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# Conformal Carrollian Spin-3 Gravity

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**Iva Lovrekovic, TU Wien**

# OUTLINE

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# INTRODUCTION

- Motivation for research of conformal carroll spin three gravity in three dimensions comes from several sides
- Conformal symmetry was been studied by 't Hooft because of it's important role at the Planck energy scale ['t Hooft 2010,2013,2015 ]. As an additional symmetry, it often brings simplifications
- In studying higher spin theories, first step is usually considering spin-3 fields
- In three dimensions one can describe consistent higher spin theory in form of Chern-Simons action [Blencowe 1988, Pope and Townsend 1989, Grigoriev et al 2019]
- Carroll limit has number of motivations, some of which are its connection with asymptotic symmetries of flat space-times described by Bondi-Metzner-Sachs (group) [Duval, Gibbons, Horvathy 2014], which is also related with near horizon boundary conditions [Afshar et al. 2016, Grumiller et al 2016].
- There is also interesting picture that relates soft theorems and asymptotic symmetries in QFT in asymptotically flat space-times with memory effects [Strominger 2017]
- [talks in past few days]

# CONSTRUCTION

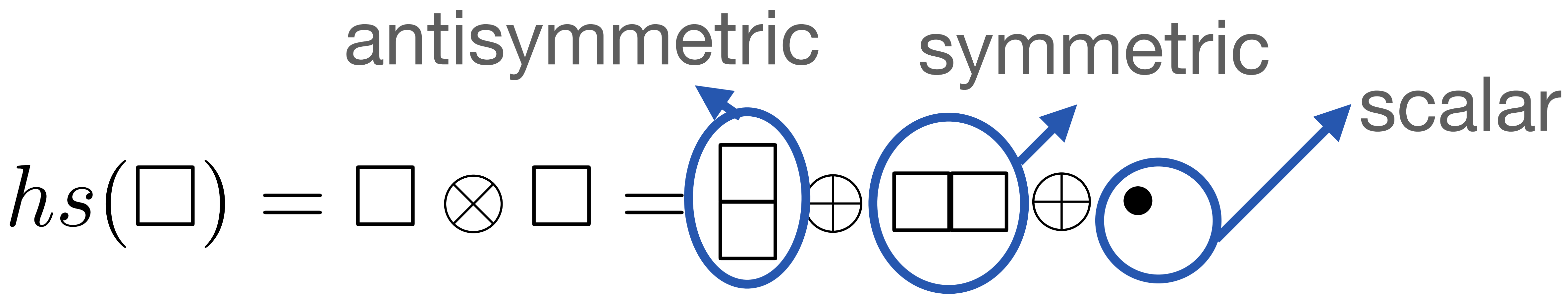
- We are going to use the action which is the same Chern-Simons action for a higher spin algebras for partially massles fields from AdS4
- To construct a finite spectrum of Fradkin-Tseytlin fields one needs to take non-trivial finite-dimensional irreducible representation V of so(3,2). Which is irreducible tensor or a spin-tensor
- Then, one needs to evaluate U(so(d,2)) in V, which means multiply the generators of so(3,2) in this representation to find the algebra "hs(V)" that they generate. The algebra is the algebra of all the the matrices of size dimV,

$$hs(V) = End(V) = V \otimes V^*$$

- We want to decompose the hs(V) into irreducible so(3,2) modules which determines the spectrum of Fradkin-Tseytlin fields

$$hs(V) = gl(V) = sl(V) \oplus u(1)$$

- Young diagram that we consider and corresponding algebra




$$hs(\square) = \square \otimes \square = \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \end{array} \oplus \bullet$$

- The algebra is matrix algebra of $(3+2)^2$ generators $t_A{}^B$
- They are decomposed with respect to so(3,2)
- They have commutation relations of $gl_{3+2}$ $[t_A{}^B, t_C{}^D] = -\delta_A{}^D t_C{}^B + \delta_C{}^B t_A{}^D$
- In the so(3,2) base one has

$$T_{AB} = t_{A|B} - t_{B|A}, \qquad S_{AB} = t_{A|B} + t_{B|A} - \frac{2}{d+2} t_C{}^C, \qquad R = t_C{}^C$$

- The commutation relations

$$[T_{AB}, T_{CD}] = \eta_{BC} T_{AD} - \eta_{AC} T_{BD} - \eta_{BD} T_{AC} + \eta_{AD} T_{BC} \quad (1)$$

$$[T_{AB}, S_{CD}] = \eta_{BC} S_{AD} - \eta_{AC} S_{BD} + \eta_{BD} S_{AC} - \eta_{AD} S_{BC} \quad (2)$$

$$[S_{AB}, S_{CD}] = \eta_{BC} T_{AD} + \eta_{AC} T_{BD} + \eta_{BD} T_{AC} + \eta_{AD} T_{BC} \quad (3)$$

$$\omega = \omega^{A,B} T_{AB} + \omega^{AB} S_{AB}$$

antisymmetric ($T_{AB}$)

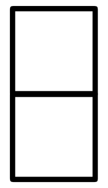
symmetric ($S_{AB}$)

$$Tr(t_A{}^B, t_C{}^D) = \delta_A{}^D \delta_C{}^D$$

- That defines invariant bilinear forms

$$Tr(T_{AB}T_{CD}) = 2(\eta_{AD}\eta_{CB} - \eta_{BD}\eta_{CA})$$

$$Tr(S_{AB}S_{CD}) = 2(\eta_{AD}\eta_{CB} + \eta_{BD}\eta_{CA} - \frac{2}{d+2}\eta_{AB}\eta_{CD})$$

- For example, for spin-2 $T_{AB}$ tensor, denoted with , we have standard conformal algebra

$$[D, P^a] = -P^a\,, \qquad [J^{ab}, P^c] = P^a\eta^{bc} - P^b\eta^{ac}\,,$$

$$[D, K^a] = +K^a\,, \qquad [J^{ab}, K^c] = K^a\eta^{bc} - K^b\eta^{ac}\,,$$

$$[P^a, K^b] = -J^{ab} + \eta^{ab}D\,, \quad [J^{ab}, J^{cd}] = J^{ad}\eta^{bc} - J^{ac}\eta^{bd} - J^{bd}\eta^{ac} + J^{bc}\eta^{ad}$$

- To identify the higher spin content for one of the higher spin algebras above we have to linearise the theory over the Minkowski vacuum.

  - Chern-Simons action

$$S[\omega] = \int Tr\left[\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right]$$

Lie algebra valued one form

$$\omega \equiv \omega^{\Lambda} t_{\Lambda}$$

eom
  - we linearise equations of motion around vacuum

$$h^a \equiv h^a{}_{\mu} dx^{\mu} \qquad h^a{}_{\mu} = \delta^a{}_{\mu} \qquad P_a \in so(3,2) \subset hs$$

$$d\omega + \omega_0 \wedge \omega + \omega \wedge \omega_0 = 0 \qquad \delta\omega = d\xi + [\omega_0, \xi]$$

gauge parameter

$$\xi \equiv \xi^{\Lambda} t_{\Lambda}$$

Lie algebra valued zero form

$$\omega_0 = h^a P_a$$

definition of vacuum

- We can write the linearised equations of motion and linearised gauge transformations around the vacuum as

$$d\omega^{\Lambda}t_{\Lambda} + h^a \wedge \omega^{\Lambda}[P_a, t_{\Lambda}] = 0 \qquad\qquad \delta\omega^{\Lambda}t_{\Lambda} = d\xi^{\Lambda}t_{\Lambda} + h^a \wedge \xi^{\Lambda}[P_a, t_{\Lambda}]$$

- IW contraction

algebra that we start with

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{i}$$

ideal

subalgebra of $\mathfrak{g}$

$$\mathfrak{i} \to \epsilon\mathfrak{i}$$

[Inönu, Wigner 1953, Bacry, LevyLeblond 1968, Bergshoeff at al. 2017]

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}\,, \qquad [\mathfrak{h}, \mathfrak{i}] \subseteq \frac{1}{\epsilon}\mathfrak{h} + \mathfrak{i}\,, \qquad [\mathfrak{i}, \mathfrak{i}] \subseteq \frac{1}{\epsilon^2}\mathfrak{h} + \frac{1}{\epsilon}\mathfrak{i}$$

$$\epsilon \to \infty$$

Example for spin-2 case:

# SPIN-2

- Conformal Carroll algebra $a = \{0, i; i = 1, 2\}$

$$P_0 \equiv H\,, \qquad K_0 \equiv K_0\,, \qquad J_{0i} \equiv B_i$$

$$\mathfrak{h} = \{P_i, K_i, D, J_{ij}\}\,, \qquad \mathfrak{i} = \{H, K_0, B_i\}$$

$$[D, H] = -H\,, \qquad [D, K_0] = K_0\,,$$
$$[B^j, P_k] = H\eta^{jk}\,, \qquad [B^j, K^k] = K^0\eta^{jk}$$
$$[H, K^j] = -B^j\,, \qquad [P^i, K^0] = B^i\,,$$
$$[J^{ij}, B^l] = B^i\eta^{jl} - B^j\eta^{il},$$

[Bagchi et al]

- Conformal Carroll algebra valued 1- and 0- form

$$\omega = e^i P_i + \tau H + \frac{1}{2}\omega^{i,j} J_{ij} + \beta^j B_j - bD + f^i K_i + \kappa K_0$$

$$\xi = e_i \xi^{i+} + \tau \xi^{0+} + \frac{1}{2}\omega_{i,j}\xi^{i,j} + \beta_j \xi^{0j} + b\xi^{+-} + f_i \xi^{i-} + \kappa \xi^{0-}$$

- components that can be gauge fixed to zero $\quad b^m = \tau^m = 0$
- remaining fields $\quad e^{mi} \equiv \phi^{mi}, e^{0i}, \tau^0, \beta^{mi}$
- dynamical equation for $\phi_{ij}$

[Hartong 2015, Herfray 2021]

$$\partial_m \partial_i \partial^i \phi^{jn} - \partial^m \partial_i \partial^j \phi^{ni} - \partial^m \partial_i \partial^n \phi^{ji} = 0$$

- Gauge transformation of the fields

$$\delta e^{mi} = \frac{1}{2}(\partial^m \xi^{i+} + \partial^i \xi^{m+}) - \frac{1}{2} h^{mi} \partial_l \xi^{l+}$$

$$\delta e^{0i} = \partial^0 \xi^{i+}$$

$$\delta \tau^0 = \partial^0 \xi^{0+} - \frac{1}{2} h_0{}^0 \partial_l \xi^{l+}$$

$$\delta \beta^{mj} = -\partial^m \partial^j \xi^{0+} + \frac{1}{2} h^{jm} \partial_l \partial^l \xi^{0+}$$

- remaining gauge parameters $\xi^{i+}$ and $\xi^{0+}$

# SPIN-3

- Spin-3 conformal Carroll algebra
- letter S denotes that generators are coming from the symmetric $S_{AB}$ generator

$$\mathfrak{h} = \{SP_j, SK_j, t_{++}, t_{--}, SJ_{jk}, SC\} \qquad \mathfrak{i} = \{SH, SK_0, SB_i\}$$

$$[P_i, SP_j] = -\eta_{ij} t_{++}\,, \quad [P_i, SK_j] = SJ_{ij} + \frac{1}{2}(SJ^k{}_k + SC)\eta_{ij}\,, \quad [P_i, t_{--}] = 2SK_{i-}\,,$$

$$[P_i, t_{++}] = 0\,, \quad [P_i, SJ_{jk}] = -\eta_{ij} SP_k - \eta_{ik} SP_j\,,$$

$$[P_i, SH] = 0\,, \quad [P_i, SK_0] = SB_i\,, \quad [P_i, SB_k] = -\eta_{ik} SH\,,$$

$$[H, t_{++}] = 0\,, \quad [H, t_{--}] = 2SK_0\,, \quad [H, SK_j] = B_j\,,$$

$$[H, t_{--}] = 2SK_0\,, \quad [H, SJ_{jk}] = 0\,. \quad [H, SP_j] = 0$$

- We expand the gauge field and gauge parameter in this algebra
- Conformal Carroll algebra valued 1- and 0- form

$$\omega = se^i SP_i + s\tau SH + \tfrac{1}{2}s\omega^{ij}SJ_{ij} + s\beta^i SB_i + \gamma SC + \frac{1}{2}\omega^{++}t_{++} + \frac{1}{2}\omega^{--}t_{--} + sf^i K_i + s\kappa K_0$$

$$\xi = se^i\xi_{i+} + s\tau S\xi^{0+} + \tfrac{1}{2}s\omega_{ij}\xi^{ij} + s\beta_i\xi^{0i} + \gamma\xi^{00} + \frac{1}{2}\omega^{++}\xi^{++} + \frac{1}{2}\omega^{--}\xi^{--} + sf_i\xi^{i-} + s\kappa\xi^{0-}$$

- We can gauge fix all the components except

$$\phi^{mni}, \omega^{0++}, s\beta^{mi}, \gamma^m, \gamma^0$$

- We obtain the constraints on the fields and a dynamical equation for the spin-3 field

$$\frac{1}{6}(-\eta^{mn}\partial_a\phi^{ija} + \eta^{jn}\partial_a\phi^{ima} + \eta^{jm}\partial_a\phi^{ina} - \eta^{in}\partial_a\phi^{jma} - \eta^{im}\partial_a\phi^{jna} + \eta^{ij}\partial_a\phi^{mna} + \partial^i\phi^{jmn} - 3\partial^j\phi^{imn} + \partial^m\phi^{ijn} + \partial^n\phi^{ijm}) = 0$$

- Gauge invariance of the spin-3 field

$$\delta\phi^{mij} = \frac{1}{2}\partial^m\partial^j\partial^i\xi^{++} - \frac{1}{10}(\partial^l\partial^j\partial_l\xi^{++}h_m{}^i + \partial^l\partial^i\partial_l\xi^{++}h_m{}^j) - \frac{1}{5}\partial_l\partial^m\partial^l\xi^{++}\eta^{ij}$$

# COMMENT ON GENERALISATION TO HIGHER SPINS

- To determine the spectrum of the theory for arbitrary spin, one has to preform “IW contraction” of the given algebra
- For the spin-s prior to contraction, the algebra will be determined by the Young Tableaux with s-1 boxes in the first row and s-t boxes in the second row
- The generators that need to be set in the h will be those with 1 and 2 components and those from the definition of the traces, to make sure algebra is closed
- The generators in 0 direction have to be part of i
- One can expect to get one dimension lower spin s field and additional undetermined fields related by constraints, and two free gauge parameters

# CONFORMAL CARROLL SPIN-2 AND SPIN-3 GRAVITY

*Spin 2*

We partially fix to radial gauge

$$\omega = b^{-1}(\rho)(d + \omega(t,\phi))b(\rho)$$

spin-2 case

$$\omega_\phi^{(s_2)} = B_i + \mathcal{H}(t,\phi)H + \mathcal{P}_i(t,\phi)P_i + \mathcal{J}_{ij}(t,\phi)J_{ij} + \mathcal{K}_i(t,\phi)K_i + \mathcal{K}_0(t,\phi)K_0 + \mathcal{D}(t,\phi)D$$

$$\omega_t^{(s_2)} = \mu(t,\phi)H$$

$$b(\rho) = e^{\rho P_2}$$

needs to be element of the group

Reading out dreibeins we find the metric

$$ds^2 = \left((\mathcal{P}_2(t,\phi) + \mathcal{D}(t,\phi)\rho)^2 + (\mathcal{P}_1(t,\phi) + \mathcal{J}_{12}(t,\phi)\rho)^2\right) d\phi^2 + 2(\mathcal{P}_2(t,\phi) + \mathcal{D}(t,\phi)\rho)d\phi d\rho + d\rho^2$$

$$d\tau = (\rho + \mathcal{H}(t,\phi))d\phi + \mu(t,\phi)dt$$

The metric is similar to the non-conformal case

*Spin 2 and Spin 3*

Follow the same procedure as above

$$\omega(t,\phi) = \omega(t,\phi)^{(s_2)} + \omega(t,\phi)^{(s_3)}$$

Minimally modify the boundary conditions $\omega_\phi^{(s_3)} = \mathcal{S}(t,\phi) t_{++}$

$$b(\rho) = e^{\rho P_2 + a_2 \rho S P_2 + a_3 B_1}$$

Group element needs to have additional dependence, which allows to eliminate the shift in $\phi$ direction that appears in Carroll gravity and its conformal version above

# CONCLUSION

- We have studied a form of IW contraction of the algebras corresponding to Young modules for conformal graviton and spin-3 field from the so(3,2) algebra
- In the first case we obtain conformal Carrollian algebra based on which we considered a gravity theory starting from Chern-Simons action.
- For the spin-3 case we followed the same line of work and constructed a gravity theory. It would be imaginable to think of it as a suitable spin-3 generalisation of Conformal gravity.
- From studying the holography of the theory we obtain the generalisation of the Carrollian gravity and for the holography of the spin-3 theory we manage to find the boundary conditions that eliminate the shift that appears in the Carrollian gravity.
- In the future study it would be interesting to consider spin-s generalisation in more detail and find a map between the conformal Carrollian algebras and the higher spin fields in the metric formulation.
- It would be also interesting to see if one could obtain additional information of the restriction of interacting vertices in higher spin gravity in this theory.

*Thank you!*