# Carrollian and celestial spaces at infinity

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CARROLL WORKSHOP
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# Plan

- Homogeneous carrollian spacetimes
  - José Figueroa-O'Farrill, Ross Grassie
  - 1809.01224, 1905.00034
  - Introduce homogeneous spaces
  - Review all homogeneous carrollian spacetimes
  - Connect to interesting recent advances
- Carrollian and celestial spaces at infinity
  - José Figueroa-O'Farrill, Emil Have, Jakob Salzer
  - 2112.03319
- Please see these papers for references
- Feel free to ask questions

# Outline

# Maximally symmetric spaces

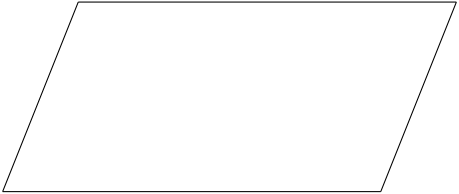
- Maximally symmetric spaces
  - Riemannian: Euclidean, Sphere, Hyperbolic
  - Lorentzian: Minkowski, de Sitter, anti-de Sitter
- Properties (intuitive)
  - Maximal amount of symmetry (e.g., Killing vectors)
  - Every point looks the same
  - Symmetry connects each point
- Properties have nice consequences
  - Because every point of the space is "the same" a lot can be learned by just analyzing any specific point
  - More complicated problems can be reduced to linear algebra
    - Similar to Lie group $G \to$ Lie algebra $\mathfrak{g}$
- Why are they important?
  - Backgrounds for physics
  - Vacuum for gravitational theories (empty universe), c.f., Carrollian gravity talks
  - Starting point for gauging

# Outline

# What is a homogeneous space?

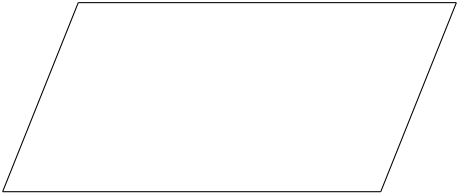
- Two elements
  1. Smooth space (Manifold)
  2. Continuous symmetries acting on the space such that every point is connected continuously to every other point using such a symmetry (transitive Lie group action)

# Example: (Plane, rotations + translations)



- Space: Two dimensional plane ($\mathbb{R}^2$)
- Symmetries: Rotations and translations ($ISO(2)$)
  - Connect all points ✓
- Space + Symmetries $\rightarrow$ Homogeneous space
- Often interested in invariants
  - Nondegenerate riemannian metric $ds^2 = dx^2 + dy^2$

# Example: (Plane, rotations + translations)



- Equivalent way: Symmetries first
- Symmetries: Rotations and translations ($ISO(2)$)

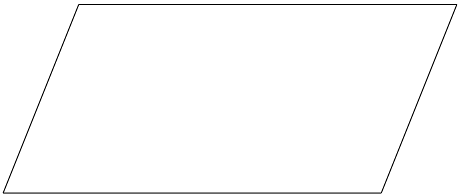
$$(R_1, \vec{a}_1) \cdot (R_2, \vec{a}_2) = (R_1 R_2, R_1 \vec{a}_2 + \vec{a}_1)$$

- Subset of symmetries that close: Rotations

$$\text{Space} = \frac{\text{Rotations + translations}}{\text{“ignore” rotations}} \left( = \frac{ISO(2)}{SO(2)} \right)$$

- “Ignore” it is important that the rotations are a subgroup

# Example: (Plane, rotations + translations)



- Something interesting has happened
- To specify the homogeneous space specify:
  - Symmetries
  - Subgroup of symmetries
- Only symmetries!
- Homogeneous space:
  - (Manifold, Symmetries) $\rightarrow$ Klein pair (Symmetries, Subgroup of symmetries)
- Equivalent because

$$\text{Space} = \frac{\text{Symmetries}}{\text{Subgroup of symmetries}}$$

# Minkowski = (Poincaré, Lorentz)

- Try to understand Minkowski space using this concept
- Symmetries: Rotations, Lorentz-boosts, spatial and time translations (Poincaré)
- Subgroup of Symmetries: Rotations, Lorentz-boosts (Lorentz)
- Homogeneous space specified: $(\text{Poincaré}, \text{Lorentz})$
- Minkowski space

$$\frac{\text{Poincaré}}{\text{Lorentz}} = \frac{\text{Rotations, boosts, spatial and time translations}}{\text{Rotations, boosts}}$$

- Sanity check: Spatial and time translations = dimension of spacetime
- Invariant: Nondegenerate lorentzian metric

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

- Symmetries of invariant structure $\mathcal{L}_\xi g = 0 \Rightarrow$ Poincaré

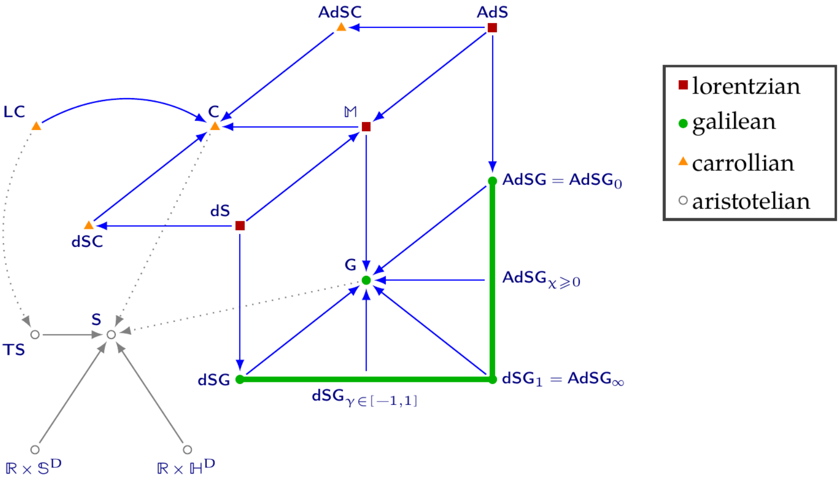
# What are spatially isotropic spacetimes?

1. Restrict the Lie algebra to the following generators
   - Rotations $\boldsymbol{J}$ ($J_{ab}$)
   - "Boosts" $\boldsymbol{B}$ ($B_a$)
   - "Spatial translations" $\boldsymbol{P}$ ($P_a$)
   - Time translations $H$

$$[\boldsymbol{J},\boldsymbol{J}] = \boldsymbol{J} \quad [\boldsymbol{J},\boldsymbol{B}] = \boldsymbol{B} \quad [\boldsymbol{J},\boldsymbol{P}] = \boldsymbol{P} \quad [\boldsymbol{J},H] = 0$$

2. Classify all possible Lie algebras satisfying this commutation relations
   - This is possible due the rotational invariance
3. Classify all homogeneous spaces for each Lie algebra
   - Search Lie subalgebras $\{\boldsymbol{J},\boldsymbol{B}\}$
   - At this step the "boosts" $\rightarrow$ boosts
4. Search for invariants $\rightarrow$ characterize the spacetime
5. Check that boosts are noncompact

# Generic dimensions: $[\boldsymbol{J}, \boldsymbol{J}] = \boldsymbol{J}$, $[\boldsymbol{J}, \boldsymbol{B}] = \boldsymbol{B}$, $[\boldsymbol{J}, \boldsymbol{P}] = \boldsymbol{P}$



# Outline

# Homogeneous Carrollian spaces

- They all share (are kinematical spacetimes)
  - Rotations $\boldsymbol{J}$ ($J_{ab}$)
  - Carroll boosts $\boldsymbol{B}$ ($B_a$)
  - Spatial translations $\boldsymbol{P}$ ($P_a$)
  - Time translations $H$
  - Carrollian invariants: degenerate riemannian metric, non-vanishing vector field
- $\mathfrak{g} = \langle \boldsymbol{J}, \boldsymbol{B}, H, \boldsymbol{P} \rangle$, subalgebra $\mathfrak{h} = \langle \boldsymbol{J}, \boldsymbol{B} \rangle$

$$[\boldsymbol{J}, \boldsymbol{J}] = \boldsymbol{J} \qquad [\boldsymbol{J}, \boldsymbol{B}] = \boldsymbol{B} \qquad [\boldsymbol{J}, \boldsymbol{P}] = \boldsymbol{P} \qquad [\boldsymbol{J}, H] = 0$$

  - AdS Carroll $\Lambda < 0$, dS Carroll $\Lambda > 0$, Carroll $\Lambda \to 0$ spacetime

$$[\boldsymbol{B}, \boldsymbol{P}] = H \qquad [H, \boldsymbol{P}] = -\Lambda \boldsymbol{B} \qquad [\boldsymbol{P}, \boldsymbol{P}] = -\Lambda \boldsymbol{J}$$

  - Lightcone (Lie algebra $\mathfrak{so}(4,1)$)

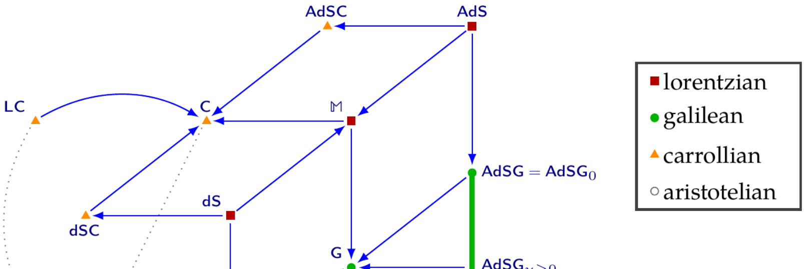
$$[\boldsymbol{B}, \boldsymbol{P}] = H + \boldsymbol{J} \qquad [\boldsymbol{B}, H] = \boldsymbol{P} \qquad [H, \boldsymbol{P}] = -\boldsymbol{P}$$

- All realized as null hypersurfaces of lorentzian manifolds

# Cosmological carrollian spaces

- The two carrollian "limits" of de Sitter space
  - Empty universe with no preferred place → Vacuum homogeneous space

1. dS → LC
   - Lightcone: No limit in the sense of a contraction
   - Underlying symmetry algebra is the same as for dS $\mathfrak{so}(4,1)$, but the homogeneous space changes
   - Easy to visualize
2. dS → dS Carroll
   - Limit in the sense of a IW contraction of symmetries



# Observations

- One Lie algebra can have various spacetimes:
  - $\mathfrak{so}(4,1)$: de Sitter $\leftrightarrow$ Light cone
  - $\mathfrak{iso}(3,1)$: AdS-Carroll $\leftrightarrow$ Minkowksi space
- The symmetries of the homogeneous space $\mathfrak{g}$ are not necessarily the same as the symmetries of the invariant structure $\mathcal{L}_\xi(\text{invariants}) = 0$
  - Carroll Lie algebra $\leftrightarrow$ infinite dimensional algebra

# Outline

# Observation

- Motivation: Holography
- $AdS_4/CFT_3$
  - Both based on $\mathfrak{so}(3,2)$ but have different dimension
- Isometries of the vacuum in the bulk
  $\leftrightarrow$ Symmetries of the vacuum of the boundary
  - Not asymptotic symmetries
  - True in $2+1$ dim.
- Simple: Same symmetries $+$ different dimension $+$ vacuum should be homogeneous

# Plan

- $\mathrm{AdS}_4$: homogeneous spaces of $\mathfrak{so}(3,2)$ with $d \leq 3$
- This means we have to search for all subalgebras of $\mathfrak{so}(3,2)$ with $d \geq 7$
- Analyze if they exist and are relevant
- Powerful:
  - Nearly no input: Only symmetries (Lie algebra + Lie subalgebra)
  - Precise (no room to wiggle, one might come back empty-handed)
  - Coordinate invariant
  - Complete with the given assumptions and if one is able to do it
- Universal: This approach is not restricted to any specific geometry
- Unifying: If there are more possible interesting options one should obtain all of them
- Provides possibly underlying symmetries and geometries, but more work is needed.
  - Necessary, not sufficient

literature

t carrollian and t celestial

Carrollian and celestial spaces at infinity

#1

José Figueroa-O'Farrill, Emil Have, Stefan Prohazka, Jakob Salzer (Dec 6, 2021)

e-Print: 2112.03319 [hep-th]

pdf cite 3 citations

A Carrollian Perspective on Celestial Holography

#2

Laura Donnay (Vienna, Tech. U.), Adrien Fiorucci, Yannick Herfray, Romain Ruzziconi (Vienna, Tech. U.) (Feb 9, 2022)

e-Print: 2202.04702 [hep-th]

pdf cite 2 citations

Scattering Amplitudes: Celestial and Carrollian

#3

Arjun Bagchi (Indian Inst. Tech., Kanpur), Shamik Banerjee, Rudranil Basu, Sudipta Dutta (Feb 16, 2022)

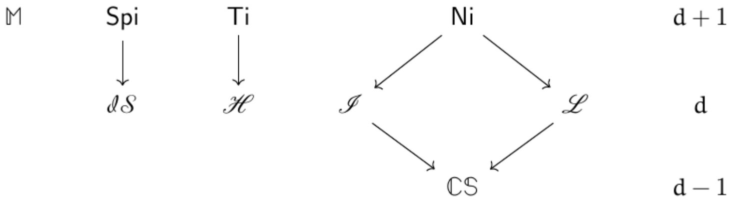
e-Print: 2202.08438 [hep-th]

# Check of proposal

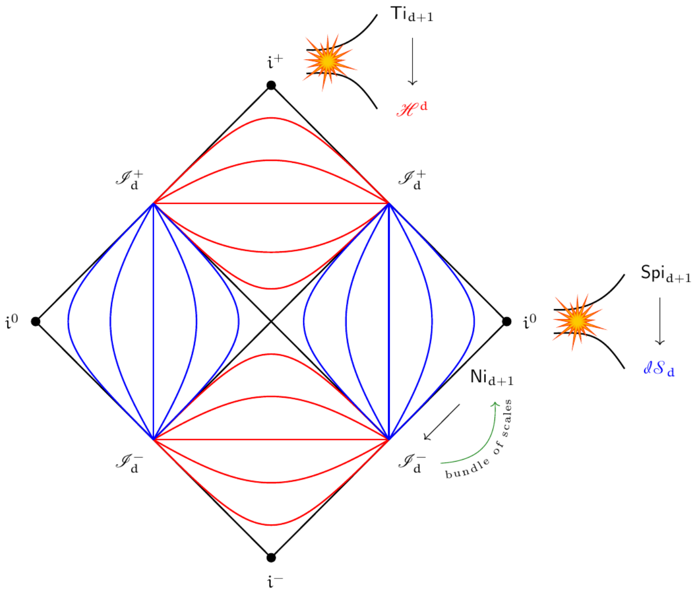
- $\mathrm{AdS}_4$: homogeneous spaces of $\mathfrak{so}(3,2)$ with $d \leq 3$
- $\mathrm{dS}_4$: homogeneous spaces of $\mathfrak{so}(4,1)$ with $d \leq 3$
- $\mathbb{M}_4$: homogeneous spaces of $\mathfrak{iso}(3,1)$ with $d \leq 3$

- This means Lie subalgebras of $d \geq 7$

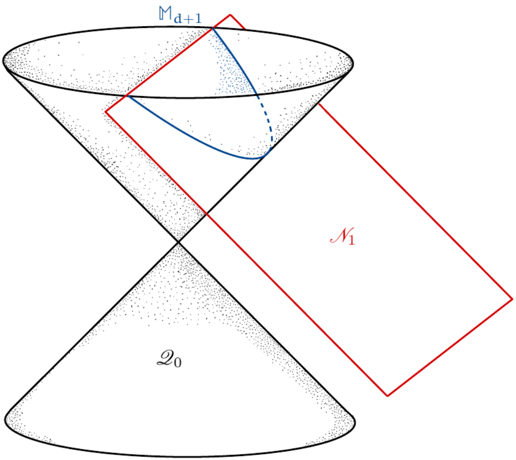
# General picture



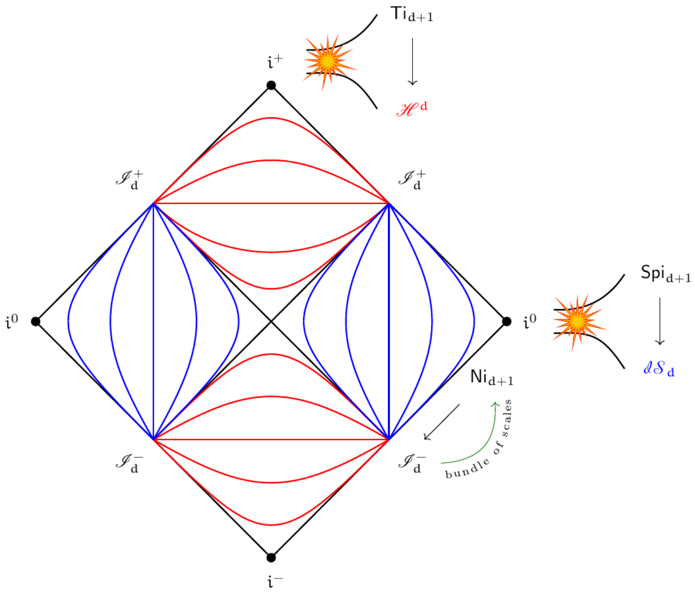
# General picture



# Embedding Minkowski space



# General picture



# Spi

To summarize, the result of the blowing up of $i^0$ is a 4-manifold which has the structure of a principal fibre bundle: The base space is the unit timelike hyperboloid in the tangent space of $i^0$, and the structure group is the additive group of reals. *This S will be called Spi—spatial infinity*. From its very construction, $S$ inherits two tensor fields: a covariant, second rank, symmetric (degenerate) tensor field $h_{ab}$, the pullback to $S$ of the natural metric on the hyperboloid $\mathcal{K}$; and a vertical vector field $v^a$, the generator of the natural, one-parameter family of diffeomorphisms on $S$ induced by its structure group.[32]

# Ni


- Invariant structure: doubly-carrollian
  - two nowhere-vanishing vectorfields
  - doubly degenerate metric
- Symmetries of the invariant structure: $BMS_4$
- Not conformal carrollian symmetries, but proper ones $\mathcal{L}_\xi(\mathrm{inv}) = 0$
- Explanation: Bundle of scales of $\mathscr{I}_3$

With $i = 1, 2$, $\eta_{ij} = \delta_{ij}$ and $\eta_{+-} = \eta_{-+} = 1$

$$\begin{aligned}
[L_{ij}, L_{k\ell}] &= \delta_{jk} L_{i\ell} - \delta_{ik} L_{j\ell} - \delta_{j\ell} L_{ik} + \delta_{i\ell} L_{jk} \\
[L_{ij}, L_{\pm k}] &= \delta_{jk} L_{\pm i} - \delta_{ik} L_{\pm j} \\
[L_{+i}, L_{-j}] &= -L_{ij} - \delta_{ij} L_{+-} \\
[L_{+-}, L_{\pm i}] &= \pm L_{\pm i} \\
[L_{ij}, P_k] &= \delta_{jk} P_i - \delta_{ik} P_j \\
[L_{\pm i}, P_j] &= \delta_{ij} P_{\pm} \\
[L_{\pm i}, P_{\mp}] &= -P_i \\
[L_{+-}, P_{\pm}] &= \pm P_{\pm}.
\end{aligned}$$

$$\mathfrak{g} = \mathfrak{iso}(3,1) = \langle L_{ij}, L_{+i}, L_{-i}, L_{+-}, P_i, P_+, P_- \rangle_{i,j=1,2}$$

$$\begin{aligned}
\mathfrak{h}_{Ni} &= \langle L_{ij}, L_{-i}, P_i, P_- \rangle \\
\mathfrak{h}_{\mathfrak{J}} &= \langle L_{ij}, L_{-i}, P_i, P_-, L_{+-} \rangle \\
\mathfrak{h}_{\mathsf{LC}} &= \langle L_{ij}, L_{-i}, P_i, P_-, P_+ \rangle \\
\mathfrak{h}_{CS} &= \langle L_{ij}, L_{-i}, P_i, P_-, L_{+-}, P_+ \rangle
\end{aligned}$$

# Outline

# Summary and outlook

- There are 4 homogeneous carrollian spaces
  - Carroll, (A)dS Carroll, Light-cone
  - Seem to be useful, see talks workshop talks
- Rich and interesting world of carrollian physics
  - Beyond the flat case
    - AdS Carroll related to Ti, carrollian limit of AdS related to timelike infinity in $\mathbb{M}$ (?)
  - Beyond purely carrollian, e.g.,
    - Spi is pseudo-carrollian
    - Ni is dubbly-carrollian
  - Not necessarily limits (in the sense of a contraction)
    - Reinterpretation, i.e., different homogeneous spaces
- Arise as bundles of scales of conformal riemannian spaces
- Carroll: More general

# Summary and outlook

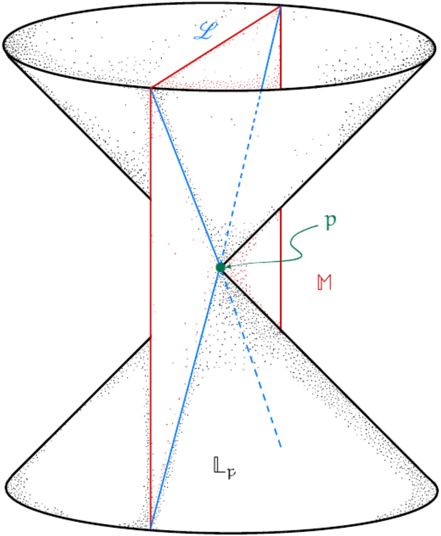
- "Algebraic holography": Search lower-dimensional spaces
- Vacuum homogeneous
- Unique $\mathsf{CS}_3$ for $\mathsf{dS}_4$ (nearly unique or $\mathsf{AdS}_4$)
- Flat space holography
  - Carroll physics derives from the Poincaré algebra with different homogeneous space
  - Unifying: Spatial, temporal and null
  - Much of what has been said generalizes to other signatures (Celestial torus)
- Uniqueness:
  - $\mathscr{J}_3$ is unique effective hom. space in $d = 3$
  - $\mathsf{CS}_2$ unique for $d = 2$
  - $\mathsf{CS}_2$: No counterpart in $\mathsf{(A)dS}_4$

# Summary and outlook

- Celestial sphere in (A)dS?
- Homogeneous spaces are the starting point for gauging
- Embedding picture interesting for holography?
  - Bulk-to-boundary propagators
  - Extremal surfaces
  - Reconstruction

- Thank you!


WE ARE THE KNIGHTS
WHO SAY "NI*", AND
WE DEMAND...
A SHRUBBERY!
* Spi, Ti, Seri



$\mathscr{L}$
$\mathfrak{p}$
$\mathbb{M}$
$\mathbb{L}_{\mathfrak{p}}$