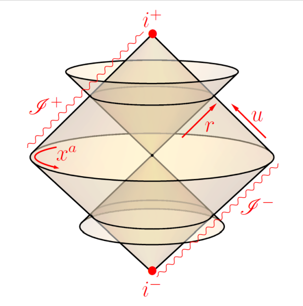
Local expression: there exist a coordinate system (u, x^a) such that

- Fundamental vector field $\xi = \frac{\partial}{\partial u}$
- Curves $x^a = x^a_0$ parametrised by u
- $\mathbb{R}$-action $u \to u - u_0$ $(u_0 \in \mathbb{R})$
- Fibration $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}} : (u, x^a) \mapsto x^a$

Fundamental vector field

Example: Future null infinity $\mathscr{I}^+_{d+1}$ at the conformal boundary of compactified Minkowski spacetime

- Coordinates (u, x^a) on $\mathscr{I}^+_{d+1} \cong \mathbb{R} \times S^d$
- Fundamental vector field $\xi = \frac{\partial}{\partial u}$ is null
- Null rays generating the cone
- $\mathbb{R}$-action $u \to u - u_0$ $(u_0 \in \mathbb{R})$
- Fibration $\pi : \mathscr{I}^+_{d+1} \twoheadrightarrow S^d : (u, x^a) \mapsto x^a$



Projection on the base manifold

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$
with fundamental vector field ξ.

- **Projectable vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = f\,\xi$
  where $f \in C^\infty(\mathscr{M})$
- **Super-projectable vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = f\,\xi$
  with $\mathcal{L}_\xi f = 0$
- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$

Remark: Vertical vector fields, i.e. $X = h\,\xi$ with $h \in C^\infty(\mathscr{M})$, are necessarily projectable.

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- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$

Remark: Projectable vector fields are infinitesimal automorphisms of the fibre bundle

$$u' = u + \epsilon\, F(u,x)\,, \quad x' = x + \epsilon\, G(x)\,.$$

The latter can be interpreted as “Carrollian diffeomorphisms”.

Projection on the base manifold

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- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$

Remark: Invariant vector fields are infinitesimal automorphisms of the principal $\mathbb{R}$-bundle,

$$u' = u + \epsilon\, F(x)\,, \quad x' = x + \epsilon\, G(x)\,.$$

Projection on the base manifold

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- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$

Example: Invariant vertical vector fields ($X = h\,\xi$ with $\mathcal{L}_\xi h = 0$) generate vertical automorphisms of the principal $\mathbb{R}$-bundle

$$u' = u + f(x)\,, \quad x' = x\,,$$

which are interpreted as "supertranslations" in the BMS context.

Pullback from the base manifold

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$ with fundamental vector field ξ.

- **Invariant differential one-form:** $A \in \Omega^1(\mathscr{M})$ such that $\mathcal{L}_\xi A = 0$
- **Horizontal differential one-form:** $A \in \Omega^1(\mathscr{M})$ such that $A \cdot \xi = 0$
- **Basic differential one-form:** invariant & horizontal
  $\Leftrightarrow A = \pi^* \bar{A}$ with $\bar{A} \in \Omega^1(\bar{\mathscr{M}})$

These definitions generalise to covariant tensor fields (e.g. the Carrollian metric).

Carrollian geometry

Carrollian structure :

Field of observers

&

Carrollian metric

Timelike metric structure

Field of observers: fundamental vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on the spacetime manifold $\mathscr{M}$ fibred over the absolute space $\bar{\mathscr{M}}$.

Provides a distinction between the type of vectors in Carroll geometry:

$$\begin{cases} V^\mu = f\,\xi^\mu \quad \text{with} \quad \begin{cases} f \neq 0 & \textbf{Timelike (or Vertical)} \\ f > 0 & \textbf{Future-oriented} \end{cases} \\ V^\mu \neq f\,\xi^\mu \qquad\qquad\qquad \textbf{Spacelike} \end{cases}$$

Timelike metric structure

Field of observers: fundamental vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on the spacetime manifold $\mathscr{M}$ fibred over the absolute space $\bar{\mathscr{M}}$.

An affine parameter u of this congruence of Carroll worldlines (i.e. $\xi = \partial/\partial u$) is a Carroll time.

Spacelike metric structure

Carrollian metric: Positive semi-definite metric γ on the spacetime $\mathscr{M}$ whose kernel is spanned by the fundamental vector field

$$\begin{cases} \gamma_{\mu\nu} V^\mu W^\nu \geqslant 0 \\ \gamma_{\mu\nu} V^\mu = 0 \quad \Leftrightarrow \quad V^\mu = f\, \xi^\mu \end{cases}$$

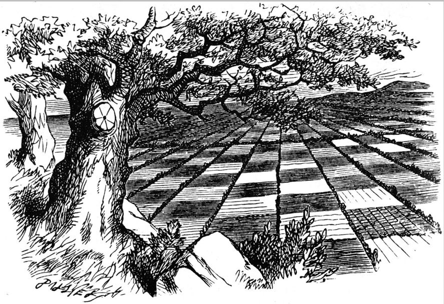
Remark: There is a one-to-one correspondence between

- *invariant* Carrollian metrics $\gamma_{\mu\nu}$ on $\mathscr{M}$ and
- Riemannian metrics $\bar{\gamma}_{ab}$ on the base $\bar{\mathscr{M}}$

since an invariant Carrollian metric is basic, $\gamma = \pi^* \bar{\gamma}$.

Spacelike metric structure

An invariant Carrollian metric allows to measure distances and angles on the base manifold $\bar{\mathscr{M}}$.



Carrollian structure

Definition (Henneaux, 1979)

(Invariant) Carrollian structure: two data

1. Field of observers
2. (Invariant) Carrollian metric

One will focus on *invariant* Carrollian structures, so this assumption will sometimes be implicitly assumed from now on.

Example : Future null infinity $\mathscr{I}^+$ in Bondi frame

- Coordinates (u, x^a) on $\mathscr{I}^+ \cong \mathbb{R} \times S^d$
- Null vector field $\xi = \frac{\partial}{\partial u}$
- Carrollian metric = pullback of the metric on the unit sphere

$$ds^2 = \gamma_{ab}(x)\, dx^a dx^b = d\ell^2_{S^d}$$

Carrollian isometries

Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$

Remark: For an invariant Carrollian structure, these Carrollian isometries project onto isometries of the Riemannian metric on the base.

Carrollian isometries

Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$

Remark: The algebra of Carrollian isometry generators has a structure of semi-direct sum

$$\mathfrak{carr\,isom}(\mathscr{M}) \cong \mathfrak{isom}(\bar{\mathscr{M}}) \ni C^\infty(\bar{\mathscr{M}})$$

Bondi-Metzner-Sachs as Conformal Carroll

Conformal Carrollian structure

Definition (Penrose, 1965; Geroch, 1977)

Conformal Carrollian structure: equivalence class of Carrollian structures with respect to the equivalence relation

1. Field of observers $\xi \sim \Omega^{-1}\xi$
2. (Invariant) Carrollian metrics $\gamma \sim \Omega^2\gamma$ (with $\mathcal{L}_\xi\Omega = 0$)

Conformal Carrollian isometries

Conformal Carrollian isometry: diffeomorphism of $\mathscr{M}$ such that

1. (Conformal rescaling) $\xi' = \Omega^{-1}\xi$
2. (Conformal isometry) $\gamma' = \Omega^2\gamma$

with $\mathcal{L}_\xi\Omega = 0$.

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Example: For null infinity $\mathscr{I}$

Theorem ((Penrose, 1965) revisited (Duval-Gibbons-Horvathy, 2014))

BMS transformations = Conformal Carrollian isometries

Conformal Carroll-Killing vector field

Conformal Carroll-Killing vector field: $X \in \mathfrak{X}(\mathscr{M})$ such that

1. (super-projectable) $\mathcal{L}_X \xi = f\,\xi$ with $\mathcal{L}_\xi f = 0$
2. (conformal Killing) $\mathcal{L}_X \gamma = -2f\,\gamma$

Conformal Carroll-Killing vector field

Consider an invariant conformal Carrollian structure.

The projection $\bar{X} = \pi_*(X)$ on the base $\bar{\mathscr{M}}$ of a conformal Carroll-Killing vector field X on $\mathscr{M}$ is a conformal Killing vector field $\bar{X}$ on $\bar{\mathscr{M}}$.

Conformal Carroll-Killing vector field: $X \in \mathfrak{X}(\mathscr{M})$ such that

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2. (conformal Killing) $\mathcal{L}_{\bar{X}} \bar{\gamma} = -2\bar{f}\bar{\gamma}$

Conformal Carroll-Killing vector field

The conformal Carroll-Killing vector fields on $\mathscr{I}_{d+1} \cong \mathbb{R} \times S^d$ *span the (extended) BMS algebra*

$$(\mathfrak{e})\mathfrak{bms}_{d+2} = \mathfrak{conf}(S^d) \in C^\infty(S^d)$$

where the elements of $C^\infty(S^d)$ *transform as densities of weight* $-1/d$ *under* $\mathfrak{conf}(S^d)$.

Generalised BMS geometry

Campiglia-Laddha structure

Let $(\mathscr{M}, \xi)$ be a principal $\mathbb{R}$-bundle and assume $\mathscr{M}$ is orientable. Then an **invariant volume form** is a nowhere-vanishing top-form $\varepsilon \in \Omega^{d+1}(\mathscr{M})$ such that $\mathcal{L}_\xi \epsilon = 0$.

Campiglia-Laddha structure: equivalence class $[\xi, \varepsilon]$ of pairs (ξ, ε) with respect to the equivalence relation

1. Field of observers $\xi \sim \Omega^{-1}\xi$
2. (Invariant) volume forms $\varepsilon \sim \Omega^{d+1}\varepsilon$ (with $\mathcal{L}_\xi \Omega = 0$)

Generalised BMS transformations

Generalised conformal maps: diffeomorphism of $\mathscr{M}$ such that

1. $\xi' = \Omega^{-1}\xi$
2. $\varepsilon' = \Omega^{d+1}\varepsilon$

with $\mathcal{L}_\xi\Omega = 0$.

Example: The generalised conformal maps on $\mathscr{I}_{d+1} \cong \mathbb{R} \times S^d$ span the generalised BMS algebra

$$\mathfrak{gbms}_{d+2} = \mathfrak{X}(S^d) \ni C^\infty(S^d)$$

where the elements of $C^\infty(S^d)$ transform as densities of weight $-1/d$.

Generalised BMS transformations

This leads to the hierarchy

$$\mathfrak{iso}(d+1,1) \subset \mathfrak{bms}_{d+2} \subseteq \mathfrak{gbms}_{d+2} \subset \mathfrak{X}_{\mathsf{spro}}(\mathscr{I}_{d+1}) \subset \mathfrak{X}_{\mathsf{pro}}(\mathscr{I}_{d+1})$$

Higher-spin extension

Towards higher-spin extension

Higher-spin recipe:

- Vector field → Differential operator

$$\mathfrak{X}(\mathscr{M}) \to \mathcal{D}(\mathscr{M})$$

- Higher-spin algebra
  ≡ universal enveloping algebra / annihilator
  of a representation

$$(\,\mathfrak{g}, V\,) \quad \longrightarrow \quad \mathfrak{hg}(V) \equiv \mathcal{U}(\mathfrak{g})\,/\,\text{Annihilator(V)}$$

Towards higher-spin extension

Main idea:

Consider Sachs representation V of BMS algebra $\mathfrak{g}$

Carrollian Hermitian form

Consider a Campiglia-Laddha structure $[\xi, \varepsilon]$. Pick a representative (ξ, ε) and introduce the Hermitian form

$$\langle\!\langle \psi_1 \,|\, \psi_2 \rangle\!\rangle = i \int_{\mathscr{M}} \psi_1^* \, d\psi_2 \wedge \mathcal{V} = i \int_{\mathscr{M}} \psi_1^* \, \mathcal{L}_\xi \psi_2 \, \varepsilon$$

where $\mathcal{V} = \pi^* \bar{\varepsilon}$ is the pullback of the volume form $\bar{\varepsilon}$ of the base $\bar{\mathscr{M}}$.

The many faces of Sachs representation

Definition (Intrinsic)

Sachs: unitary representation spanned by square-integrable densities $\psi \in C^\infty(\mathscr{I}_{d+1})$ of weight $1/2$ and positive Carrollian energy endowed with the Hermitian product

$$\langle\!\langle \psi_1 \,|\, \psi_2 \rangle\!\rangle = i \int_{\mathscr{I}} \psi_1^*\, d\psi_2 \wedge \mathcal{V} = i \int_{\mathscr{I}} \psi_1^*\, \mathcal{L}_\xi \psi_2\, \varepsilon$$

Local expression: Sachs (1962)

$$\langle\!\langle \psi_1 \,|\, \psi_2 \rangle\!\rangle = i \int du\, d^d x \,\sqrt{\gamma}\; \psi_1^* \,\frac{\partial \psi_2}{\partial u} = \langle \psi_1 \,|\, \hat{H} \,|\, \psi_2 \rangle$$

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$$\langle\!\langle \psi_1 \mid \psi_2 \rangle\!\rangle = i \int_{\mathscr{I}} \psi_1^* \, d\psi_2 \wedge \mathcal{V} = i \int_{\mathscr{I}} \psi_1^* \, \mathcal{L}_\xi \psi_2 \, \varepsilon$$

Carrollian physics interpretation:

Matrix element of Carroll Hamiltonian $\langle\!\langle \psi_1 \mid \psi_2 \rangle\!\rangle = \langle \psi_1 \mid \hat{H} \mid \psi_2 \rangle$

The many faces of Sachs representation

Definition (Asymptotic)

Sachs: representation spanned by boundary data of radiation solutions to d'Alembert equation in position space

$$\begin{cases} \Box\,\Phi(r,u,\mathbf{x}) = 0 \\ \psi(u,\mathbf{x}) = \lim_{r\to\infty}\left[\, r^{\frac{d}{2}}\,\Phi(r,u,\mathbf{x})\,\right] \end{cases}$$

where (r,u,x^a) are Bondi coordinates on Minkowski spacetime $\mathbb{R}^{d+1,1}$.

The many faces of Sachs representation

Definition (Celestial/Carrollian)

Sachs: Fourier transform over the energy of solutions to d'Alembert equation in impulsion space

$$\psi(u,\mathbf{x}) = \int_{-\infty}^{+\infty} \frac{dE}{\sqrt{2\pi}}\, E^{\frac{d}{2}-1}\, e^{-iEu}\, \Phi(E,\mathbf{x})\,,$$

where $E = |\mathbf{q}| > 0$ and $\mathbf{x} = \mathbf{q}/|\mathbf{q}| \in S^d$.

The many faces of Sachs representation

Remarks: In this way, the usual Hermitian form for the spin-zero massless UIR of Poincaré

$$\langle\Phi_1|\Phi_2\rangle = \int_{\mathbb{R}^{d+1}} \frac{d^{d+1}\mathbf{q}}{|\mathbf{q}|}\, \Phi_1^*(\mathbf{q})\Phi_2(\mathbf{q})$$

identifies with the Carrollian Hermitian form

$$\langle\Phi_1|\Phi_2\rangle = \langle\!\langle \psi_1 \mid \psi_2 \rangle\!\rangle = i \int du\, d^d x\, \sqrt{\gamma}\, \psi_1^* \, \frac{\partial \psi_2}{\partial u}$$

The many faces of Sachs representation

Note that the Carrollian Hermitian form is manifestly invariant under the groups:

- Poincaré: Obvious from the identification
- (Extended) BMS: This confirms that any UIR of Poincaré lifts to an UIR of BMS [corollary of McCarthy (1974-1975)].
- Generalised BMS: The only necessary data is a Campiglia-Laddha structure $[\xi, \varepsilon]$.
- Carrollian diffeomorphisms: The kinematic structure on null infinity is weak enough to automatically admit this huge symmetry enhancement.

Symmetries of Sachs representation

Consider a Campiglia-Laddha structure $[\xi,\varepsilon]$. Pick a representative (ξ,ε) and introduce the non-degenerate Hermitian form

$$\langle\psi_1|\psi_2\rangle := \int_{\mathscr{M}} \psi_1^*\,\psi_2\,\varepsilon \qquad \forall\psi\in C^\infty(\mathscr{M})$$

which is nothing but the standard inner product of complex-valued "wavefunctions" on $\mathscr{M}$ with volume form ε.

$\Longrightarrow$ Hermitian conjugation with respect to this Hermitian form

$$\langle D^\dagger\psi_1,\psi_2\rangle := \langle\psi_1, D\psi_2\rangle$$

for any differential operator $D\in\mathcal{D}(\mathscr{M})$. In particular for vector field $X\in\mathfrak{X}(\mathscr{M})$

$$X^\dagger = -X^* - \text{div}(X^*)\,, \quad \text{with} \quad \mathcal{L}_X\varepsilon = \text{div}(X)\,\varepsilon\,.$$

Symmetries of Sachs representation

The Carrollian Hermitian form can then be written as

$$\langle\!\langle \psi_1 | \psi_2 \rangle\!\rangle = i \langle \psi_1 | \xi | \psi_2 \rangle \,.$$

Definition (Hermiticity condition)

Symmetries of the Carrollian Hermitian form: differential operator $D \in \mathcal{D}(\mathscr{M})$ such that the infinitesimal transformations $\delta\psi = iD\psi$ preserve the Carrollian Hermitian form for all ψ, i.e. a differential operator that is Hermitian in the sense

$$\xi \circ D = D^\dagger \circ \xi \,.$$

Symmetries of Sachs representation

Examples:

- a zeroth order operator is Hermitian in the above sense iff it is a real invariant function.
- a first-order operator $\hat{X} = X + h$ is a Hermitian symmetry iff X is a purely imaginary projectable vector field while $\text{Im}(h) = \frac{1}{2}\,\text{div}\,X$ and $\text{Re}(h)$ is an invariant function.

Remark: Thus we recover that *all* bundle automorphisms preserve the Carrollian Hermitian form provided wavefunctions are Carrollian densities with weight $1/2$. (This includes all super-projectable vector fields, which in turn includes all generalised conformal vector fields and of course all Carrollian conformal vector fields.)

Symmetries of Sachs representation

Theorem (Algebra of Hermitian symmetries)

The Lie algebra $\mathcal{H}_{\text{sym}}(\mathscr{M})$ of all symmetries of the Carrollian Hermitian form is isomorphic to the semi-direct sum

$$\mathcal{H}_{\text{sym}}(\mathscr{M}) \cong \mathcal{H}(\bar{\mathscr{M}}) \in \mathcal{H}_{\text{vsym}}(\mathscr{M})$$

of the Lie algebra $\mathcal{H}(\bar{\mathscr{M}})$ of differential operators on the base that are Hermitian with respect to

$$(\bar{\psi}_1|\bar{\psi}_2) := \int_{\bar{\mathscr{M}}} \bar{\psi}_1^* \, \bar{\psi}_2 \, \bar{\varepsilon} \qquad \forall \bar{\psi} \in C^\infty(\bar{\mathscr{M}}),$$

and the Lie ideal $\mathcal{H}_{\text{vsym}}(\mathscr{M})$ of all vertical higher symmetries.

Symmetries of Sachs representation

Theorem (Algebra of super-projectable Hermitian symmetries)

The Lie algebra $\mathcal{H}_{\mathsf{spro}}(\mathscr{M})$ of super-projectable higher symmetries of the Carrollian Hermitian form is a tensor product

$$\mathcal{H}_{\mathsf{spro}}(\mathscr{M}) \cong \mathcal{H}(\bar{\mathscr{M}}) \otimes U_+\big(i\mathfrak{gl}(1)\big)$$

where $U_+\big(i\mathfrak{gl}(1)\big)$ is the real form of the universal enveloping algebra which is spanned by Weyl-ordered polynomials in $i\,\xi$ and $i\,\eta$ where η is an Euler vector field.

Higher-spin extension of each symmetry algebra

Higher-spin recipe:

higher-spin algebras ≡ universal enveloping algebra / annihilator

$$\mathfrak{hg} = \mathcal{U}(\mathfrak{g})\,/\,\text{Ann}(\text{Sachs})$$

Higher-spin extension of each symmetry algebra

Any real Lie algebra $\mathfrak{g}$ in the hierarchy

$$\mathfrak{iso}(d+1,1) \subset \mathfrak{bms}_{d+2} \subseteq \mathfrak{gbms}_{d+2}$$

admits a higher-spin extension $\mathfrak{hg}$, built as the real Lie subalgebra of higher symmetries of the Sachs representation spanned by Weyl-ordered products of the generators of $\mathfrak{g}$.

This leads to the table of inclusions

$$\begin{array}{ccccccc} \mathfrak{hiso}(d+1,1) & \subset & \mathfrak{hbms}_{d+2} & \subseteq & \mathfrak{hgbms}_{d+2} & \subset & \mathcal{D}_{\text{spro}}(\mathscr{I}_{d+1}) \\ \cup & & \cup & & \cup & & \cup \\ \mathfrak{iso}(d+1,1) & \subset & \mathfrak{bms}_{d+2} & \subseteq & \mathfrak{gbms}_{d+2} & \subset & \mathfrak{X}_{\text{spro}}(\mathscr{I}_{d+1}) \end{array}$$

Higher-spin algebras as asymptotic symmetries

Our two candidate higher-spin algebras of asymptotic symmetries are

- the partially-massless Minkowski algebra $\mathfrak{hiso}(d+1,1)$
- the generalised BMS higher-spin algebra $\mathfrak{hgbms}_{d+2}$

They are, respectively, higher-spin extensions of Poincaré and generalised BMS algebras based on Sachs representation.

Higher-spin algebras as asymptotic symmetries

Remark: Trace conditions are a recurring problem of tentative higher-spin algebras in Minkowski spacetime, because they preclude the interpretation of traceful tensors as algebras of global symmetries of massless gauge fields.

This suggests to look for "exotic" higher-spin gravity theories, e.g. "partially-massless-like" gauge fields on Minkowski spacetime (Campoleoni-Pekar, 2021).

Unfortunately, exotic theories on Minkowski spacetime including such partially-massless-like fields do not seem to be unitary (in fact, partially-massless fields are only unitary on dS, not on AdS).

Higher-spin algebras as asymptotic symmetries

Nevertheless, it is tempting to conjecture that an exotic higher-spin gravity around Minkowski space, whose spectrum is a tower of partially-massless-like fields of all spins and all odd depths, admits as algebra of asymptotic symmetries

- the partially-massless Minkowski algebra
- the generalised BMS higher-spin algebra

for suitable fall-off conditions that generalise (respectively) to higher depths

- the strong fall-offs in (Campoleoni-Francia-Heissenberg, 2017)
- the weak ones in (Campoleoni-Francia-Heissenberg, 2020)

of the massless case.

Conclusion

Summary of results

- Identification of Sachs representation
  as a possible analogue (in Minkowski spacetime)
  to the scalar singleton (in Anti de Sitter spacetime)
- Sachs Hermitian product
  - Geometric and manifestly BMS-invariant formulation
  - Relation with Wigner Hermitian product
- Definition of possible higher-spin extensions
  of generalised BMS algebra
- Relation with known asymptotic symmetries
  of higher-spin gauge fields

Thank you for your attention

Thank you to the organizers for this wonderful Carrollian workshop!

Credits

All illustrations of Alice are from
John Tenniel (1820-1914)