Carroll symmetry in field theory and gravity

Carroll workshop, Vienna, February 16, 2022

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DET FRIE FORSKNINGSRÅD
DANISH COUNCIL FOR
INDEPENDENT RESEARCH

THE VELUX FOUNDATIONS
VILLUM FONDEN VELUX FONDEN

based on work

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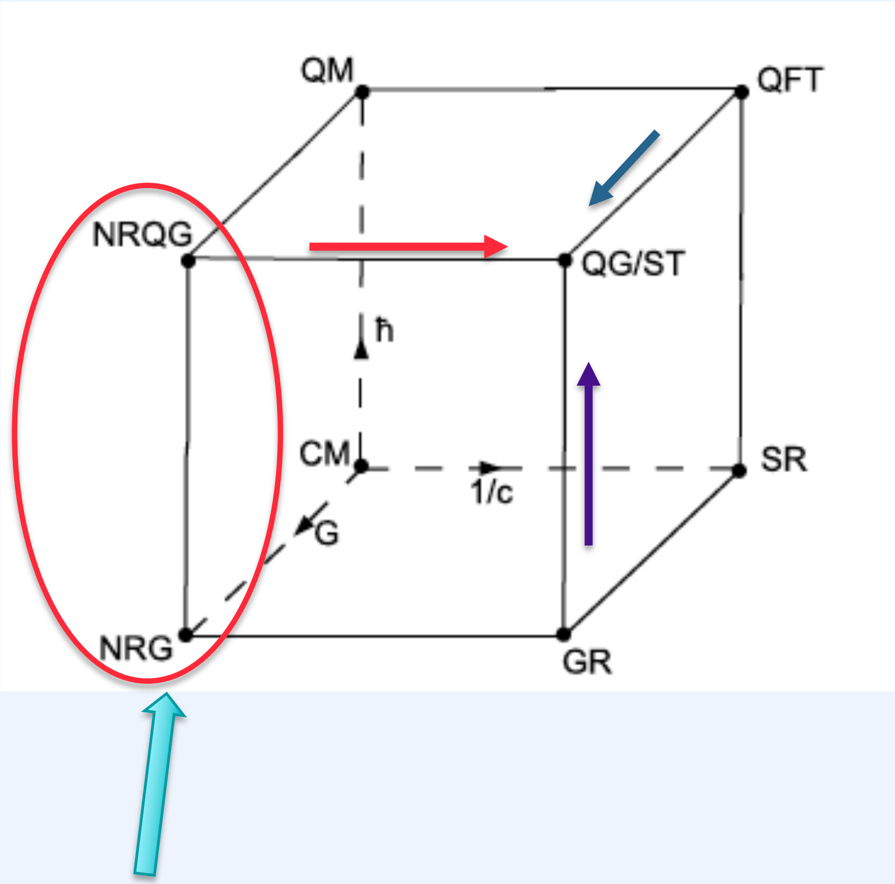
Cube of physical theories

$(\hbar, G_N, 1/c)$



a third route towards (relativistic) quantum gravity

how does this fit with string theory/holography ?

already (classical) non-relativistic gravity (NRG) is more than just Newtonian gravity

-→ uses Newton-Cartan geometry (and torsional generalization)

What about small speed of light limit ?

Carroll limit:

zero speed of light contraction of Poincare

running (boosting) without moving: Levy-Leblond(1965)

Red Queens race of L. Carroll's Through the looking glass

→ expand relativistic theories around c=0:

what do we get ? what can we use it for ?

(Carroll symmetry as organizing principle for perturbative expansion around c = 0)

terminology: this is not ultra-relativistic limit ($v/c \rightarrow 1$)

rather ultra-local limit

Intro

Carroll group is kinematical group → Carrollian manifolds

(timelike and spacelike) vielbeine transforming under local Carroll boosts

Bekaert,Morand,2015/Hartong,2015/Figueroa-O'Farrill,Prohazka,2018

example: null hypersurfaces

e.g. null infinity of assymptotically flat spacetime

Duvall,Gibbons,Horvathy,2014

Examples

- black hole membrane paradagim
  [Donnay,Marteau,2019],[Penna,2018]

- (3D) flat space holography
  [Bagchi,Detournay,Fareghbal,Simon,2012],[Hartong,2015],
  [Ciambella,Marteau,Petkou,Petropoulos,Siampos,2018],
  & 4D: [Donnay,Fiorucci,Herfray,Ruzziconi]

- tensionless limits of strings
  [Bagchi,2013]

- limits of GR
  [Henneaux,1979],[Bergshoeff,Gomis,Rollier,ter Veldhuis,2017],
  [Henneaux,Salgado-Rebolledo,2021],[Perez,2021],[Hansen et al, 2021]

- inflationary cosmology
  [de Boer,Hartong,Obers,Sybesma,Vandoren,2021]

- setups with: effective speed of light (characteristic speed) << velocity

Plan

- Carroll symmetries, Carroll particles and Ward identity
- Carroll field theories
- c=infty/0 expansion of GR and NC/Carroll geometry
- Carroll perfect fluids on Carroll geometry
- Outlook

Carroll algebra

- Lorentz transformations of energy & momentum:

$$\vec{p}_{\|}{}' = \gamma_\beta \left(\vec{p}_{\|} - \vec{\beta}\, \frac{E}{c} \right) \,, \qquad \vec{p}_{\perp}{}' = \vec{p}_{\perp} \,, \qquad \frac{E'}{c} = \gamma_\beta \left(\frac{E}{c} - \vec{\beta} \cdot \vec{p} \right) \,,$$

under Carroll limit:

$$\vec{p}\,' = \vec{p} - \vec{b}\, E \,, \qquad E' = E \,.$$

Carroll algebra: generators H, P_i, C_i, J_{ij}

$$[P_i \,, C_j] = \delta_{ij} H \,, \quad \leftarrow \text{central element}$$

$$[J_{ij} \,, P_k] = \delta_{ik} P_j - \delta_{jk} P_i \,,$$

$$[J_{ij} \,, C_k] = \delta_{ik} C_j - \delta_{jk} C_i \,,$$

$$[J_{ij} \,, J_{kl}] = \delta_{ik} J_{jl} - \delta_{jk} J_{il} + \delta_{jl} J_{ik} - \delta_{il} J_{jk} \,,$$

Energy momentum tensor & Zero energy flux

- on-shell conserved currents:

$$\partial_\mu (T^\mu{}_\nu K^\nu) = 0$$

EM tensor → $T^\mu{}_\nu$; ↖ generators: (Killing vectors of flat Carroll spacetime)

$$H = \partial_t \ , \quad P_i = \partial_i \ , \quad C_i = x^i \partial_t \ , \quad J_{ij} = x^i \partial_j - x^j \partial_i$$

$$\Rightarrow \quad \partial_\mu T^\mu{}_\nu = 0 \ , \quad \boxed{T^i{}_t = 0} \ , \quad T^i{}_j = T^j{}_i$$

energy flux must vanish ↗

$$\frac{1}{c} T^i{}_t + c T^t{}_i = 0 \ ,$$

↖ follows also from $c \to 0$ limit of relativistic property

Carroll particles

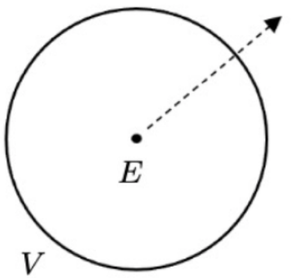
two types of particles (from limit of free relativistic particles)

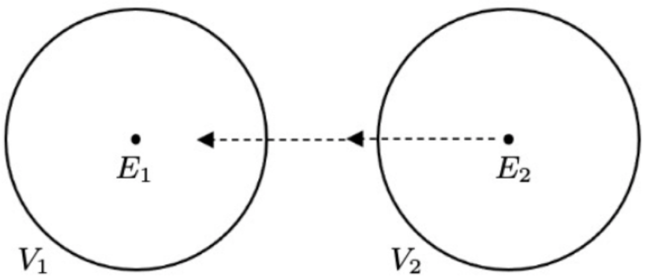
- $E = E_0 = mc^2 \, , \qquad \vec{p} = 0 \, .$

  mc^2 fixed

- $E = 0 \, , \qquad \vec{p} = \pm p_0 \vec{n} \, . \qquad p_0 \equiv -imc \, ,$

  mc fixed. (from tachyon)





- $E \neq 0 \longrightarrow$ cannot move
- if it can move $\longrightarrow E = 0$

Carroll (scalar) field theory

χ = Lagrange mult.

equivalent

$$\mathcal{L} = \frac{1}{2c^2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - V(\phi)\,. \Longleftrightarrow \mathcal{L} = -\frac{c^2}{2}\chi^2 + \chi\partial_t\phi - \frac{1}{2}(\partial_i\phi)^2 - V(\phi)\,.$$

↓ Carroll limit ↓ Carroll limit

- $$\mathcal{L} = \frac{1}{2}\dot\phi^2 - \tilde V(\phi)$$

missing gradient term
⇒ zero energy flux:

$$T^t{}_t = -\left(\frac{1}{2}\dot\phi_0^2 + V\right), \quad \boxed{T^i{}_t = 0,}$$

$$T^t{}_i = -\dot\phi_0\partial_i\phi_0\,, \quad T^i{}_j = \left(\frac{1}{2}\dot\phi_0^2 - V\right)\delta^i{}_j\,.$$

⇝ $E \neq 0$ irrep

- $$\mathcal{L} = \chi\dot\phi - \frac{1}{2}(\partial_i\phi)^2 - V(\phi)\,.$$

$$\delta\phi = \vec b\cdot\vec x\,\dot\phi\,, \qquad \delta\chi = \vec b\cdot\vec x\,\dot\chi + \vec b\cdot\vec\partial\phi\,.$$

$\delta\chi \to \dot\phi = 0$

⇒ energy flux = 0

⇝ $E = 0$ irrep

2-point functions

General solution of WI's for 2-pt. fn.

two classes:

- $$\langle O(t,\vec{x})O(0,0)\rangle = F(t)\delta(\vec{x})$$ $\rightsquigarrow$ $E \neq 0$

- $$\langle O(t,\vec{x})O(0,0)\rangle = f(|\vec{x}|),$$ $\rightsquigarrow$ $E = 0$

Carroll E&M (electric limit)

$$\mathcal{L} = \frac{1}{2c^2}(E^i)^2 - \frac{1}{4}(F_{ij})^2 \,, \quad \overset{\text{Carroll}}{\Longrightarrow} \quad \boxed{\mathcal{L} = \frac{1}{2}E_iE_i} \,, \qquad E_i = \partial_i E_t - \partial_t E_i$$

$$\delta A_t = \vec{b}\cdot\vec{x}\,\partial_t A_t \quad , \qquad \delta A_i = \vec{b}\cdot\vec{x}\,\partial_t A_i + b_i A_t$$

- energy-momentum tensor:

$$T^t{}_t = \; -\frac{1}{2}(E^i)^2 \,, \quad T^i{}_t = \; 0 \,,$$

$$T^t{}_j = (\vec{E}\times\vec{B})_j \,, \quad T^i{}_j = \; -E^iE_j + \frac{1}{2}(E^i)^2\delta^i{}_j \,,$$

- EOM

$$\partial_i B_i = 0 \,, \quad \partial_t B_i + (\vec{\nabla}\times\vec{E})_i = 0 \,.$$

$$\partial_i E_i = 0 \,, \quad \partial_t E_i = 0 \,, \quad \leftarrow \text{Ampère law without } \vec{\nabla}\times\vec{B}$$

Carroll E&M (magnetic limit)

$$\mathcal{L} = -\frac{c^2}{2}\chi_i\chi_i + \chi_i E_i - \frac{1}{4}(F_{ij})^2 \,.$$

Carroll $\Rightarrow$

$$\mathcal{L} = \chi_i E_i - \frac{1}{4}(F_{ij})^2 \,.$$

$$\delta\chi_j = \vec{b}\cdot\vec{x}\partial_t\chi_j + \left(\vec{b}\times\vec{B}\right)_j \,,$$

- energy-momentum tensor:

$$T^t{}_t = -\frac{1}{2}(B_i)^2 \,, \quad T^i{}_t = 0 \,,$$

$$T^t{}_j = (\vec{\chi}\times\vec{B})_j \,, \quad T^i{}_j = -B_iB_j + \frac{1}{2}(B_k)^2\delta^i{}_j \,,$$

- EOM

$$\partial_i B_i = 0 \,, \quad \partial_t B_i = 0 \,,$$

$$\partial_i\chi_i = 0 \,, \quad -\partial_t\chi_i + (\vec{\nabla}\times\vec{B})_i = 0 \,.$$

← Faraday's law without $\vec{\nabla}\times\vec{E}$

Carroll fields in 2D

- 1 + 1 dimensions: Carroll algebra allows central extension
- Let $i, j = 1 \ldots 2n$ & $\omega_{ij} = -\omega_{ji}$ constant invertible matrix

$$\mathcal{L} = \frac{1}{2}\partial_\tau X^i \partial_\tau X^i - \omega_{ij} X^i \partial_\sigma X^j$$

Carroll invariant: $\delta X^i = b\sigma\partial_\tau X^i - b\tau\omega_{ij}X^j$

- This model can be obtained as gauged fixed version of Polyakov-type theory for closed string with Carrollian world-sheet

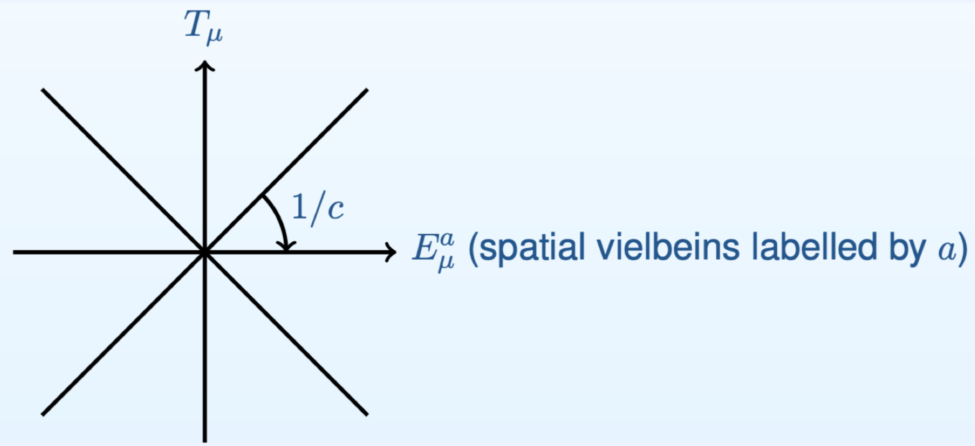
Bidussi,Harmark,Hartong,NO,Oling, to appear

- non-relativistic gravity from 1/c expansion
- ultralocal/Carroll gravity from c expansion

speed of light dependence in GR

metric: $$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$$

T = time-like vielbein
Pi = spatial metric



large speed of light → light-cone opens up

expand in $$\sigma = 1/c^2$$

small speed of light → light-cone closes up

pre-non-relativistic GR

rewrite GR in terms of T and Pi:

new choice of connection:

$$C^{\rho}_{\mu\nu} = -T^{\rho}\partial_{\mu}T_{\nu} + \frac{1}{2}\Pi^{\rho\sigma}\left(\partial_{\mu}\Pi_{\nu\sigma} + \partial_{\nu}\Pi_{\mu\sigma} - \partial_{\sigma}\Pi_{\mu\nu}\right)$$

has torsion: proportional to:

$$T_{\mu\nu} \equiv \partial_{\mu}T_{\nu} - \partial_{\nu}T_{\mu} \,.$$

analogue of metric compatibility

$$\overset{(C)}{\nabla}_{\mu}T_{\nu} = 0 \,, \qquad \overset{(C)}{\nabla}_{\mu}\Pi^{\nu\rho} = 0 \,,$$

EH Lagrangian in pre-non-relativistic form

$$\tilde{\mathcal{L}} = E\left[\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma} + \sigma\Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu} - \sigma^{2}T^{\mu}T^{\nu}\overset{(C)}{R}_{\mu\nu}\right]$$

recent progress: understand this in 1st order formulation of GR

Hansen,Hartong,OlingNO(2020)

NR gravity from large c expansion of GR

metric in GR depends on speed of light c:

expand in 1/c

Dautcourt (1996)
van den Bleeken(2017)
Hansen,Hartong,NO(2018,2020)

$$g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + h_{\mu\nu} - m_\mu\tau_\nu - m_\nu\tau_\mu + \frac{1}{c^2}(\cancel{B_\mu}\tau_\nu + \cancel{B_\nu}\tau_\mu - \Phi_{\mu\nu}) + O(c^{-4})$$

- LO and NLO fields define a novel version of NC geometry

LO fields:	$\tau_\mu \quad h_{\mu\nu}$
NLO fields:	$m_\mu \quad \Phi_{\mu\nu}$

- expand Einstein-Hilbert action of GR:

$$S_{\mathrm{EH}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

Lagrangian expansion

- basic expansion structure (for any type of field)

$$\phi^I(x;\sigma) = \phi^I_{(0)}(x) + \sigma\phi^I_{(2)}(x) + \sigma^2\phi^I_{(4)}(x) + \mathcal{O}(\sigma^3)$$

$$\sigma = 1/c^2$$

Lagrangian: factor out overall c-power and expand:

$$\mathcal{L}(c^2,\phi,\partial_\mu\phi) = c^N\tilde{\mathcal{L}}(\sigma) = c^N \overset{(-N)}{\mathcal{L}}_{\mathrm{LO}} + c^{N-2}\overset{(2-N)}{\mathcal{L}}_{\mathrm{NLO}} + c^{N-4}\overset{(4-N)}{\mathcal{L}}_{\mathrm{NNLO}} + \mathcal{O}(c^{N-6}),$$

- first terms:

$$\overset{(-N)}{\mathcal{L}}_{\mathrm{LO}} = \tilde{\mathcal{L}}(0) = \overset{(-N)}{\mathcal{L}}_{\mathrm{LO}}(\phi_{(0)},\partial_\mu\phi_{(0)})$$

$$\overset{(2-N)}{\mathcal{L}}_{\mathrm{NLO}} = \left.\frac{\partial\tilde{\mathcal{L}}}{\partial\sigma}\right|_{\sigma=0} + \phi_{(2)}\frac{\delta\overset{(-N)}{\mathcal{L}}_{\mathrm{LO}}}{\delta\phi_{(0)}}$$

$$\sigma = 1/c^2$$

EOM of the NLO field of NLO Lagrangian
= EOM of LO field of LO Lagrangian
(cascading structure repeats at every order)

→ at particular order: take action at that order and forget about the previous orders

NR expansion of EH Lagrangian

expansion of EH: $$\mathcal{L}_{\mathrm{EH}} = \frac{c^6}{16\pi G}\left[\mathcal{L}_{\mathrm{LO}} + \sigma\mathcal{L}_{\mathrm{NLO}} + \sigma^2\mathcal{L}_{\mathrm{N^2LO}} + O(\sigma^3)\right]$$

$$\mathcal{L}_{\mathrm{LO}} = \frac{e}{4}h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}$$

$$\tau_{\mu\nu} = \partial_\mu\tau_\nu - \partial_\nu\tau_\mu$$

EOM enforce TTNC (causality)

$$\mathcal{L}_{\mathrm{NLO}} = eh^{\mu\nu}\check{R}_{\mu\nu} + \frac{\delta\mathcal{L}_{\mathrm{LO}}}{\delta\tau_\mu}m_\mu + \frac{\delta\mathcal{L}_{\mathrm{LO}}}{\delta h_{\mu\nu}}\Phi_{\mu\nu}$$

Galilean gravity

(see also Bergshoeff et al (2017) for 1st order formalism)

Non-relativistic Gravity Action

for EOM of NNLO involving only NLO fields we can use TTNC off-shell (using Lagrange mulitiplier)

$$\mathcal{L} = e\Big[-v^\mu v^\nu \check{R}_{\mu\nu} - 2m_\nu \check{\nabla}_\mu \left(h^{\mu\rho}h^{\nu\sigma} - h^{\mu\nu}h^{\rho\sigma}\right) K_{\rho\sigma} + \Phi h^{\mu\nu}\check{R}_{\mu\nu} + \frac{1}{4}h^{\mu\rho}h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$$
$$-\Phi_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma}\left(\check{R}_{\mu\nu} - \check{\nabla}_\mu a_\nu - a_\mu a_\nu - \frac{1}{2}h_{\mu\nu}h^{\kappa\lambda}\check{R}_{\kappa\lambda} + h_{\mu\nu}e^{-1}\partial_\kappa\left(eh^{\kappa\lambda}a_\lambda\right)\right)\Big], \quad (27)$$

- resulting action (unique 2-derivative action of fields $\tau_\mu,\ h_{\mu\nu},\ m_\mu,\ \Phi_{\mu\nu}$ respecting all invariances)

can be rewritten in manifest (Milne) boost invariant quantities:

Action of non-relativistic gravity (NRG)

expand Einstein-Hilbert action of GR

leading order → EOMs imply causality

next-to-leading order: bifurcation in either → absolute time / → hypersurface orthogonality

(depends on the matter sources)

- NNLO gives NRG action:

$$\mathcal{L} = -\frac{1}{16\pi G} e \Big[\hat{v}^\mu \hat{v}^\nu \bar{R}_{\mu\nu} - \tilde{\Phi} h^{\mu\nu} \bar{R}_{\mu\nu} - \Phi_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} \left(\bar{R}_{\rho\sigma} - a_\rho a_\sigma - \bar{\nabla}_\rho a_\sigma \right) + \frac{1}{2} \Phi_{\mu\nu} h^{\mu\nu} \left[h^{\rho\sigma} \bar{R}_{\rho\sigma} - 2 e^{-1} \partial_\rho \left(e h^{\rho\sigma} a_\sigma \right) \right] \Big]$$

- for the first time an action principle for Newtonian gravity !
- goes beyond by allowing for strong gravity (gravitational time dilation)

Coupling of matter to NRG

coupling of matter: perform 1/c expansion of relativistic matter
e.g. point particles, scalar/vector fields, fluids, …

- simplest case: non-relativistic point particle

$$-m\int \mathrm{d}\lambda\, \delta(x - x(\lambda))\tau_\mu \dot{x}^\mu$$

EOMs of the total action give:

- time = absolute
- Newton's Poisson equation

Punchline (for large speed of light expansion)

new version of TNC (type II) is what the large c expansion of GR tells us to do !

What does it achieve ?

- while Cartan's original geometry can geometrize EOMS, it cannot be used to define the theory off-shell
  → needed for the action (analogue of EH action for NRG)

- the NRG action can then simply be obtained by doing the (right) expansion of GR (see below)

- what replaces Poincare invariance ?
  → new symmetry algebra that follows from Poincare from a well-defined procedure (Lie algebra expansion of Poincare)

(principle which can be used to geometrize any Post-Newtonian order)

- opposite case: Carrollian expansion of GR

Hansen,NO,Oling,Søgaard (2112.12684)

see also recent works: Henneaux,Salgado-Rebolledo/Perez (2021)

→ talk later today by Gerben Oling

Carroll geometry

- PUR (pre-"ultra-relativistic") expansion of pseudo-Riem. geometry:

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \qquad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}.$$

- $c \to 0$ expansion:

NLO fields

$$V^\mu = v^\mu + c^2 M^\mu + \mathcal{O}(c^4), \qquad T_\mu = \tau_\mu + \mathcal{O}(c^2)$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \mathcal{O}(c^4), \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \mathcal{O}(c^2),$$

↳ LO fields: Carrollian geometry.

$$\tau_\mu v^\mu = -1, \quad \tau_\mu h^{\mu\nu} = 0, \quad h_{\mu\nu} v^\nu = 0, \quad \delta^\mu_\nu = -v^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu}.$$

$$h^{\mu\nu} = \delta^{ab} e^\mu_a e^\nu_b.$$

- transformation of Carrollian geometry fields under Carrol boosts:

$$\delta\tau_\mu = \lambda_a e^a_\mu \qquad \delta h^{\mu\nu} = v^\mu e^\nu_a \lambda^a + v^\nu e^\mu_a \lambda^a \qquad \delta M^\mu = \lambda^a e^\mu_a$$

$h_{\mu\nu}$, v^μ are left invariant.

Preferred connection and EH action

- Convenient PUR connection:

$$\tilde{C}^{\rho}_{\mu\nu} = -V^{\rho}\partial_{(\mu}T_{\nu)} - V^{\rho}T_{(\mu}\mathcal{L}_V T_{\nu)} + \frac{1}{2}\Pi^{\rho\lambda}\left[\partial_\mu \Pi_{\nu\lambda} + \partial_\nu \Pi_{\lambda\mu} - \partial_\lambda \Pi_{\mu\nu}\right] - \Pi^{\rho\lambda}T_\nu \mathcal{K}_{\mu\lambda},$$

$$\mathcal{K}_{\mu\nu} = -\tfrac{1}{2}\mathcal{L}_V \Pi_{\mu\nu}.$$

extrinsic curvature.

- yields the Carrollian ($c \to 0$) (minimal torsion) connection:

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \tilde{C}^{\rho}_{\mu\nu}\Big|_{c=0} = -v^{\rho}\partial_{(\mu}\tau_{\nu)} - v^{\rho}\tau_{(\mu}\mathcal{L}_v \tau_{\nu)} + \frac{1}{2}h^{\rho\lambda}\left[\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right] - h^{\rho\lambda}\tau_\nu K_{\mu\lambda},$$

$$\tilde{\nabla}_\mu v^\mu = 0, \qquad \tilde{\nabla}_\rho h_{\mu\nu}.$$

← 'metric' compatibility.

- Einstein-Hilbert in the PUR variables:

$$R \approx \frac{1}{c^2}\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \mathcal{K}^2\right) + \Pi^{\mu\nu}\overset{(\tilde{C})}{R}_{\mu\nu} + \frac{c}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}\left(dT\right)_{\mu\rho}\left(dT\right)_{\nu\sigma},$$

LO and NLO action

- expand R for c→0 ; LO action :

$$\overset{(2)}{\mathcal{L}}_{\mathrm{LO}} = \frac{e}{16\pi G_N}\left[K^{\mu\nu}K_{\mu\nu} - K^2\right],$$

Henneaux,1979

$$\delta\overset{(2)}{\mathcal{L}}_{\mathrm{LO}} = \frac{e}{8\pi G_N}\left[\overset{(2)}{G}{}^{v}_{\mu}\delta v^{\mu} + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\delta h^{\mu\nu}\right],$$

$$\overset{(2)}{G}{}^{v}_{\mu} = -\frac{1}{2}\tau_{\mu}(K^{\rho\sigma}K_{\rho\sigma} - K^2) + h^{\gamma\lambda}\tilde{\nabla}_{\lambda}(K_{\mu\gamma} - Kh_{\mu\gamma}),$$
$$\overset{(2)}{G}{}^{h}_{\mu\nu} = -\frac{1}{2}h_{\mu\nu}(K^{\rho\sigma}K_{\rho\sigma} - K^2) + K(K_{\mu\nu} - Kh_{\mu\nu}) - v^{\rho}\tilde{\nabla}_{\rho}(K_{\mu\nu} - Kh_{\mu\nu}).$$

- NLO action :

$$\overset{(4)}{\mathcal{L}}_{\mathrm{NLO}} = \frac{e}{8\pi G_N}\left[\frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu} + \overset{(2)}{G}{}^{v}_{\mu}M^{\mu} + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\Phi^{\mu\nu}\right].$$

see also: Bergshoeff et al,2017

- Carroll fluids

de Boer et al (2021 & to appear)

see also:

de Boer et al (2017)
Ciambelli,Marteau,Petkou,Petropoulos,Siampos (2018, 2018)
Donnay,Marteau (2019)
Ciambelli,Marteau,Petropoulos,Ruzziconi (2020)

Carroll perfect fluids

- Most general perfect fluid (in LAB frame):

$$T^t{}_t = -\mathcal{E}\ , \qquad T^i{}_t = -(\mathcal{E}+P)v^i\ , \qquad T^t{}_j = \mathcal{P}_j\ , \qquad T^i{}_j = P\delta^i{}_j + v^i\mathcal{P}_j\ .$$

de Boer,Haratong,NO,Sybesma,Vandoren,2017

momentum density: $\mathcal{P}_i = \rho v_i$

fluid variables: $T,\ v_i$

transforms under Carrollian boosts.

$$\mathcal{P}_i' = \rho' v'^i = \rho' \frac{v^i}{1-\vec{b}\cdot\vec{v}} = \rho v^i(1-\vec{b}\cdot\vec{v}) - b_i(\mathcal{E}+P)\,,$$

$$\boxed{\mathcal{E}+P=0\,.}$$

for any Carroll fluid!

reminiscent of dark energy eq. of state $w=-1$

Carroll perfect fluids on curved spacetime

three derivations:

- expand relativistic perfect fluid

  in analogy with the two types of Carroll particles:
  → can take timelike or spacelike fluid velocity vector

- **hydrostatic partition function**

- null hypersurfaces

Hydrostatic partition function

most elegant/highlights the power of (Carrroll) geometry

1st law of thermodynamics (general perfect fluid)

$$\tilde{\mathcal{E}} = \mathcal{E} - \rho u^2 = Ts - P\,, \quad d\tilde{\mathcal{E}} = Tds - \frac{1}{2}\rho du^2$$

define: $\tilde{T} = T/\sqrt{1-u^2/c^2}$ and $\tilde{s} = s\sqrt{1-u^2/c^2}$,

→ $$\tilde{\mathcal{E}} = \tilde{s}\tilde{T} - P\,, \quad \tilde{s} = \frac{dP}{d\tilde{T}}$$

Energy momentum tensor

general variation of Lagrangian coupling to Carroll geometry

$$\delta \mathcal{L} = e\left(-T^{\mu}\delta\tau_{\mu} + \frac{1}{2}T^{\mu\nu}\delta h_{\mu\nu}\right)$$

EM tensor

$$T^{\mu}{}_{\nu} = -T^{\mu}\tau_{\nu} + T^{\mu\rho}h_{\rho\nu}\,.$$

Carroll boost invariant
(zero energy flux)

$$(T_{\rm Car})^{\mu}{}_{\nu}v^{\nu}e^{a}_{\mu} = 0$$

diffeo Ward identity

$$e^{-1}\partial_{\mu}\left(eT^{\mu}{}_{\rho}\right) + T^{\mu}\partial_{\rho}\tau_{\mu} - \frac{1}{2}T^{\mu\nu}\partial_{\rho}h_{\mu\nu} = 0\,,$$

Two Carroll perfect fluid stress tensor(s)

simplest hydrostatic partition function
(in derivative expansion)

$$\mathcal{L} = eP(\tilde{T}).$$

choice 1) Killing vector

$$\beta^\mu = \frac{u^\mu}{T}$$

$$\tau_\mu u^\mu = 1.$$

$$\beta^\mu \beta^\nu h_{\mu\nu} = \frac{u^2}{T^2} = \frac{1}{\tilde{T}^2}$$

from action
(using 1st law)

$$T^\mu = P v^\mu , \qquad T^{\mu\nu} = P h^{\mu\nu} - \frac{\tilde{\mathcal{E}} + P}{u^2} u^\mu u^\nu$$

→

$$(T_{\mathrm{Car}})^\mu{}_\nu = P \delta^\mu{}_\nu - \frac{\tilde{\mathcal{E}} + P}{u^2} u^\mu u^\rho h_{\rho\nu} .$$

choice 2)

$$\beta^\mu = -\frac{1}{\tilde{T}} v^\mu$$

→

$$(T_{\mathrm{Car}})^\mu{}_\nu = P \delta^\mu{}_\nu .$$

physically distinct; both satisfy

$$\mathcal{E} := (T_{\mathrm{Car}})^\mu{}_\nu v^\nu \tau_\mu = -P$$

Outlook

- Carroll strings
- Tensionless strings
- Flat space Holography
- solutions of small speed of light GR (see talk Gerben Oling)
- Cosmology and Carroll gravity
- Carroll fluids in curved spacetime

  [de Boer et al, in progress]
- applications to supersonic behavior ?

The End

Thank you for your attention !

Null hypersurfaces

- Carrollian geometry can also be obtained from metric structure on null hypersurface. $u = \text{cst.}$

$$ds^2 = 2du\left(\bar{\Phi}du - \hat{\tau}_\mu dx^\mu\right) + h_{\mu\nu}dx^\mu dx^\nu\,,$$

$$\bar{\Phi} = -\tau_\rho M^\rho + \tfrac{1}{2}h_{\rho\sigma}M^\rho M^\sigma \text{ and } \hat{\tau}_\mu = \tau_\mu - h_{\mu\nu}M^\nu$$

- Local Lorentz trafo's that leave $\perp$ 1-form du invariant (i.e. null rotations) $\Rightarrow$ Carroll transformations

Carroll in cosmology

- Hubble law: $v = Hd.$

- Hubble radius: $R_H = cH^{-1},$

for $d \gg R_H \Rightarrow v \gg c$

$\Rightarrow$ super Hubble scales are Carrollian.

$c \to 0$: Hubble radius vanishes $\rightsquigarrow$ entire universe is super-Hubble

(ultra-local limit)

- expanding away from $c = 0$:

Hubble cells grow containing more and more d.o.f.

Freedman equations and dark energy

• from Fr. eqs:

$$\mathcal{E} + P = \frac{3c^2}{8\pi G_N}(1+w)H^2(t) \quad \to \quad \rho = \frac{3}{8\pi G_N}(1+w)H^2(t) \ ,$$

• cosmological model based on scalar field:

$$w = \frac{\frac{1}{2c^2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2c^2}\dot{\phi}^2 + V(\phi)}$$

Carroll perspective

$$w = \frac{\frac{1}{2}c^2\pi_\phi^2 - V(\phi)}{\frac{1}{2}c^2\pi_\phi^2 + V(\phi)} = -1 + \frac{\pi_\phi^2}{V}c^2 + \mathcal{O}(c^4) \ .$$

$$\begin{cases} \pi_\phi = \frac{1}{c^2}\dot{\phi}, & = \text{finite} \ (\dot{\phi} \to 0) \\ \text{assume } V \neq 0 & \end{cases}$$

slow roll

• Λ = fixed so:

$$G_C \equiv \frac{G_N}{c^2} \ , \qquad H^2 = \frac{8\pi G_C}{3}\Lambda \ , \qquad G_C \ \text{fixed} \ .$$

• Hubble radius

$$R_H = cH^{-1} = \frac{c}{\sqrt{8\pi G_C \Lambda}} \to 0,$$

de Sitter patch

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + e^{2Ht}\mathrm{d}\vec{x}^2 \ ,$$

$\Rightarrow$ conformal to $\mathbb{R}^3$

$$\mathrm{d}s^2 = e^{Ht}\mathrm{d}\vec{x}^2.$$

$$\begin{cases} T = \text{const.} \\ S \sim c^3 \to 0 \end{cases}$$

Scalar field with w=1

• free scalar field with $V=0$: take $c \to 0$ limit

from KG eq:

$$\pi_\phi = \sqrt{\frac{1}{12\pi G_N}}\frac{1}{(t + a_1/a_0)}\,,$$

$$\Longrightarrow \quad \phi(t) = \frac{c^2}{\sqrt{12\pi G_N}}\ln\left(t + a_1/a_0\right) + \phi_0\,.$$

→ constant in $c \to 0$ limit

energy density:

$$\mathcal{E} = \frac{1}{2}c^2\pi_\phi^2 \;\to 0\,.$$

still

$$\mathcal{E} + P = 0$$

but both $\mathcal{E} = P = 0$

(even for non-inflationary metrics : ∃ super Hubble scales)

Inflation

- w time dependent: consider chaotic inflation

$$H^2 = \frac{4\pi G_N}{3}\left(\pi_\phi^2 + \frac{m^2\phi^2}{\hbar^2}\right) , \qquad \dot{\pi}_\phi + 3H\pi_\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 .$$

F. eq. — Scalar field eq.

expand:

$$\phi = \phi_0 + c^2\phi_1 + \cdots , \qquad \pi_\phi = \frac{1}{c^2}\dot{\phi}_0 + \dot{\phi}_1 + \cdots , \qquad H = H_0 + c^2 H_1 + \cdots ,$$

$$G_C \equiv \frac{G_N}{c^2} , \qquad \mu \equiv \frac{mc}{\hbar} ,$$

- LO & NLO solution reproduce (early time) inflationary solutions:

$$H_0^2 = \frac{4\pi G_C}{3}\mu^2\phi_0^2 , \qquad \phi_1(t) = -\frac{\mu^2}{3H_0}\phi_0 t .$$

$$H_1 = \frac{\mu^2}{18H_0} - \frac{1}{3}\mu^2 t ,$$

- Inflationary solutions are attracted to $w=-1$ in Carroll lim.!
- Freeze out of scalar perturbations in 2-pt. fn

Irreps of Carroll algebra (d=3)

• eigenstates of Casimirs: H and $W_i W_i$:

$$W_i = HS_i + \epsilon_{ijk} C_j P_k$$

• Consider energy-momentum eigenstates (E, p_i)

When $E \neq 0$ we can always go to a frame where $p_i = 0$ by performing a Carroll boost. In this case the little group is $SO(3)$ and the eigenvalues of $W_i W_i$ are $E^2 s(s+1)$ with $s = 0, 1/2, 1, \ldots$.

When $E = 0$ the momentum p_i is Carroll boost invariant. Using a rotation we can WLOG set $\vec{p} = p\hat{e}_3$. On such states $W_i = \varepsilon_{ijk} C_j P_k$ so that $W_3 = 0$. The little group is $ISO(2)$ generated by W_1, W_2, L where $L = P_i S_i$ (helicity).

Field theory on curved spacetime

Coupling to a background metric is powerful tool in relativistic theories (QED, QCD ,..)

→ putting the field theory on a curved spacetime

$$\delta S_{\mathrm{rel.matter}} \sim \int d^4x \, T_{\mu\nu} \, \delta g^{\mu\nu}$$

- responses to varying metric gives energy-momentum tensor

- can find Ward identities as consequence of symmetries

- organizing principle for effective theories/hydrodynamics

Schroedinger field coupled to Newton-Cartan

simplest case: complex non-relativistic scalar field (Schroedinger field)

$$S_{\text{Schr.}} \sim \int d^4x [i\psi^\dagger \partial_t \psi - \frac{1}{2m} \nabla\psi \nabla\psi^\dagger] + \text{h.c.}$$

covariant coupling to background Newton-Cartan geometry:

$$S^{\text{NC}}_{\text{Schr.}} \sim \int d^4x e \left[i(\tau^\mu - h^{\mu\nu} m_\nu)\psi^\dagger \partial_\mu \psi - \frac{1}{2m} h^{\mu\nu} \partial_\mu \psi^\dagger \partial_\nu \psi \right] . + \text{h.c}$$

- enters Son's description of FQHE & interactions to EM field

Hydrodynamics

Navier Stokes equation

→ covariant formulation in terms of hydrodynamics on curved Newton-Cartan spacetime

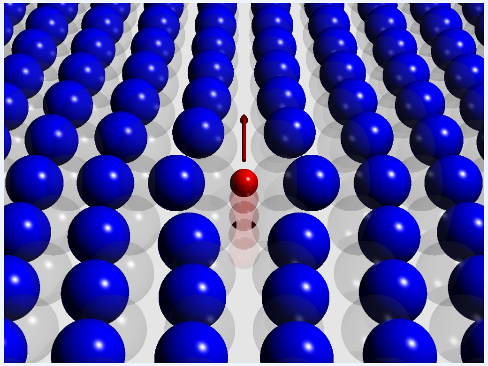
- has furthermore led to formulation of hydrodynamics for systems without any boost symmetries

[de Boer et al (SciPost 2018)]

bird flocks in air

[e.g. J. Toner, Y. Tu, and S. Ramaswamy 2005]

electron gas moving in lattice of atoms



Newton-Cartan geometry

Cartan (1923): Newtonian gravity written in frame-independent way using Newton-Cartan geometry

local symmetries of space and time $\leftrightarrow$ geometry of space and time

Cartan: Galilean $\leftrightarrow$ Newton-Cartan geometry

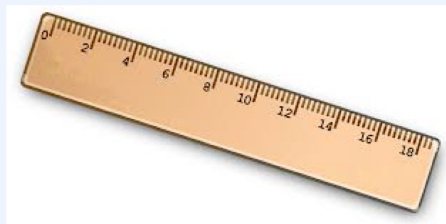
[Eisenhart,Trautman,Dautcourt,Kuenzle,Duval,Burdet,Perrin,Gibbons,Horvathy,Nicolai,Julia…] ..
[Andriga,Bergshoeff,Panda,deRoo(CQG 2011)]

Equivalence principle: freely falling observers see Galilean laws of physics

Newton-Cartan geometric data

clock form:  τ_μ

$\tau_\mu h^{\mu\nu} = 0$

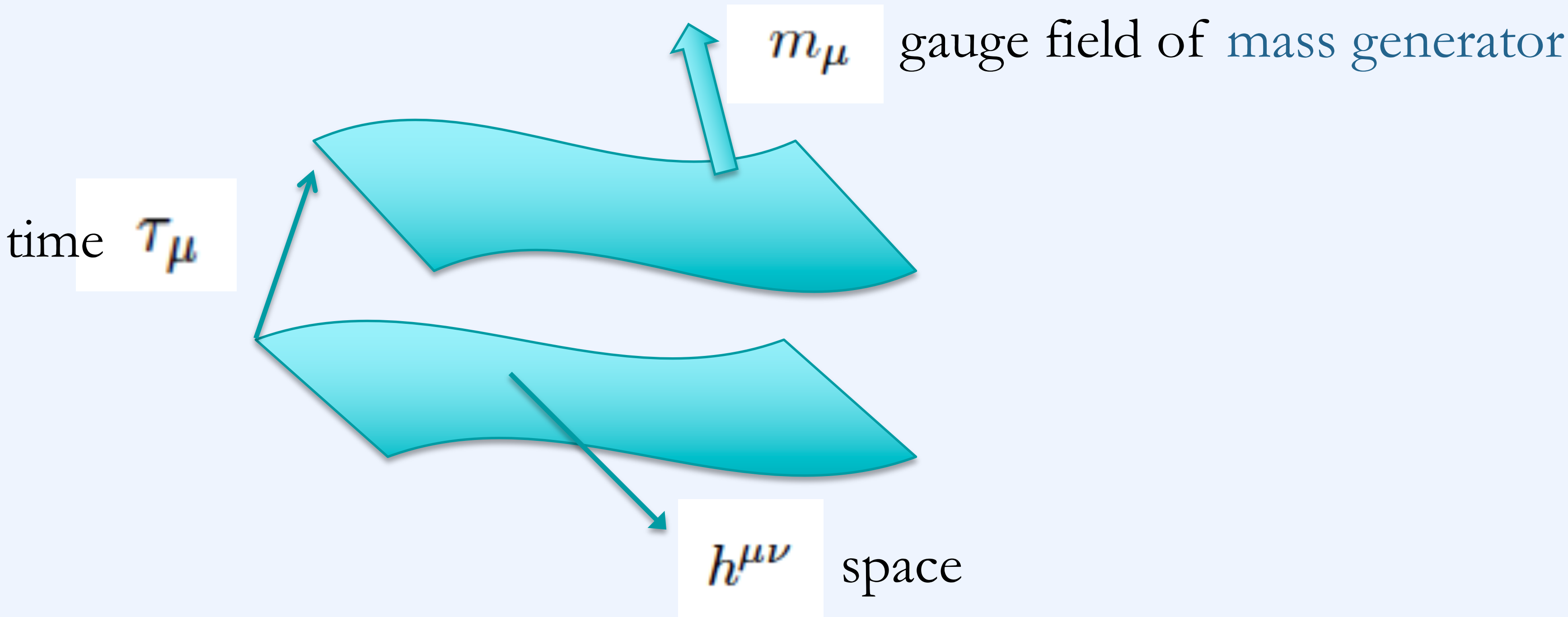
spatial metric: (ruler)  $h^{\mu\nu}$

vector field: m_μ
(encodes Newtonian potential) .

- in Newtonian gravity time is absolute: $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0$

- geometrizes Poisson equation of Newtonian gravity:

Torsional Newton-Cartan geometry



NC = no torsion → $\tau_\mu = \partial_\mu t$ absolute time

TTNC = twistless torsion → $\tau_\mu = \mathrm{HSO}$ preferred foliation equal time slices

TNC no condition on τ_μ

Christensen,Hartong,NO,Rollier (PRD, 2013)

Poisson equation of Newtonian gravity in NC form

$$\bar{R}_{\mu\nu} = 4\pi G \rho_{\rm m} \tau_\mu \tau_\nu$$

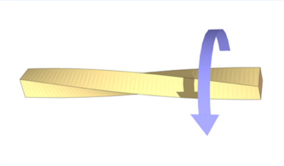
- with $d\tau = 0$ (abs. time)

recent new insights: [Hansen,Hartong,NO (PRL 2019)]

- need different version of NC geometry to find action for Newtonian gravity

- follows from (careful) large speed expansion of GR
- goes beyond Newtonian gravity

inclusion of torsion essential



- new symmetry principle which can be used at any order in 1/c

weak NR limit of Schwarzschild

Schw with factors of c reinstated

$$ds^2 = -c^2\left(1 - \frac{2Gm}{c^2 r}\right) dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

weak limit: m independent of c

$$\begin{aligned} \tau_\mu dx^\mu &= dt\,, \qquad h_{\mu\nu} dx^\mu dx^\nu = dr^2 + r^2 d\Omega_{S^2} \\ m_\mu dx^\mu &= -\frac{Gm}{r} dt\,, \qquad \Phi_{\mu\nu} dx^\mu dx^\nu = \frac{2Gm}{r} dr^2 \end{aligned}$$

point mass in flat space with Newtonian potential: Phi = -Gm/r

absolute time: tau is exact

strong NR limit of Schwarzschild

m = c^2 M;   M independent of m (VdB, 2017)

$$\tau_\mu dx^\mu = \sqrt{1 - \frac{2GM}{r}}dt\,, \qquad h_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

$$m_\mu dx^\mu = 0 = \Phi_{\mu\nu}dx^\mu dx^\nu$$

this strong expansion of Schw is not captured by Newtonian gravity:

- still described by NC geometry

different approx. of GR as compared to Post-Newtonian expansion (strong field)

tau no longer exact but: hypersurface orthogonal $\tau \wedge d\tau = 0$

strong limit captures gravitational time dilaton:
clocks tick faster/slower depending on position on constant time slice

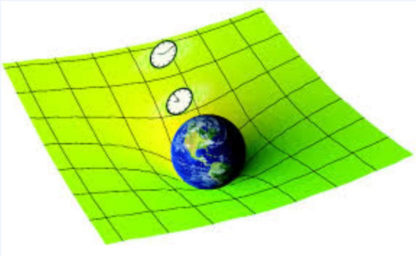
Strong gravity in NRG

Strong gravity regime:
close to compact object with Schwarzshild radius R_s

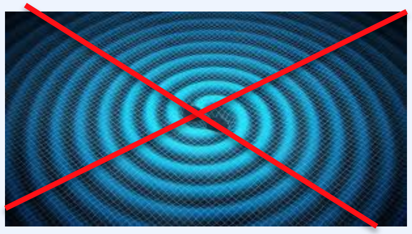
warping of time → spacetime with torsion

$$\tau_t = -\left(1 - \frac{R_s}{r}\right)$$

NR geodesics pass 3 classical tests of GR:
- precession perihelium
- bending of light
- gravitational redshift

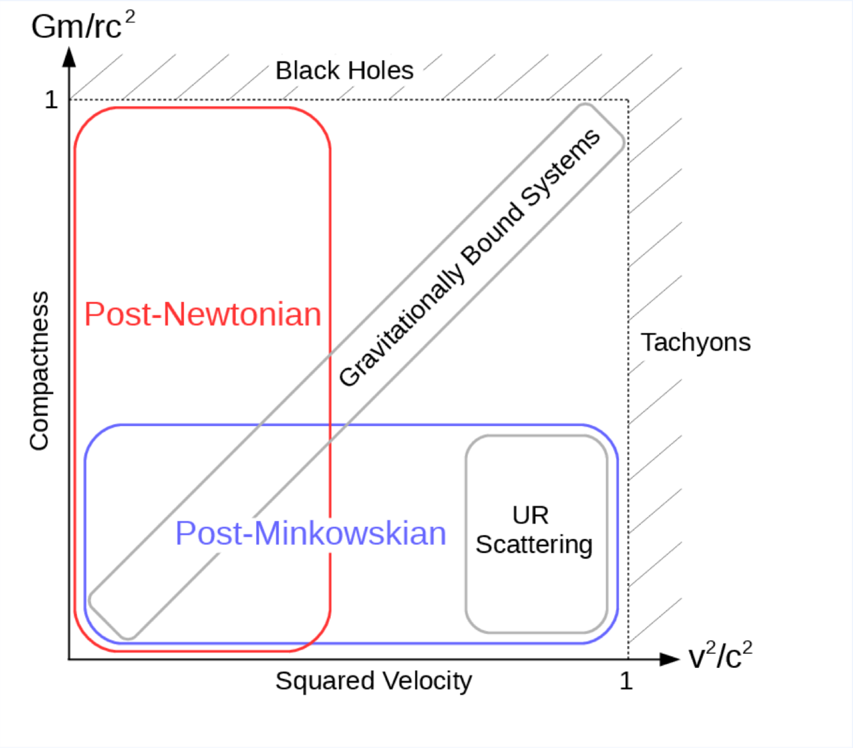


but: no gravitational waves



Post-Newtonian expansion

covariant treatment of PN physics
- application to (early phase of) binary inspirals ?

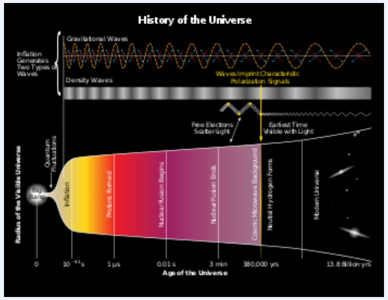


Post-Minkowskian:
resum effects in v/c at give order in G

Post-Newtonian:
use NRG to resum effects of G at given order in 1/c ?

Further properties of NRG

- cosmology: FRW solutions



- Newton-Schroedinger theory:

Coupling of non-relativistic field (electron/neutron) to NRG

- well-defined framework to treat PN corrections
- possible useful starting point to further analyze QM effects (gravitationally induced quantum interference with neutron beams)