Planar Carrollian dynamics

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State of the art of Carrollian dynamics

Two main references on Carrollian dynamics:

- C. Duval, G. Gibbons, P. Horváthy, PM. Zhang, Class. Quant. Grav 31 (2014)
- E. Bergshoeff, J. Gomis, G. Longhi, Class. Quant. Grav 31 (2014)

$\Rightarrow$ A Carrollian elementary particle moves (locally) along the time direction

These results apply to general elementary Carroll particles...
but only when the spatial dimension d is 3 or higher.

The physical interests of $2+1$ Carroll structures

A Carroll structure is a triple (M, g, ξ) with $\dim M = d+1$, a degenerate "metric" g with $\dim \ker g = 1$, a nowhere vanishing vector field $\xi \in \ker g$, and $L_\xi g = 0$. Locally, coordinates $(\boldsymbol{x}, s)$ with $g = g_0 d\boldsymbol{x} \otimes d\boldsymbol{x}$, and $\xi = \partial_s$.

It is possible to add a (non unique connection) to obtain a strong Carroll structure (M, g, ξ, ∇).

Often present in General Relativity:

- Conformal null infinity
- Black holes horizon [L. Donnay, C. Marteau, '19]
- Any null hypersurface in a Lorentzian spacetime [L. Ciambelli, et. al. '19]

All these examples are $2+1$ dimensional.

Isometries of a flat strong Carroll structure: the Carroll group.

The subtlety of Carroll planar dynamics: central extension

The dynamics of an elementary particle can be obtained from the considered group *or* a central extension of this group.

Example: the Galilei group only describes massless particles. The description of massive systems requires its non-trivial central extension.

	Dimension of non-trivial central extensions	
spatial dimension	Galilei	Carroll
$d \geq 3$	1	0
$d = 2$	2	2

$\Rightarrow$ The Carroll group potentially has a richer structure in $d = 2$.

Carroll's double central extension in 2+1 dimensions

The algebra was computed in [de Azcarraga, et. al. '98; Ngendakumana, et. al. '14],

$$[J_3, P_i] = \epsilon_{ij} P_j, \quad [J_3, K_i] = \epsilon_{ij} K_j, \quad [K_i, P_j] = M\delta_{ij},$$
$$[P_i, P_j] = \epsilon_{ij} A_1, \quad [K_i, K_j] = \epsilon_{ij} A_2.$$

Elements of the group can be represented as [L. M. '21],

$$a = \begin{pmatrix} R & 0 & \boldsymbol{c} & 0 & \epsilon \boldsymbol{b} \\ -\overline{\boldsymbol{b}} R & 1 & f & 0 & a_2 \\ 0 & 0 & 1 & 0 & 0 \\ -\overline{\epsilon \boldsymbol{c}} R & 0 & a_1 & 1 & -f - \langle \boldsymbol{b}, \boldsymbol{c} \rangle \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R \in \mathrm{O}(2)$

$\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^2$

$f, a_1, a_2 \in \mathbb{R}$

$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

This is the group one should consider when working out Carrollian dynamics in the plane.

Carroll's double central extension in 2+1 dimensions (cont.)

The physical quantities dual to the group elements:

$$J = (\ell, \boldsymbol{g}, \boldsymbol{p}, m, q_1, q_2)$$

Coadjoint action of the group: $\mathrm{Coad}(a)J = (\ell', \boldsymbol{g}', \boldsymbol{p}', m, q_1, q_2)$, with,

$$\begin{aligned} \ell' &= \ell + \boldsymbol{b} \times A\boldsymbol{g} - \boldsymbol{c} \times A\boldsymbol{p} + m\boldsymbol{b} \times \boldsymbol{c} + q_1\boldsymbol{c}^2 - q_2\boldsymbol{b}^2 \\ \boldsymbol{g}' &= A\boldsymbol{g} + m\boldsymbol{c} + 2q_2\epsilon\boldsymbol{b} \\ \boldsymbol{p}' &= A\boldsymbol{p} + m\boldsymbol{b} + 2q_1\epsilon\boldsymbol{c} \end{aligned}$$

$\Rightarrow$ 4 Casimir invariants ($m \neq 0$):

$$\begin{aligned} C_1 &= m \qquad \text{(convention: mass, not energy)} \\ C_2 &= \left(1 + 4\frac{q_1 q_2}{m^2}\right)\ell + \frac{\boldsymbol{g} \times \boldsymbol{p}}{m} + \frac{q_1}{m^2}\boldsymbol{g}^2 - \frac{q_2}{m^2}\boldsymbol{p}^2 \\ C_3 &= q_1 \quad \Rightarrow \quad [q_1] = MT^{-1} \\ C_4 &= q_2 \quad \Rightarrow \quad [q_2] = MT \end{aligned}$$

Dynamics out of a symplectic model

If a dynamical system is G-invariant, one can (locally) build its phase space as a coadjoint orbit of G (or of a central extension of G).

1. Build the "evolution space" V out of parameters of the group (*e.g.* spatial translations $\sim$ position, boosts $\sim$ momentum, etc)
2. Choose the Casimir invariants that describe the elementary particle, and pick a J_0 with such invariants
3. Endow V of a presymplectic 2-form σ out of the Maurer-Cartan Θ form on G:
$$\sigma = d\left(J_0 \cdot \Theta\right)$$
4. The equations of motion are then spanned by the kernel of σ

A free massive Carroll particle in 3+1 dimensions

From the usual Carroll group, consider a massive spinless elementary particle (Casimirs $m \neq 0$, and $\ell = 0$).

The evolution space $V \ni y = (\boldsymbol{x}, \boldsymbol{v}, s)$ is endowed with the 2-form,

$$\sigma = m d\overline{\boldsymbol{v}} \wedge d\boldsymbol{x}$$

The equations of motions are then:

$$\frac{d\boldsymbol{x}}{ds} = 0, \qquad \text{(Carrollian velocity)}$$
$$m\frac{d\boldsymbol{v}}{ds} = 0.$$

$\Rightarrow$ We recover the well known property that Carroll particles do not move.

Free planar Carroll particles

The evolution space $V \ni y = (\boldsymbol{x}, \boldsymbol{v}, s, w, z)$ is endowed with the 2-form,

$$\sigma = md\overline{\boldsymbol{v}} \wedge d\boldsymbol{x} - q_1\epsilon_{ij}dx^i \wedge dx^j + q_2\epsilon_{ij}dv^i \wedge dv^j$$

The equations of motions depend on an *effective mass*

$$\widetilde{m}^2 = m^2 + 4q_1q_2$$

$$\boxed{\widetilde{m}^2 \neq 0}$$

Free particles do not move:

$$\frac{d\boldsymbol{x}}{ds} = 0$$
$$m\frac{d\boldsymbol{v}}{ds} = 0$$

$$\boxed{\widetilde{m}^2 = 0}$$

The equations degenerate:

$$\frac{d\boldsymbol{x}}{ds} = -2\frac{q_2}{m}\epsilon\frac{d\boldsymbol{v}}{ds}$$

$\Rightarrow$ not localized?

3+1 Carroll particles with spin in an EM field

For a massive particle with spin and electric charge e, the equations of motion are,

$$\begin{aligned} \frac{d\boldsymbol{x}}{ds} &= 0, \\ m\frac{d\boldsymbol{v}}{ds} &= e\boldsymbol{E} + \mu\boldsymbol{\nabla}_{\boldsymbol{x}}\langle\boldsymbol{u}, \boldsymbol{B}\rangle, \\ \frac{d\boldsymbol{u}}{ds} &= \mu\, \boldsymbol{u} \times \boldsymbol{B}, \end{aligned}$$

where $\boldsymbol{u} \in S^2$ represents the direction of the particle's spin.

$\Rightarrow$ no actual motion, but precession of the spin around the magnetic field

Planar Carroll particles in an EM field – anyons

A particle described by the Casimirs: $m \neq 0$, $\ell \neq 0$, $q_1 \neq 0$, $q_2 \neq 0$.

There is again an effective mass,

$$\widetilde{m}^2 := m^2 + 4\left(q_1 - \frac{1}{2}eB\right)q_2,$$

There equations of motion are, for $\widetilde{m}^2 \neq 0$,

$$\begin{aligned} \frac{d\boldsymbol{x}}{ds} &= -\frac{2q_2}{\widetilde{m}^2}\epsilon\left(e\boldsymbol{E} + \mu\ell\boldsymbol{\nabla}_{\boldsymbol{x}}B\right), \\ m\frac{d\boldsymbol{v}}{ds} &= \frac{m^2}{\widetilde{m}^2}\left(e\boldsymbol{E} + \mu\ell\boldsymbol{\nabla}_{\boldsymbol{x}}B\right). \end{aligned}$$

$\Rightarrow$ We see actual motion!

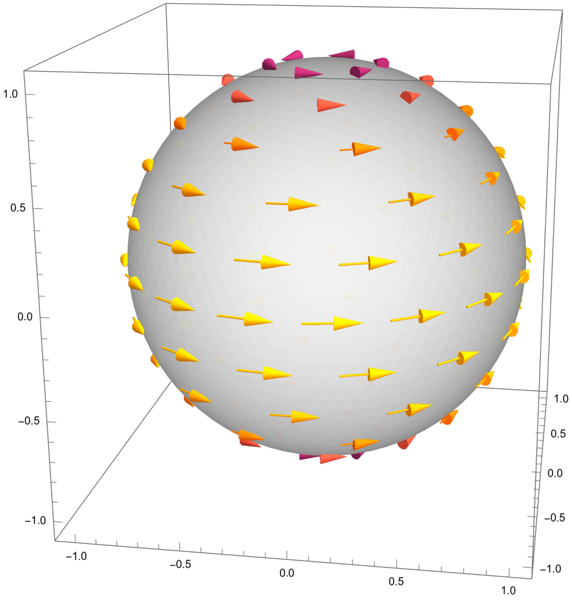
A planar Carrollian photon ($m = 0$) in an EM field

Assuming the photon has an anyon spin that couples to the magnetic field,

$$\frac{d\boldsymbol{x}}{ds} = -\frac{\mu\ell}{2q_1}\epsilon\,\boldsymbol{\nabla}_{\boldsymbol{x}}B,$$
$$q_2\frac{d\boldsymbol{v}}{ds} = 0.$$

$\Rightarrow$ We have a velocity transverse to the gradient of the magnetic field.

Velocity drift on the horizon of a Kerr-Newmann BH



Conclusion

- In $2+1$ dimensions, one needs to consider the double central extension of the Carroll group to be the most general
- Two new Casimirs to describe elementary particles: q_1 and q_2 $\Rightarrow$ Physical interpretation?
- These two exotic charges couple to the EM field to bring actual motion in Carrollian dynamics
- No exotic coupling to the gravitational field
- What about causality?

Backup slides

Carrollian time

Historical definition of the Carrollian time: $x^0 := s/C$ instead of $x^0 := ct$. While $[t] = T$, we have $[s] = L^2 T^{-1}$.

The link between the Galilei, Bargmann, and Carroll groups:

$$
\underset{\text{Carroll}}{\begin{pmatrix} R & \boldsymbol{b} & 0 & \boldsymbol{c} \\ 0 & 1 & 0 & 0 \\ -\overline{\boldsymbol{b}}R & -\|\boldsymbol{b}\|^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}}
\hookrightarrow
\overset{\text{Bargmann}}{\begin{pmatrix} R & \boldsymbol{b} & 0 & \boldsymbol{c} \\ 0 & 1 & 0 & e \\ -\overline{\boldsymbol{b}}R & -\|\boldsymbol{b}\|^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}}
\longrightarrow
\overset{\text{Galilei}}{\begin{pmatrix} R & \boldsymbol{b} & \boldsymbol{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}}
$$

The horizon of Kerr-Newman black holes

The Kerr-Newman metric,

$$g = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\varphi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(a\,dt - (r^2+a^2)d\varphi\right)^2 + \Sigma d\theta^2 + \frac{\Sigma}{\Delta}dr^2$$

The induced metric on this horizon, at $\Delta = 0$ with $r = \text{const}$, is,

$$\widetilde{g} = \frac{\sin^2\theta}{\Sigma}\left(a\,dt - (r^2+a^2)d\varphi\right)^2 + \Sigma d\theta^2$$

The vector field ξ such that $g(\xi) = 0$ and $L_\xi g = 0$ is,

$$\xi = \partial_t + \frac{a}{r^2+a^2}\partial_\varphi$$

Change of coordinates $(\theta, \varphi, t) \mapsto (\theta, \widetilde{\varphi} = \varphi - \frac{a}{r^2+a^2}s, s = t)$,

$$\widetilde{g} = \frac{(r^2+a^2)\sin^2\theta}{\Sigma}d\widetilde{\varphi}^2 + \Sigma d\theta^2 \quad \& \quad \xi = \partial_s$$