Symmetries of non-expanding horizons and corresponding charges.

Jerzy Lewandowski
Uniwersytet Warszawski,
Instytut Fizyki Teoretycznej

A. Ashtekar, N. Khera, M. Kolanowski and J.L.,
***Non-expanding horizons: multipoles and the symmetry group,* JHEP 01 (2022) 028 [2111.07873],**
***Charges and Fluxes on (Perturbed) Non-expanding Horizons,* JHEP ? (2022) ? [2112.05608]**

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Theoretical frameworks typically used:
either Noether charges and currents, or Hamiltonian charges and currents.
The latter ones are more suitable for the dynamics of GR.

On what surfaces do mathematical relativists define and describe gravitational radiation

Asymptotic boundaries

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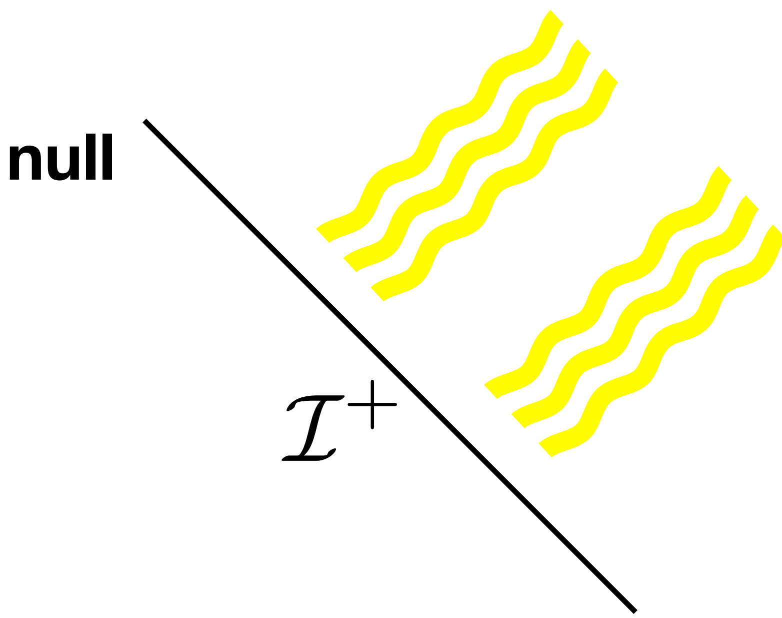
The remark of the British Prime Minister Chamberlain in 1939 on Czechoslovakia is relevant here: "This is a far far away country about which we know very little."

On what surfaces do mathematical relativists define and describe gravitational radiation

Asymptotic boundaries

$$\Lambda = 0$$



BMS group of symmetries
uniquely defined subgroup
of translations: 4-momentum

ENERGY > 0

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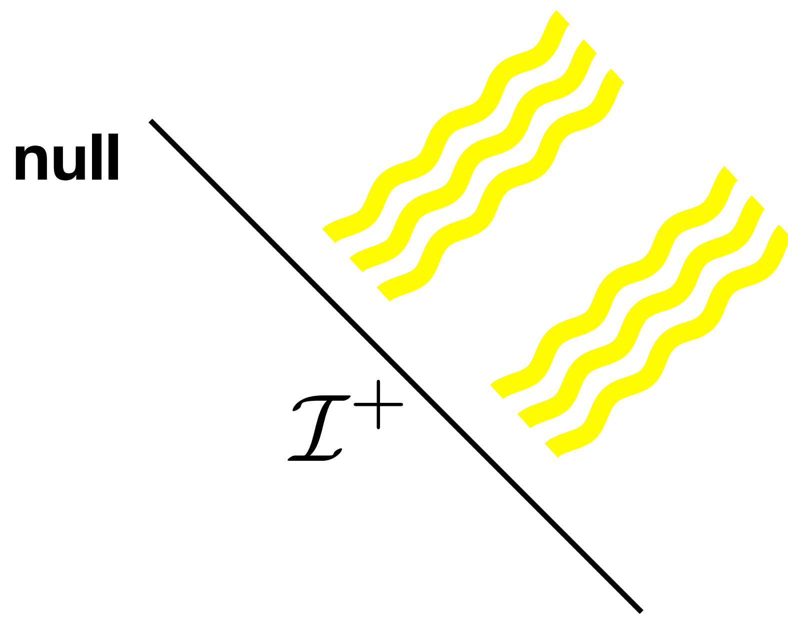

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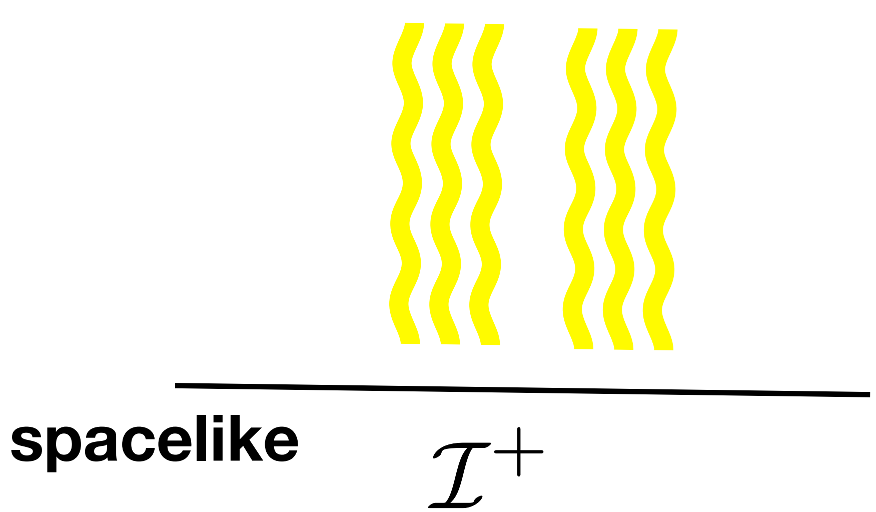
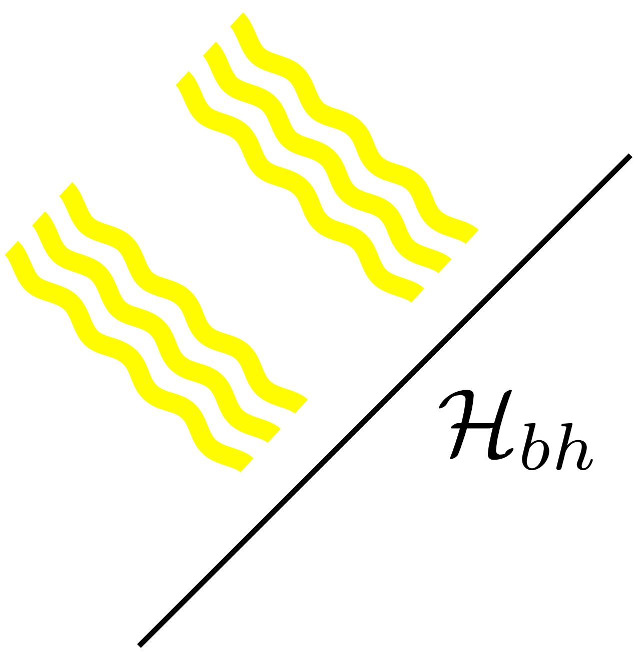


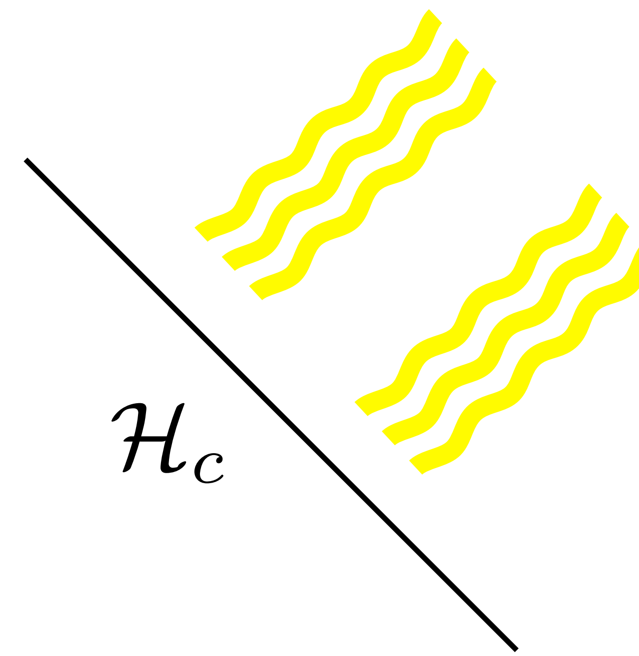
Symmetries:

all the diffeomorphisms of $\mathcal{I}^+$
Lack of unique notion of energy
or momentum, lack of the positivity.

What other surfaces?

Horizons: black hole, cosmological





Robust generalisation:

Non-expanding Horizons

Non-expanding horizon: un-embedded

Non-expanding surface is a manifold Δ

endowed with a symmetric tensor q

of signature $0+...+$

and a torsion free covariant derivative D

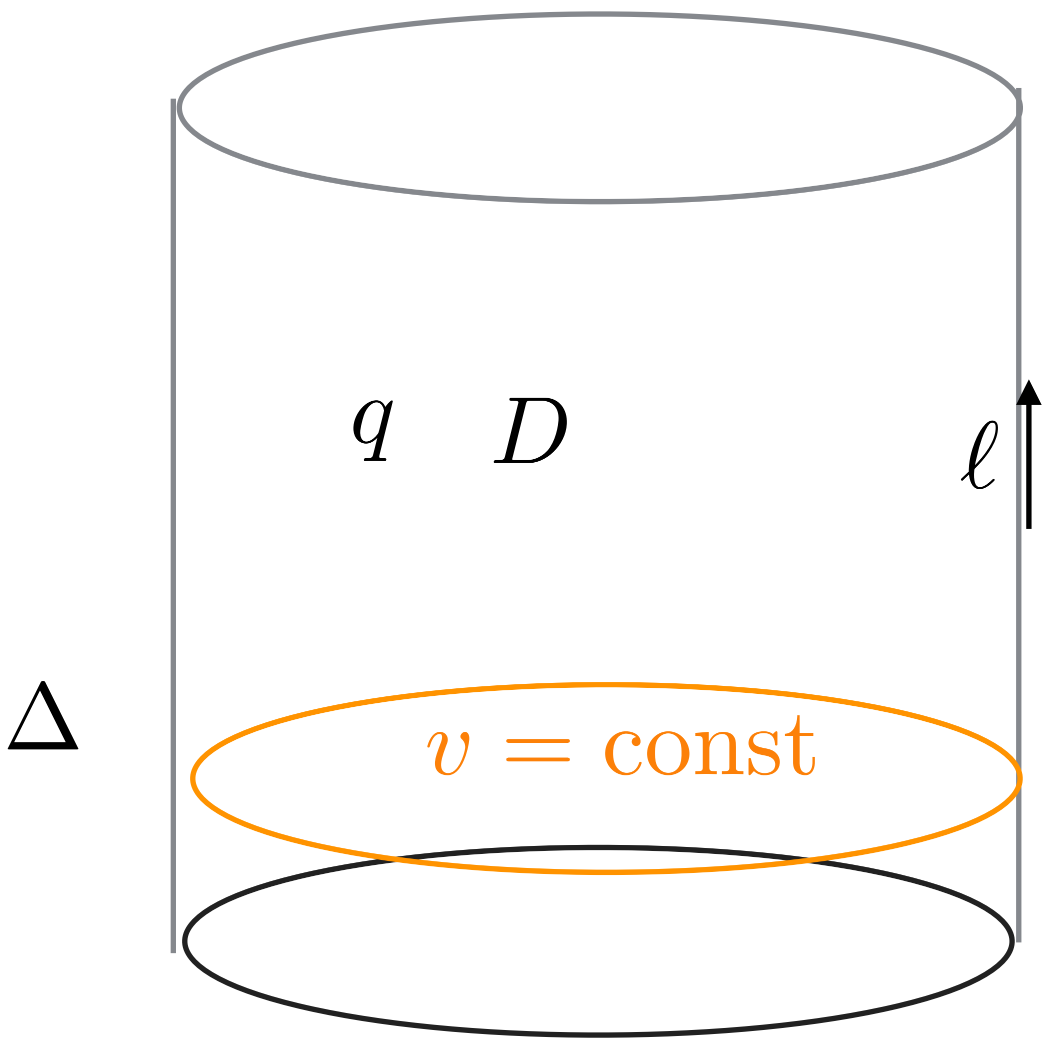
such that $Dq = 0$

A neet Lemma:

For every null vector field, that is such that : $\ell \lrcorner q = 0$

The following is true: $\mathcal{L}_\ell q = 0$

Ingredients of the non-expanding structure



The rotation 1-form potential ω

$$D_a \ell^b = \omega_a \ell^b$$

$$\ell' = f\ell \qquad \omega' = \omega + d\ln f$$

The transversal expansion-shear S_{ab}

$$\ell^a D_a v = 1$$

$$D_a D_b v =: -S_{ab}$$

Surface gravity κ

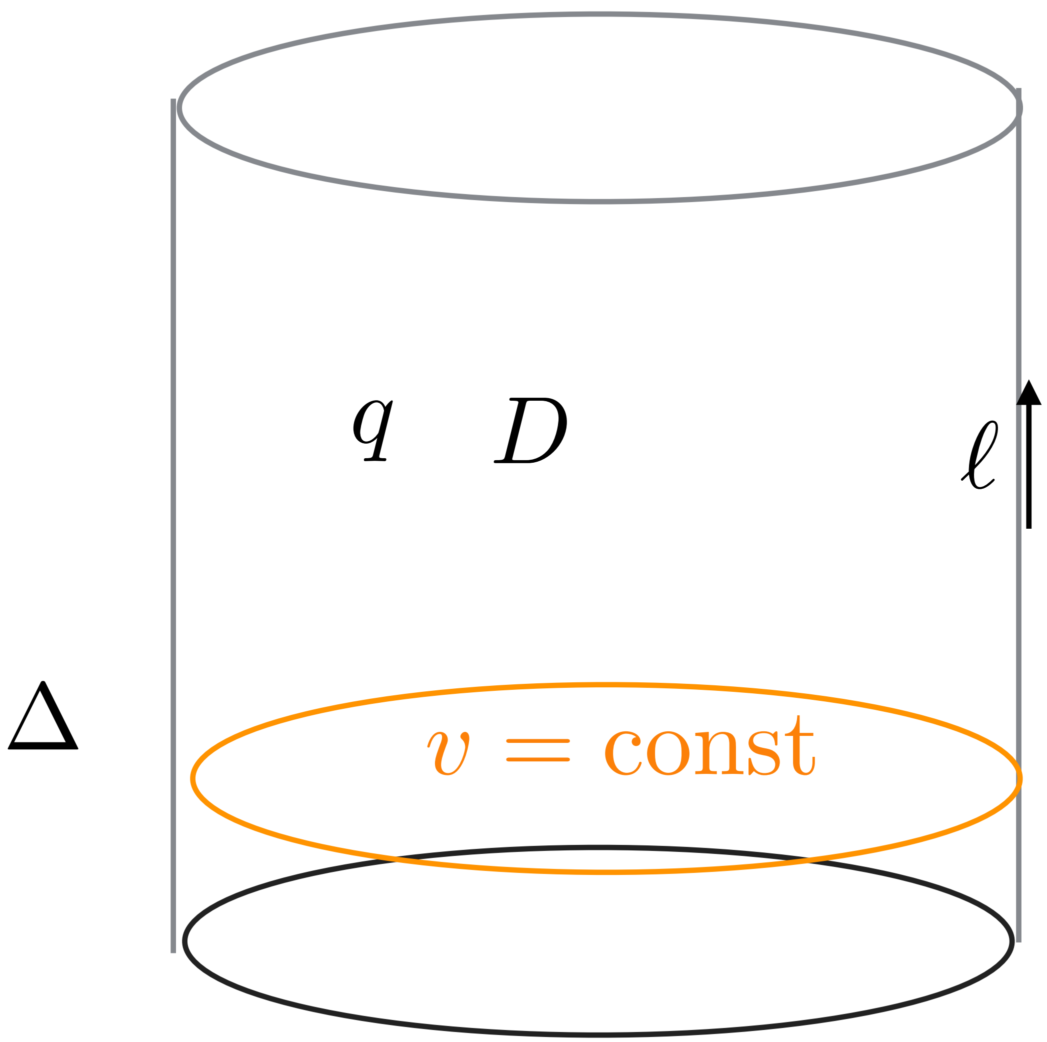
$$\kappa = \ell^a \omega_a$$

$$\ell^a D_a \ell = \kappa \ell$$

Usual assumption: trivial topology: $\Delta = \bar{\Delta} \times \mathbb{R}$

Embedded non-expanding horizon

$\Delta \subset M$ **becomes a co-dim 1 null surface**



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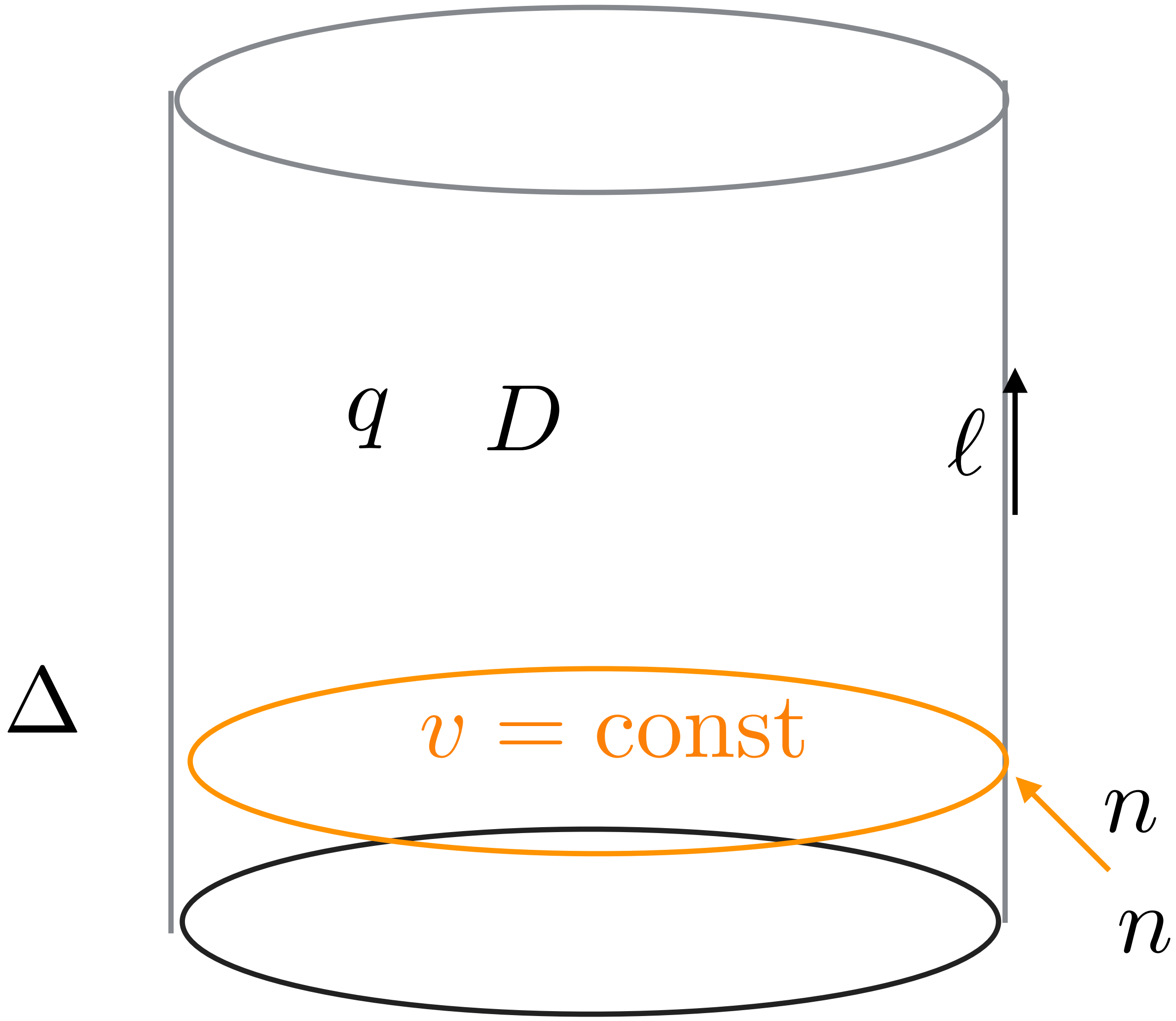

$$n^a n_a = 0 \qquad \ell^a n_a = -1$$

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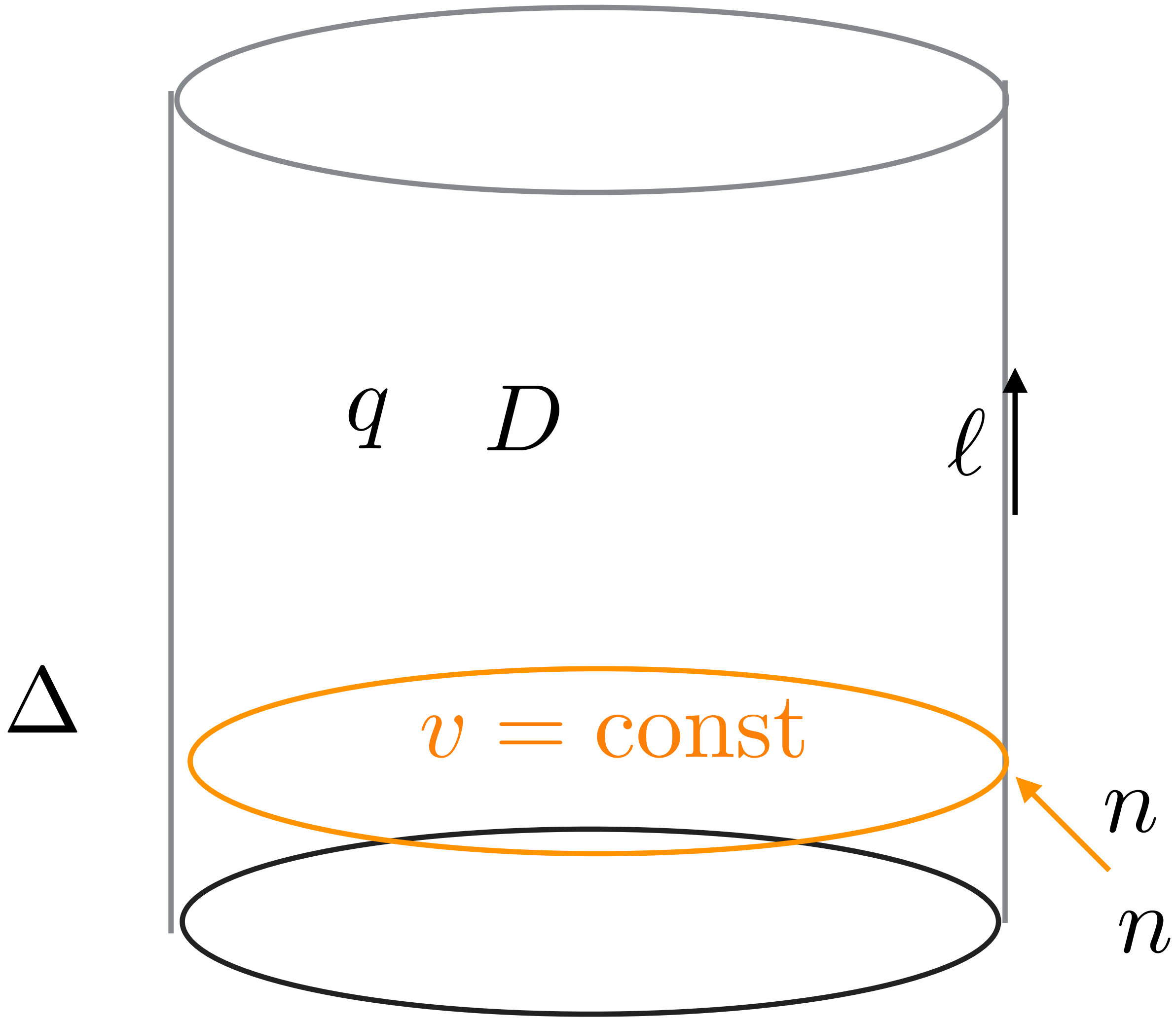

$\Delta \subset M$ **becomes a co-dim 1 null surface**

the spacetime Ricci

$$D_a \kappa = \mathcal{L}_\ell \omega_a + R_{ab} \ell^b$$

positivity assumptions and Raychaudhuri

$$\Rightarrow \quad R_{ab}\ell^b = 0 \qquad D_a \kappa = \mathcal{L}_\ell \omega_a$$

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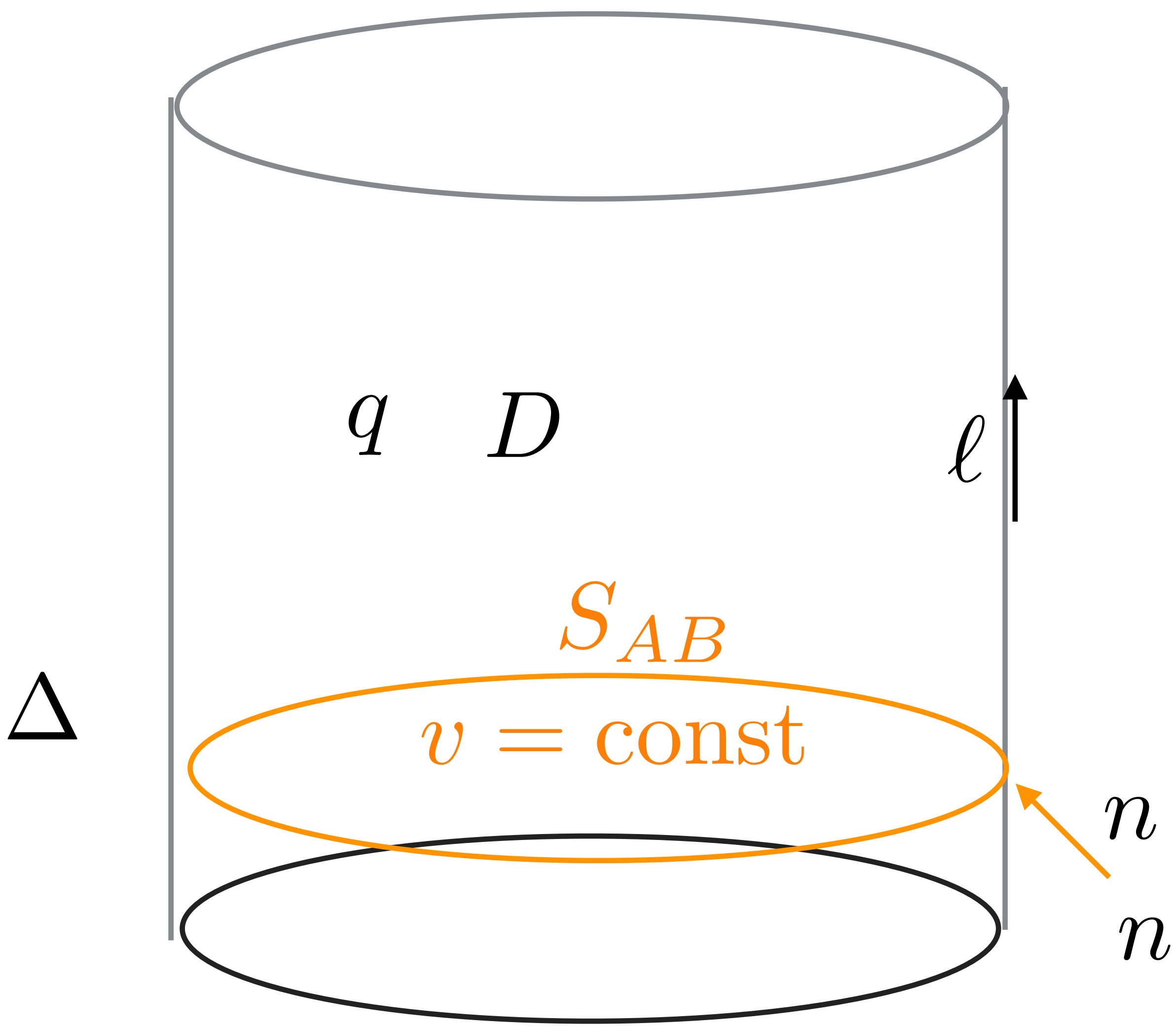

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$$D_a D_b v =: -S_{ab}$$

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$$\frac{d}{dv} S_{AB}(v) \;=\; -\kappa^\ell S_{AB}(v) \;+ \nabla_{(A}\omega_{B)} \;+\; \omega_A \omega_B \;-\; \frac{1}{2} R_{AB} \;+\; \frac{1}{2}\Lambda g_{AB},$$

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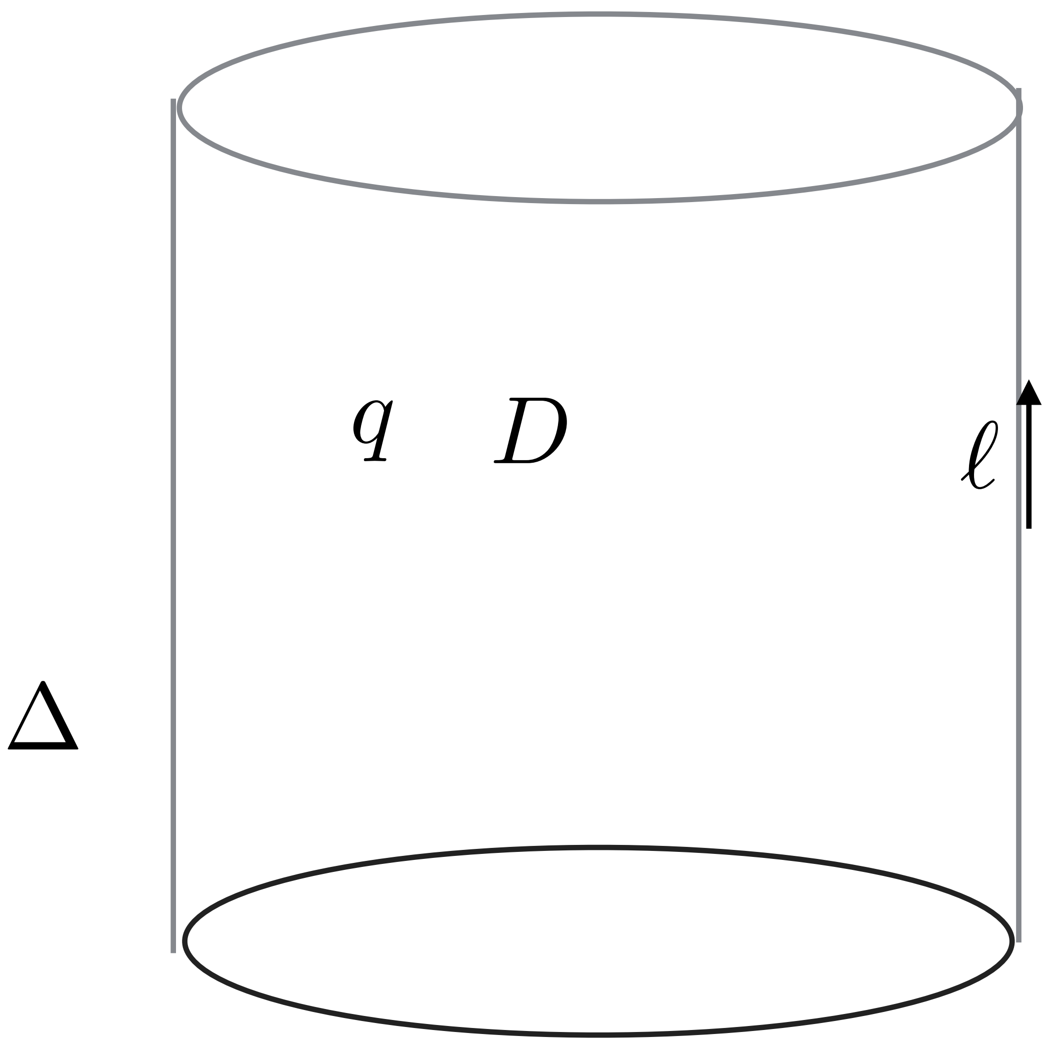

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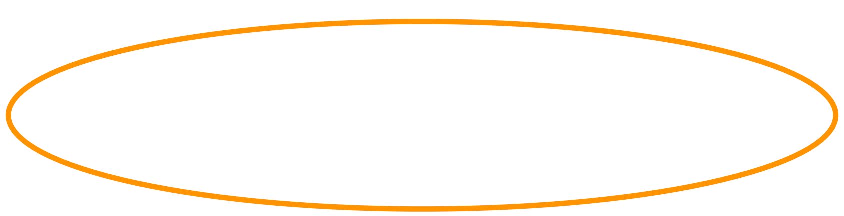
$$\Rightarrow \quad R_{ab}\ell^b = 0 \qquad D_a \kappa = \mathcal{L}_\ell \omega_a$$

via:

$$\ell' = f\ell \qquad \omega' = \omega + d\ln f$$

we can make: $\kappa' = 0$

$$\mathcal{L}_{\ell'} \omega'_a = 0$$



q_{AB} ω'_{AB}

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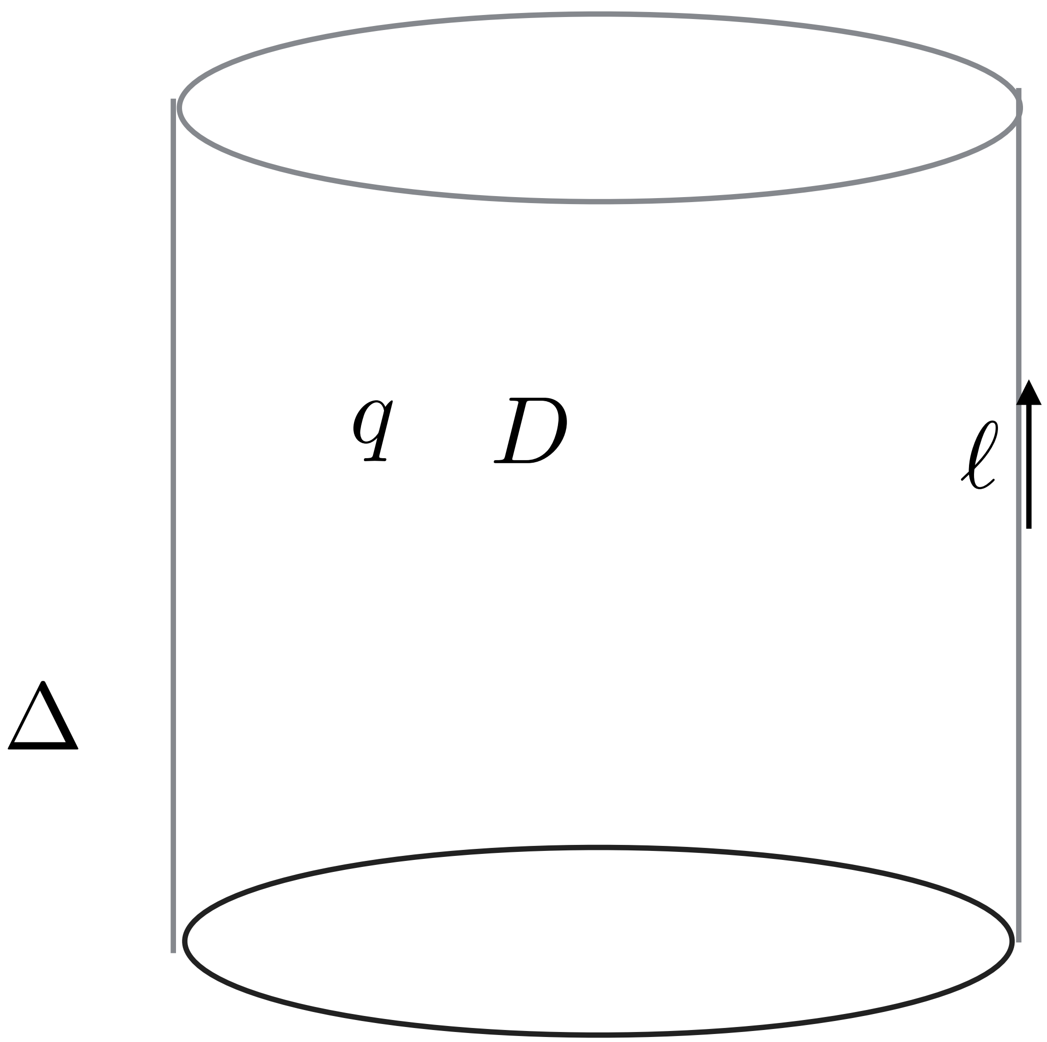

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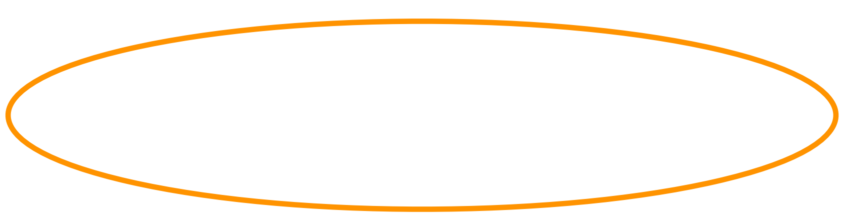
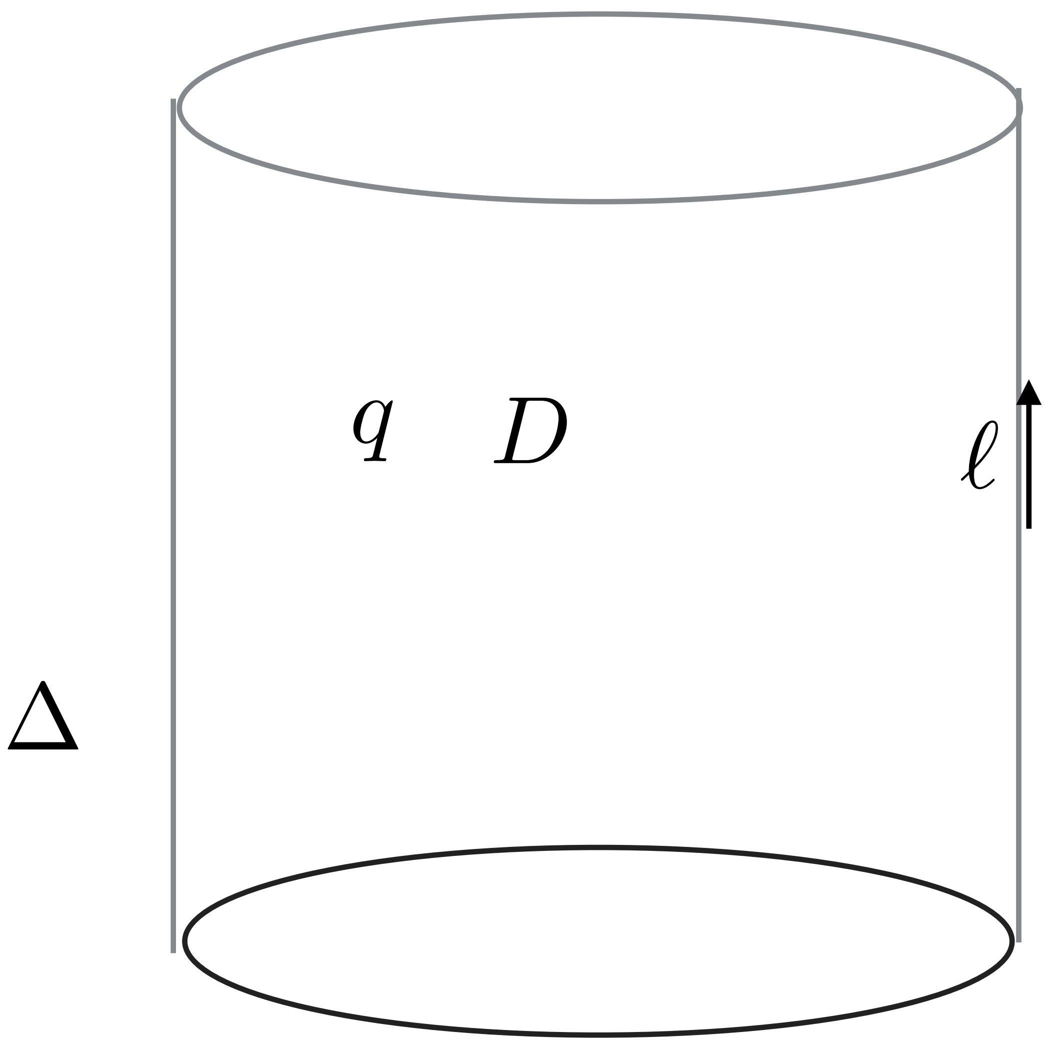
and even:



$$q_{AB} \quad \omega'_{AB} \qquad q^{AB}\nabla_A \omega'_B = 0$$

Embedded non-expanding horizons in 4d

$$\Delta = S_2 \times \mathbb{R}$$



The extended BMS

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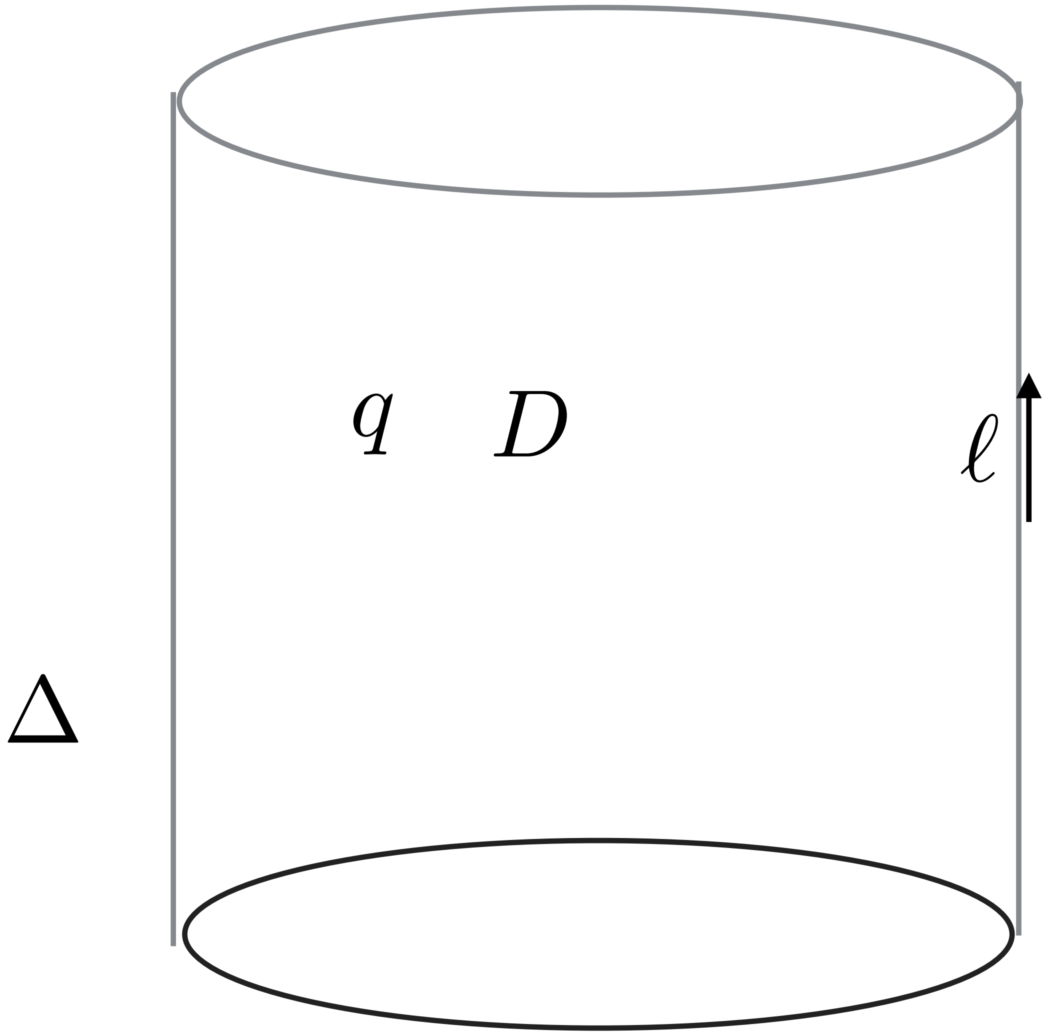

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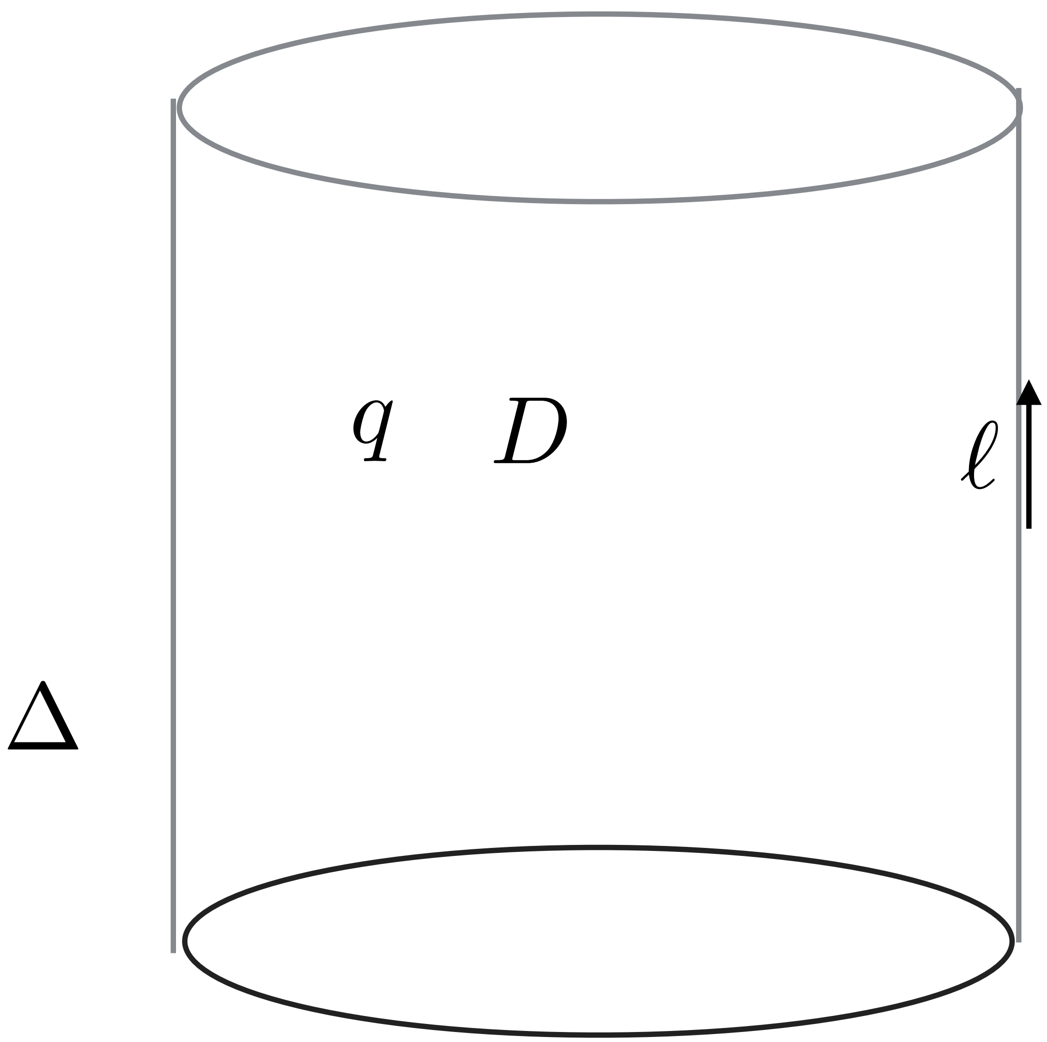

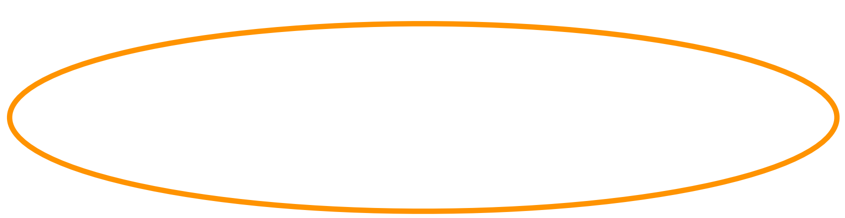

$$q_{AB} = \frac{1}{\psi^2}\mathring{q}_{AB}$$ ← **the round sphere metric**

ω_{AB}

$$q^{AB}\nabla_A \omega_B = 0$$

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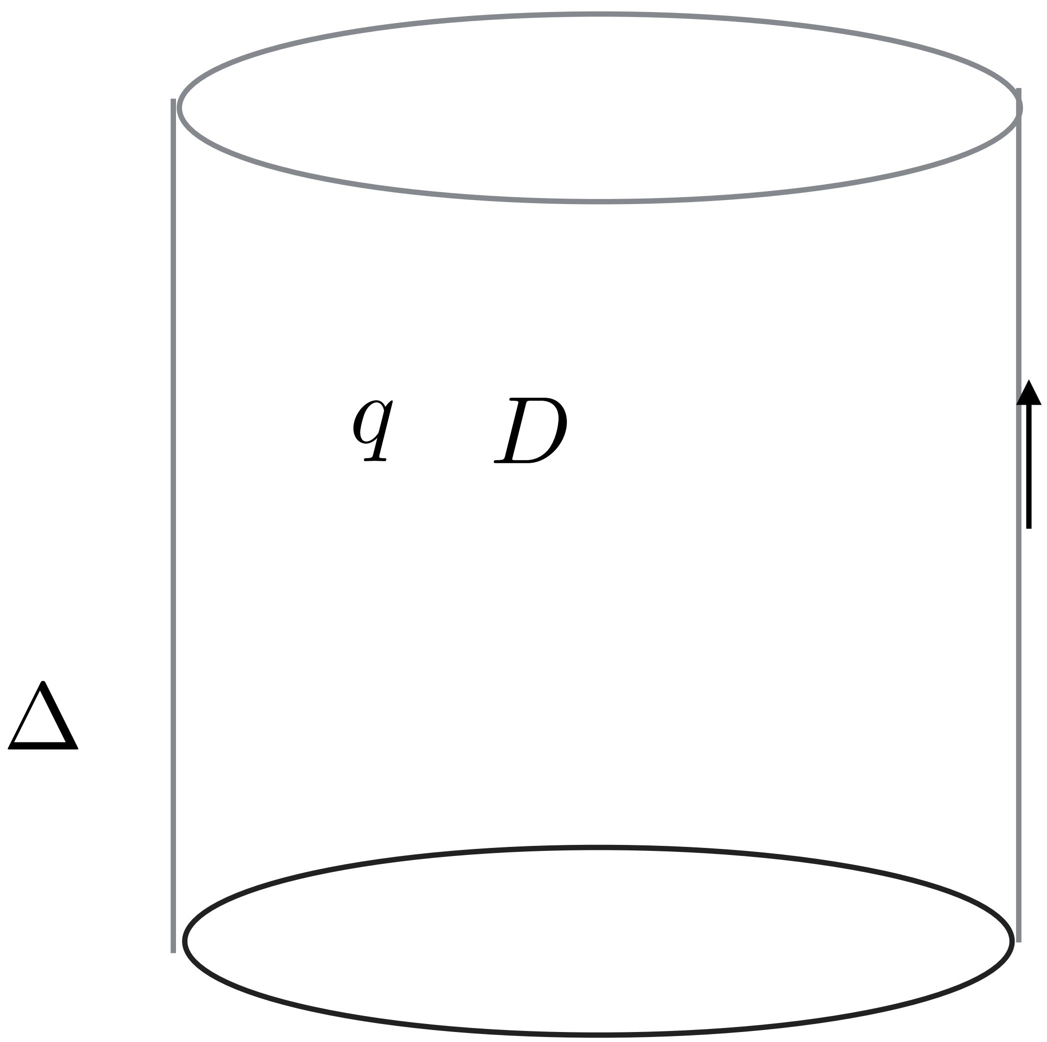

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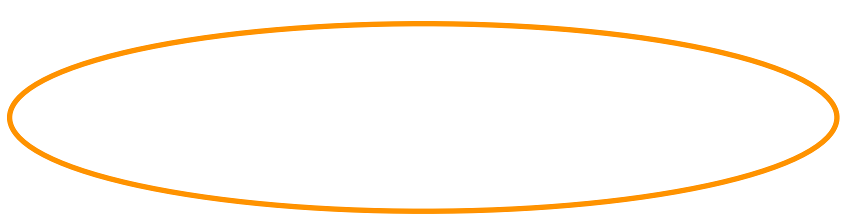

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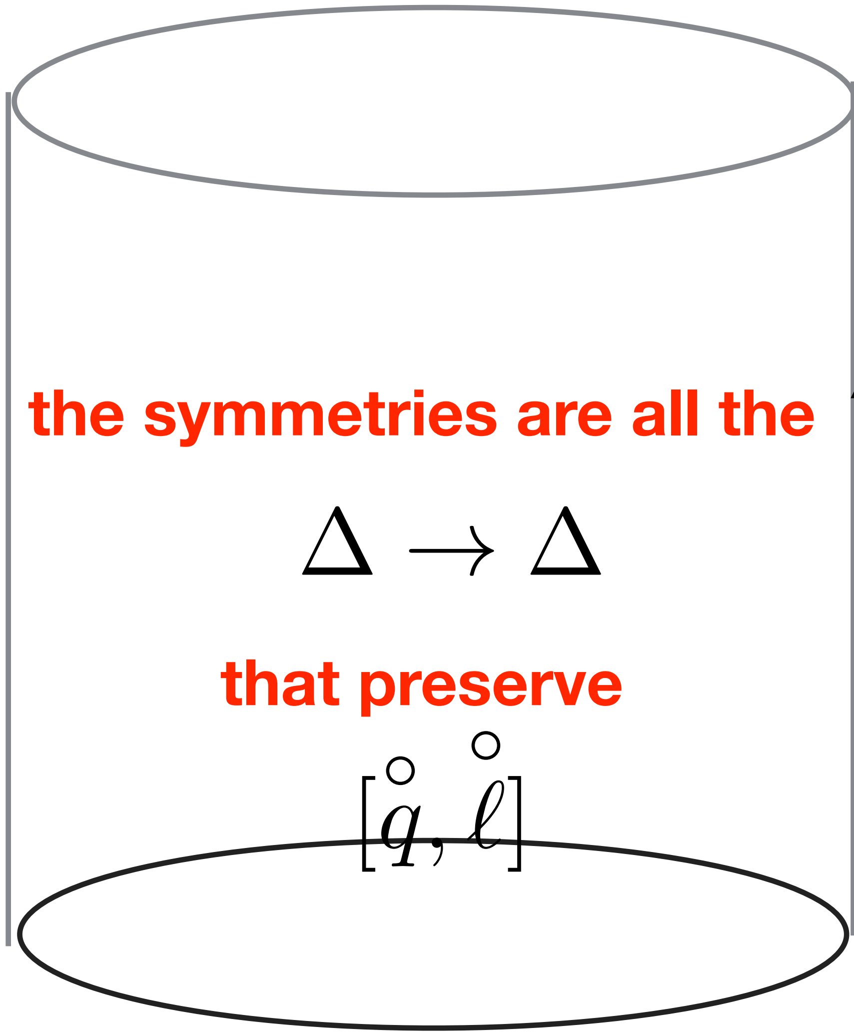


the symmetries are all the

$$\Delta \to \Delta$$

that preserve

$$[\mathring{q}, \mathring{\ell}]$$

$$\ell = \psi \mathring{\ell}$$

Δ

a conformal isometry

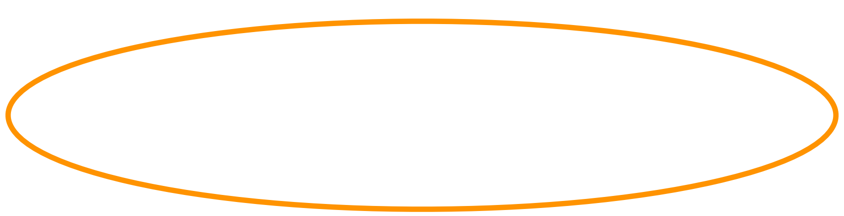
$$\mathring{q} \mapsto f^* \mathring{q} = \alpha^2 \mathring{q}$$

a constant

$$\mathring{\ell} \mapsto \frac{a}{\alpha} \mathring{\ell}$$

the underlying structure

$$[\mathring{q}, \mathring{\ell}]$$



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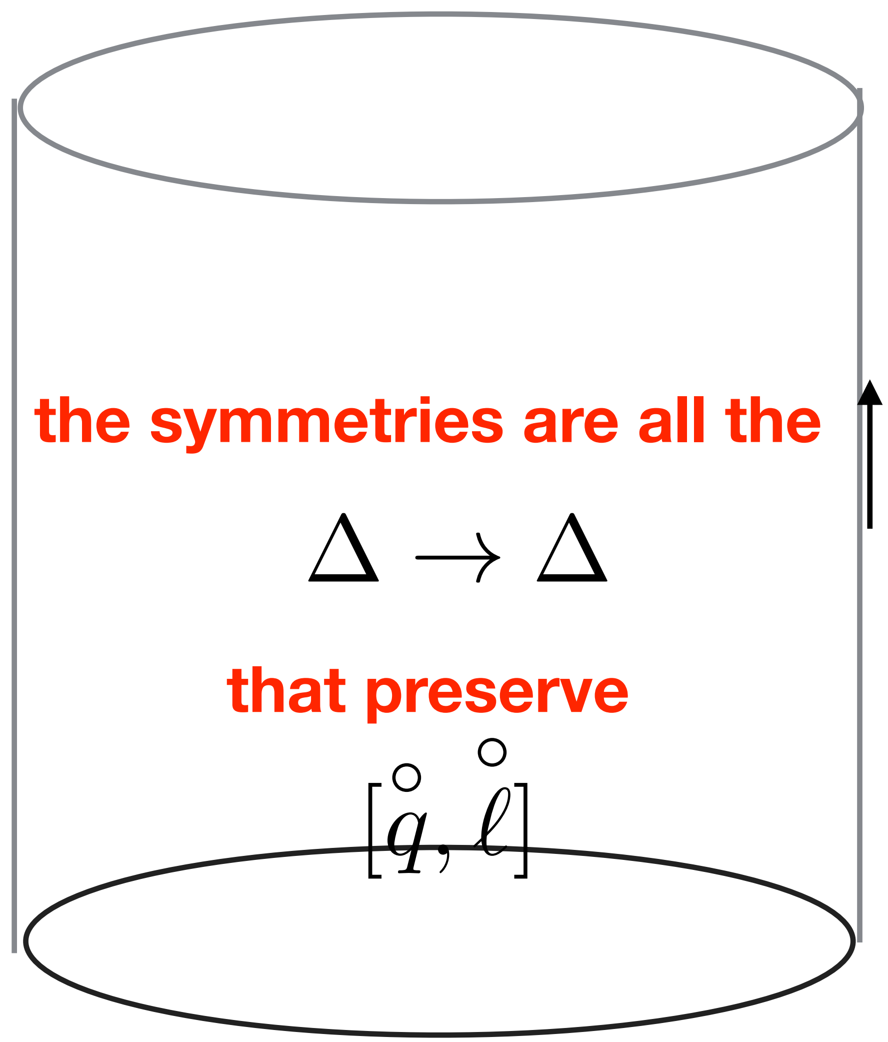

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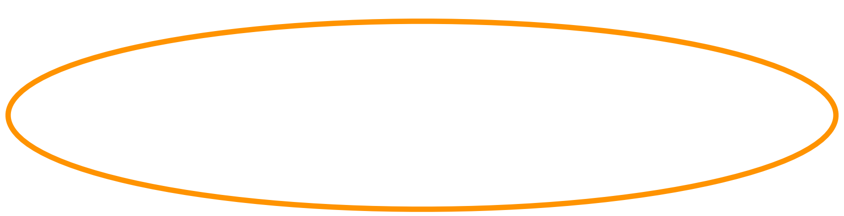

$$q^{AB} \nabla_A \omega_B = 0 \qquad \omega_{AB} \qquad \omega = -dE + *dB = *dB$$

The extended BMS

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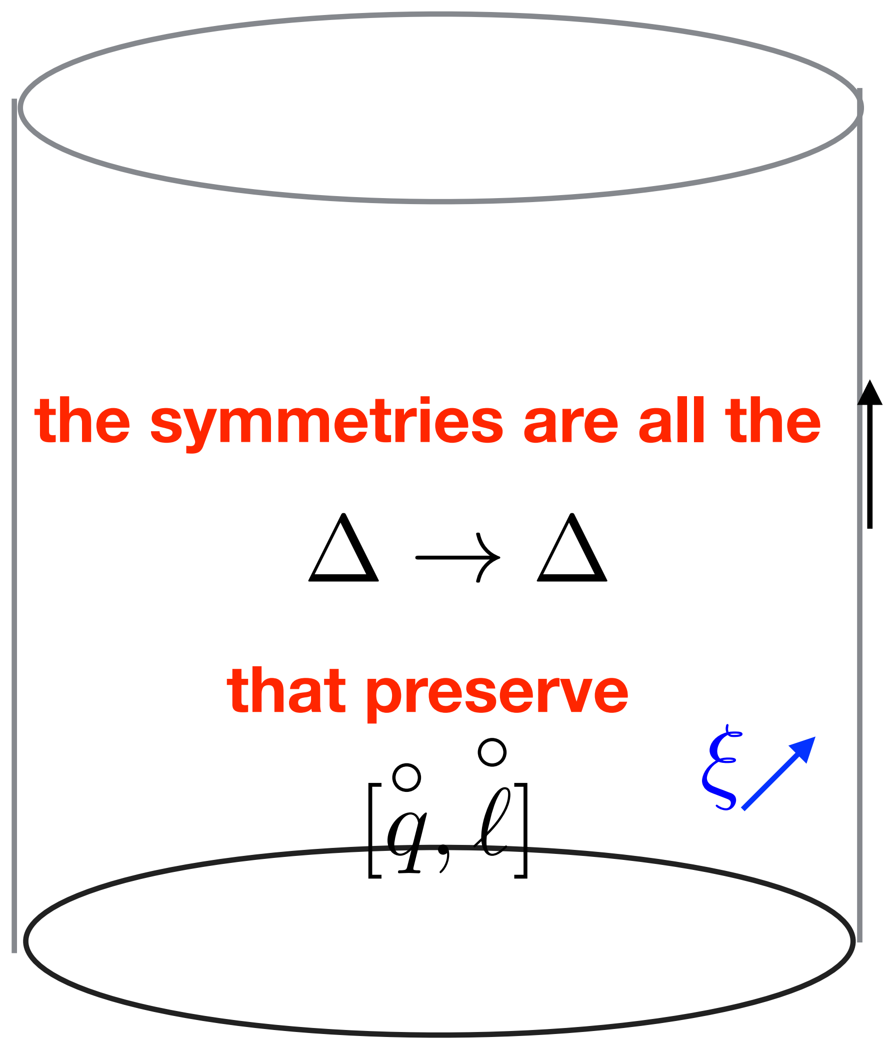
Δ



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$$q_{AB} = \frac{1}{\psi^2} \mathring{q}_{AB}$$

the round sphere metric

a symmetry generator

$$\mathcal{L}_\xi \mathring{q}_{ab} = 2\mathring{\phi}\, \mathring{q}_{ab} \qquad \mathcal{L}_\xi \mathring{\ell}^a = -(\mathring{\phi} + k)\, \mathring{\ell}^a$$

k is a constant

$\mathring{\phi} = \sum_m a_m Y_{1,m}$

$k = 0$ corresponds to the BMS

The symmetry generators

$$\xi^a = d^a + S^a + R^a + B^a$$

dilation

super translation

rotation

boost

$$d^a = \kappa \mathring{d}^a$$

surface gravity

BMS generators

additional generator

$$\xi^a = V^a_{(\xi)} + H^a_{(\xi)}, \quad \text{with} \quad V^a_{(\xi)} = (\mathring{v}\,(k + \mathring{\phi}) + \mathring{s})\mathring{\ell}^a \quad \text{and} \quad H^a_{(\xi)} = \mathring{\epsilon}^{ab}\mathring{D}_b\mathring{\chi} + \mathring{q}^{ab}\mathring{D}_b\mathring{\phi}$$

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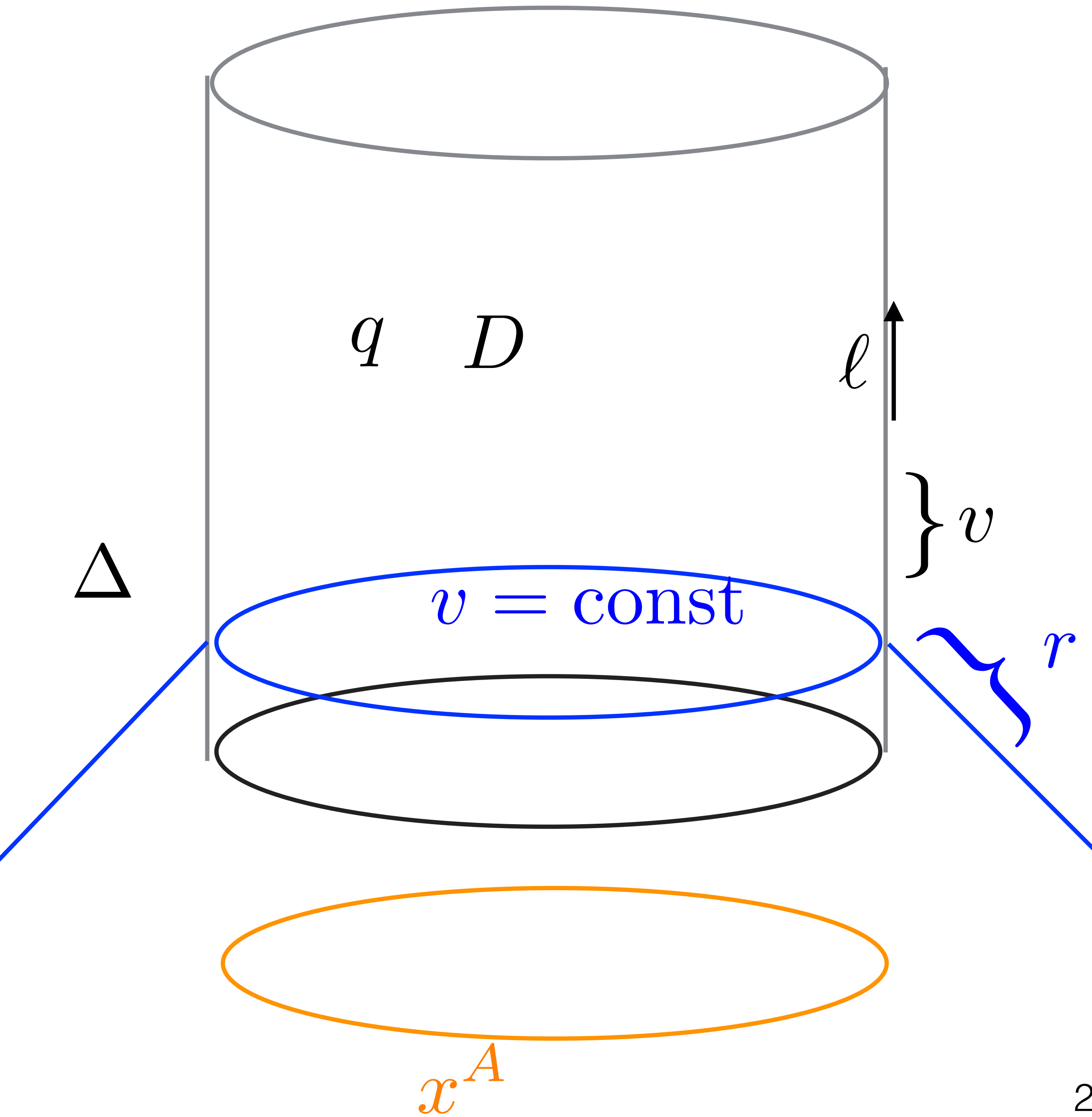
BMS generators

Example: one of the horizons in the Kerr spacetime.

$$\kappa \overset{\circ}{d} = \partial_t + \Omega \partial_\phi$$

Spacetime extension important for the charges

$$X = \overbrace{(vf_1 + f_2)\partial_v + H^A\partial_A}^{\xi} - rf_1\partial_r - r\tilde{X}^v\partial_v + r\tilde{X}^A\partial_A + r^2\tilde{\tilde{X}}^r\partial_r \,.$$



The Newman-Unti coordinates

The currents and charges

$$\boldsymbol{L} = \boldsymbol{L}(\phi) \qquad \delta \boldsymbol{L} = \boldsymbol{E}(\phi) \cdot \delta\phi + d\boldsymbol{\theta}(\phi, \delta\phi),$$

the fields

symplectic potential

$$\omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\left(\theta(\phi, \delta_2\phi)\right) - \delta_2\left(\theta(\phi, \delta_1\phi)\right)$$

symplectic current

The currents and charges

$$L(g) = \frac{1}{16\pi}(R - 2\Lambda)\epsilon$$

the volume element

$$\Theta_{abc}(g;\delta g) = \frac{1}{16\pi G}\,\epsilon_{abc}{}^{d}\left(g^{ef}\nabla_d\,\delta g_{ef} - \nabla^e\delta g_{ed}\right),$$

$$\Theta \mapsto \Theta + d() + \delta()$$

on a null surface the right choice is:

$$\underline{\Theta}_{abc}(g;\,\delta g) \;=\; \Theta_{\underleftarrow{abc}}(g;\delta g) - \frac{1}{8\pi G}\,\delta\big((\theta_{(l)}\,\epsilon_{abc})(g)\big)$$

Wald, Zupas 2008, Chandrasekaran, Flanagan, Prabhu 2018

$$\epsilon_{abc} := n^d\epsilon_{dabc}$$

The currents and charges

$$\underline{\Theta}_{abc}(g;\, \delta g) \;=\; \Theta_{\underleftarrow{abc}}(g; \delta g) - \frac{1}{8\pi G}\,\delta\big((\theta_{(l)}\,\epsilon_{abc})(g)\big)$$

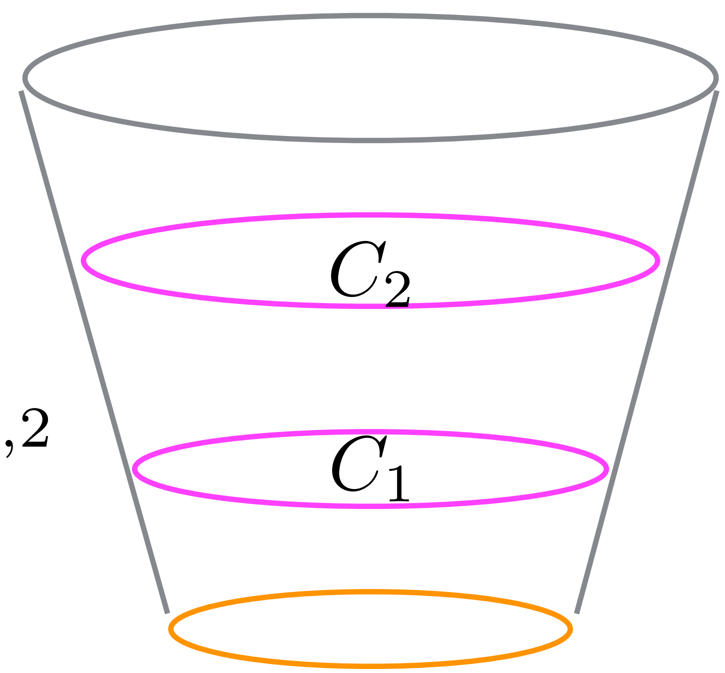
a generator of symmetry of $\mathcal{H}$

$$\omega(g, \delta g, \mathcal{L}_\xi g) = \delta\left(\underline{\Theta}(g, \mathcal{L}_\xi g)\right) - \mathcal{L}_\xi\left(\underline{\Theta}(g, \delta g)\right)$$

upon the pullback onto the null surface

$$= d\left(\xi \lrcorner \underline{\Theta}(g, \delta g)\right)$$

$\mathcal{H}_{1,2}$



$$\mathcal{F}_\xi[\mathcal{H}_{1,2}] = \int_{\mathcal{H}_{1,2}} \underline{\Theta}(g, \mathcal{L}_\xi g) = Q_\xi[C_2] - Q_\xi[C_1]$$

The currents and charges

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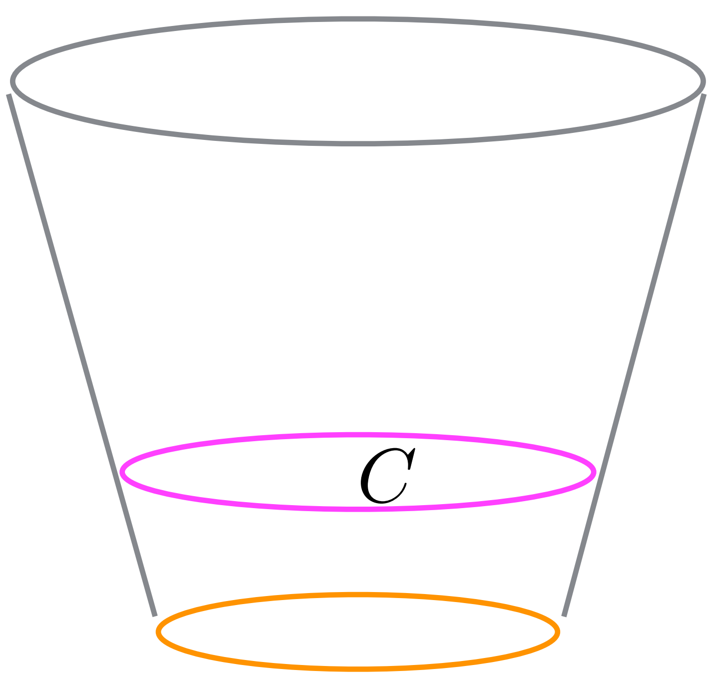
the Noether charge:

$$Q^N_{ab}[\xi] = -\frac{1}{16\pi}\epsilon_{abcd}\nabla^c\xi^d$$

the symplectic charge:

$$Q_\xi[C] = Q^{\mathrm{N}}_\xi[C] - \frac{1}{8\pi G}\oint_C \theta_{(\ell)}\,\xi^a\,\epsilon_{abc}$$



The metric perturbations

$$g_{ab}(\lambda) = {}^{\circ}g_{ab} + \lambda \frac{\mathrm{d}g_{ab}(\lambda)}{\mathrm{d}\lambda}|_{\lambda=0} + \frac{\lambda^2}{2}\frac{\mathrm{d}^2 g_{ab}(\lambda)}{\mathrm{d}\lambda^2}|_{\lambda=0} + \dots$$
$$=: {}^{\circ}g_{ab} + \lambda\, {}^{1}h_{ab} + \frac{\lambda^2}{2}\, {}^{2}h_{ab} + \dots$$

such that:

Δ is null to all the orders

the expansion and shear of ℓ vanish to the 0th order

the expansion of ℓ vanishes to the 1st order due to the the future boundary condition: the perturbed horizon approaches a NEH

$$\ell^a D_a(\theta) = -\frac{1}{2}\theta^2 - \sigma_{AB}\sigma^{AB} - R_{ab}\ell^a\ell^b$$

$$\ell^a D_a(\delta\theta) = 0 \qquad \delta\theta = \mathrm{const} = 0$$ to the 1st order

The currents and charges

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -r^2\,\gamma\,\mathrm{d}v^2 + 2\,\mathrm{d}v\mathrm{d}r + 2r\,\beta_A\,\mathrm{d}v\mathrm{d}x^A + q_{AB}\,\mathrm{d}x^A\mathrm{d}x^B\,, \qquad \Delta : r = 0$$

$$Q_d^{(0)}\,[C] = \frac{1}{8\pi G}\,k\,A[C] \qquad \text{and} \qquad Q_S^{(0)}\,[C] = 0\,,$$

$$Q_R^{(0)}\,[C] = -\frac{1}{16\pi G}\oint_C R^c\beta_c\,\epsilon_{ab} \qquad \text{and} \qquad Q_B^{(0)}\,[C] = -\frac{1}{16\pi G}\oint_C \left(2\mathring{\phi} - B^c\beta_c\right)\epsilon_{ab}$$

for Kerr: $Q_d^{(0)}[C] := \frac{kA_\Delta}{8\pi G} = \frac{M}{2} - \Omega_\Delta\,J = Q_{\bar{l}}^{(0)} + \Omega_\Delta Q_{\bar{R}}^{(0)}.$

$$Q_d^{(1)}\,[C] = \frac{k}{8\pi G}\,A'\,[C] = \frac{k}{16\pi G}\oint_C {}^1\underline{\mathrm{h}}\;{}^o\epsilon_{ab} \qquad \text{and} \qquad Q_S^{(1)} = 0$$

$$Q_R^{(1)}\,[C] = -\frac{1}{16\pi G}\oint_C R^c(\beta'_c + \tfrac{1}{2}\;{}^1\underline{\mathrm{h}}\,\beta_c)\;{}^o\epsilon_{ab}$$

$$Q_B^{(1)}\,[C] = \frac{1}{16\pi G}\oint_C \left(2\mathring{\phi}\;{}^1\underline{\mathrm{h}} - \tilde{B}^c(\beta'_c + \tfrac{1}{2}\;{}^1\underline{\mathrm{h}}\,\beta_c)\right)\;{}^o\epsilon_{ab}\,,$$

$$\ell^a D_a Q_{\cdot}^{(1)} = 0 \qquad \mathcal{F}_{\cdot}^{(1)} = 0$$

The currents and charges

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -r^2\,\gamma\,\mathrm{d}v^2 + 2\,\mathrm{d}v\mathrm{d}r + 2r\,\beta_A\,\mathrm{d}v\mathrm{d}x^A + q_{AB}\,\mathrm{d}x^A\mathrm{d}x^B\,, \qquad \Delta : r = 0$$

$$Q_d^{(2)}\,[C] = \frac{1}{8\pi G}\oint_C k\;\epsilon''_{ab} - \theta''_{(\ell)}\,d^c\,{}^o\epsilon_{abc} \qquad \text{and} \qquad Q_S^{(2)}\,[C] = -\frac{1}{8\pi G}\oint_C \theta''_{(\ell)}\,S^c\,{}^o\epsilon_{cab}$$

$$Q_R^{(2)}\,[C] = -\frac{1}{16\pi G}\oint_C R^c\,(\beta_c\,\epsilon_{ab})''$$

$$Q_B^{(2)}\,[C] = \frac{1}{16\pi G}\oint_C 2\mathring{\phi}\,\epsilon''_{ab} - \tilde{B}^c\,(\beta_c\,\epsilon_{ab})''$$

$$\mathcal{F}_d^{(2)}[\mathcal{N}_{1,2}] = -\frac{1}{4\pi G}\int_{\mathcal{N}_{1,2}} |\sigma'^{(\ell)}_{mn}|^2\,(d^d n_d)\,{}^o\epsilon_{abc}$$

$$\mathcal{F}_S^{(2)}[\mathcal{N}_{1,2}] = -\frac{1}{4\pi G}\int_{\mathcal{N}_{1,2}} |\sigma'^{(\ell)}_{mn}|^2\,(S^d n_d)\,{}^o\epsilon_{abc}$$

$$\mathcal{F}_R^{(2)}[\mathcal{N}_{1,2}] = \frac{1}{16\pi G}\int_{\mathcal{N}_{1,2}} \Big[(\mathcal{L}_R\,{}^1h_{mn})\,({}^1\dot{h}^{mn}) - \tfrac{1}{2}\,(D_m R^m)\,({}^2\dot{\underline{\mathrm{h}}})\Big]\,{}^o\epsilon_{abc}$$

$$\mathcal{F}_B^{(2)}[\mathcal{N}_{1,2}] = \frac{1}{16\pi G}\int_{\mathcal{N}_{1,2}} \Big[(\mathcal{L}_B\,{}^1h_{mn})\,({}^1\dot{h}^{mn}) - \tfrac{1}{2}\,(D_m \tilde{B}^m)\,({}^2\dot{\underline{\mathrm{h}}})\Big]\,{}^o\epsilon_{abc}$$

- Universal structure of NEHs was derived
- Their symmetry group were investigated
- Monopole moments for *all* NEHs were introduced
- Charges and fluxes associated with the symmetry group were calculated

Application: binary black holes coalescence, after a common horizon has been formed it may be well approximated by our perturbed NEH. Then our fluxes (times 2) describe absorbed energy, momentum, …

Extremal Killing horizon to the 2nd order

$$q_{AB} \equiv g_{AB}$$

$\nabla_a \ell^b =: \omega_a \ell^b$ **rotation 1-form**

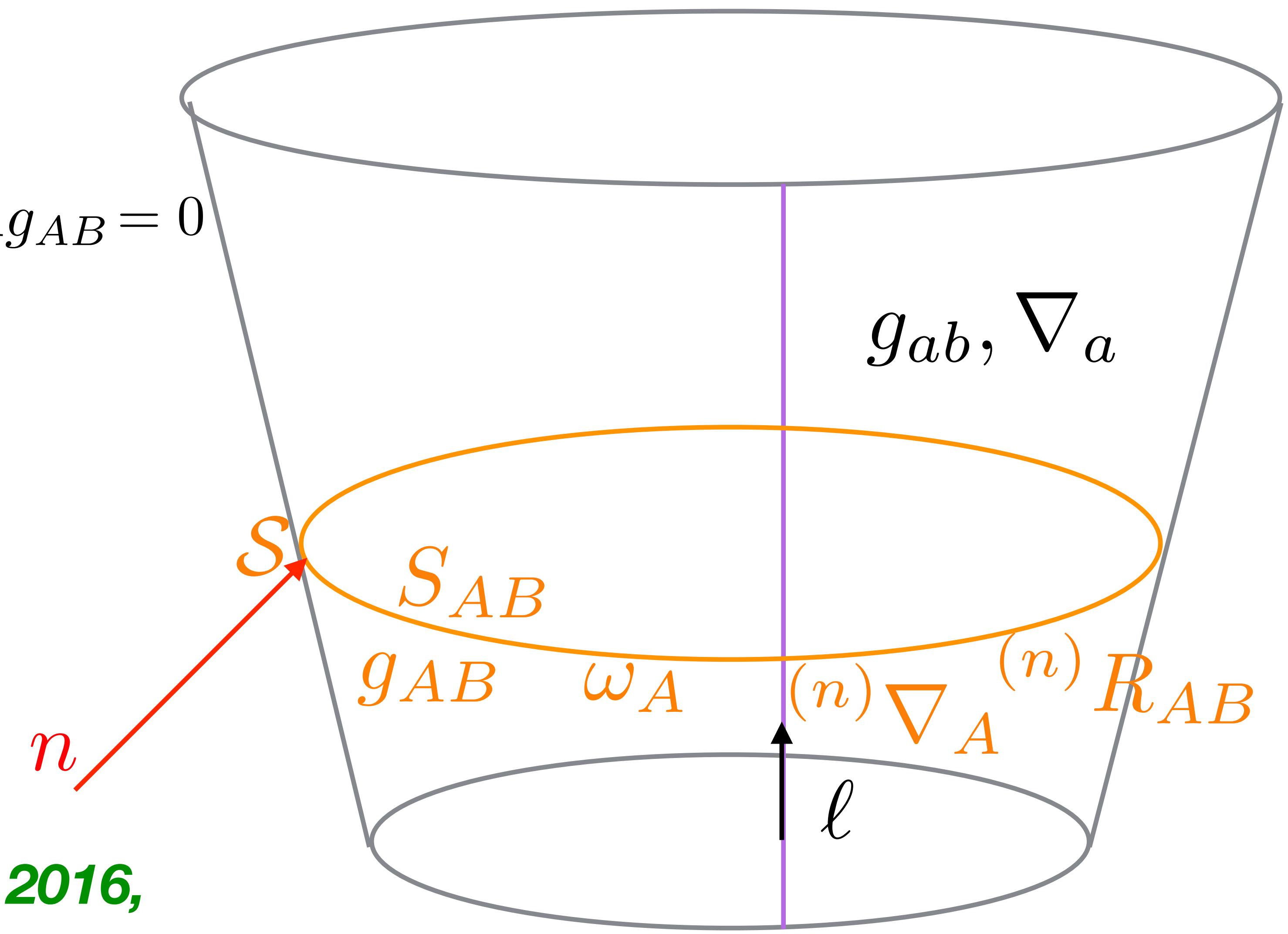
$n+1$-dim null surface

the first extremity equation:

$$^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0$$

Hajicek 1970's,
Isenberg, Moncrief 1983,
Ashtekar, Beetle, JL 2001,
JL, Pawlowski 2004
Kunduri, Lucietti 2013 (Liv. Rev.)

$$S_{AB} := \frac{1}{2}\mathcal{L}_n g_{AB}$$



second extremity equation: ***Lucietti, Li 2016,***
Kolanowski, Lewandowski, Szereszewski 2019

$$S_{AB;C}{}^{C} - S_{;AB} - 2S_{(B}{}^{C}R_{AC)} + 2S^{CD}R_{ACBD} + 2\omega^C S_{C;(AB)} + 3\omega_{(A}S_{;B)} - 3\omega^C S_{AB;C}$$
$$-2\omega_{(A}S_{B)C}{}^{;C} + 2S_{C(A}\omega^{C}_{B);} - 2\omega^C{}_{;B}S_{AC} - \omega_A\omega_B S + \omega_C\omega^C S_{AB} = 0$$

The Near Horizon Geometry spacetime

Pawłowski, JL, Jezierski 2004, Kundt 1961, Real 2003,

Given n dimensional manifold $\mathcal{S}$ endowed with g_{AB}, ω_A such that

$$ {}^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0 $$

Define on $\mathcal{S}\times\mathbb{R}\times\mathbb{R}$

$$ g_{\mu\nu}dx^\mu dx^\nu := g_{AB}dx^A dx^B - $$

$$ 2du\left[dv - 2v\omega_A dx^A - \frac{1}{2}v^2\left({}^{(n)}\nabla_A\omega^A + 2\omega^A\omega_A + \frac{2}{n}\Lambda\right)du\right] $$

Then

$$ {}^{(n+2)}G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 $$ **and** $$ S_{AB} = 0 $$

and $H = S\times\mathbb{R}\times\{v=0\}$ **is an extremal Killing horizon**

$K = v\partial_v - u\partial_u,\quad L = \partial_u$ **and non-extremal at the same time**

The Near Horizon Geometry equation in 4d

$\mathcal{S}$ - a compact 2-manifold equipped with:

$g_{AB}dx^A dx^B$ - a metric tensor, $\omega_A dx^A$ - a 1-form

$$ {}^{(2)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0 $$

K - the Gauss curvature

Λ - the cosmological constant

The integrability conditions

$$d\omega =: \Omega \; d\mathrm{Area}$$ rotation pseudo scalar

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad d\mathrm{Area}_{BC} = i(\bar{m}_B m_C - \bar{m}_C m_B)$$

First integrability condition:

$$\bar{m}^A \left({}^{(2)}\nabla_A + 3\omega_A \right) \left(K - \frac{\Lambda}{3} + i\Omega \right)^{-\frac{1}{3}} = 0$$

Second integrability condition:

$$\bar{m}^A \bar{m}^B \, {}^{(2)}\nabla_A \, {}^{(2)}\nabla_B \left(K - \frac{\Lambda}{3} + i\Omega \right)^{-\frac{1}{3}} = 0$$

Dobkowski-Ryłko, Kamiński, JL, Szereszewski 2018

NHG equation for genus > 0

Theorem *Dobkowski-Ryłko, Kamiński, JL, Szereszewski 2018*

Suppose (g_{AB}, ω_A) **are defined on a compact 2-manifold** $\mathcal{S}$

and satisfy the NHG equation:

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0 \ ;$$

Suppose

$$\chi_E(S) \leq 0$$

Then $$K = \Lambda \leq 0, \qquad \omega_A = 0$$

NHG solutions for genus =0

$$\mathcal{S} = S_2$$

axial symmetry $\Rightarrow g_{AB},\ \omega_A = g_{AB}^{\text{extremal Kerr}},\ \omega_A^{\text{extremal Kerr}}$

JL, Pawłowski 2002,
JL, Buk 2020,

uniqueness! no more solutions!

generalized to the Einstein-Maxwell case

generalized to the Einstein-Yang-Mills case

JL, Pawłowski 2002,
Kunduri, J. Lucietti 2009

Existence of non-symmetric solutions ?

only partial results known:

a unique candidate for the symmetry generator: $i\left(X^{(1,0)} - X^{(0,1)}\right)$

$${}^{(n)}\nabla_{[A}\omega_{B]} = 0 \qquad \Rightarrow K = \Lambda \geq 0,\ \omega_A = 0$$

Chruściel, Reall, Tod 2005

(non-rotating)

the linearised equation about axisymmetric solution admits only axisymmetric solutions - partly numeric

Chruściel, Szybka, Tod 2017

Applications to filing gaps in the BH uniqueness theorems

Chruściel, Costa, Heusler 2012

Uniqueness of the extremal Kerr horizon to the second order

Suppose $\mathcal{S} = S_2$ **and** g_{AB}, ω_A, S_{AB} **satisfy all the following as.:**

axial symmetry,
the first and the second extremality equation with $\Lambda = 0$**,**

and $S_A{}^A > 0$ **.**

Then, g_{AB}, ω_A, S_{AB} **correspond to the extremal Kerr solution.**

Generalisation to the Kerr-Newman is also available.

JL, Pawłowski 2003,
Lucietti, Li 2016
Kolanowski, JL, Szereszewski 2019

The integrability condition in a new role

$$\Psi_2 = K - \frac{\Lambda}{3} + i\Omega \neq 0 \quad \textbf{and} \quad \bar{m}^A \bar{m}^B {}^{(2)}\nabla_A {}^{(2)}\nabla_B \left(K - \frac{\Lambda}{3} + i\Omega \right)^{-\frac{1}{3}} = 0$$

$$\Leftrightarrow$$

the non-extremal Killing horizon to the 2nd order is of the Petrov type D

Dobkowski-Ryłko, Pawłowski, JL 2018

Therefore we call it: the type D equation.

recall: $d\omega =: \Omega \; d\mathrm{Area} \qquad g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B$

$$d\mathrm{Area}_{BC} = i(\bar{m}_B m_C - \bar{m}_C m_B)$$

Gauge transformations: $\omega'_A = \omega_A + f_{,A}$

The type D equation on g>0 compact surfaces

Theorem 1 *A pair (g, ω) is a solution to the Petrov type D equation with a cosmological constant Λ on a compact, orientable 2-surface of genus ≥ 1 if and only if g has constant Gauss curvature (Ricci scalar)*

$$K = \text{const} \neq \frac{\Lambda}{3}$$

and ω is closed

$$d\omega = 0.$$

Dobkowski-Ryłko, Kamiński, JL, Szereszewski 2018

No-hair theorem for axisymmetric solutions to the type D equation on topological sphere S.

Dobkowski-Ryłko, JL, Pawłowski 2018

Theorem 2 (no-hair):

The family of axisymmetric solutions of the type D equation with (or without) cosmological constant defined on a topological sphere can be parametrized by two numbers *(A,J)*:

the area and angular momentum, respectively. They take the following values in $\mathbb{R}^+ \times \mathbb{R}$:

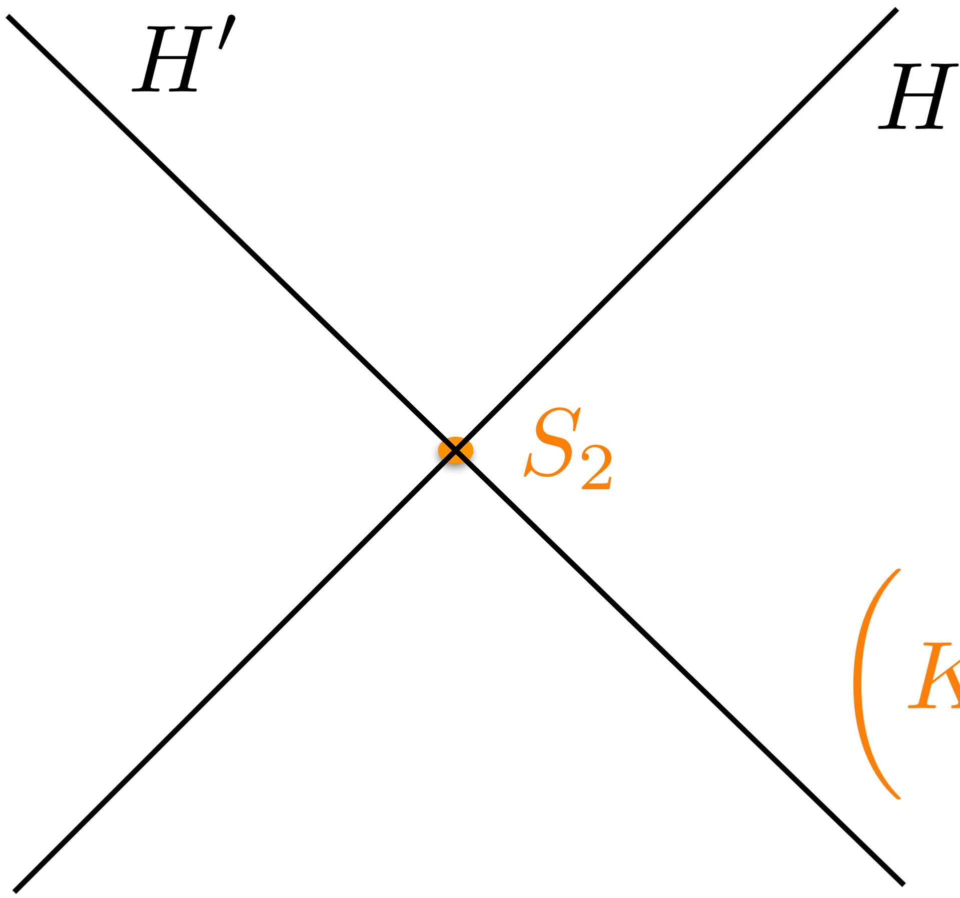
$$\Lambda > 0$$

$$J \in \Big(-\infty, \infty\Big) \text{ for } A \in \Big(0, \frac{12\pi}{\Lambda}\Big) \text{ and } |J| \in \Big[0, \frac{A}{16\pi}\sqrt{\frac{\Lambda\ A}{12\pi} - 1}\Big) \text{ for } A \in \Big(\frac{12\pi}{\Lambda}, \infty\Big)$$

$$\Lambda \leq 0$$

$$J \in \Big(-\infty, \infty\Big) \text{ and } A \in \Big(0, \infty\Big)$$

Rigidity of a bifurcated Petrov type D horizon



$$g'_{AB} = g_{AB}$$

$$\omega'_A = -\omega_A$$

$$\left(K - \frac{\Lambda}{3} + i\Omega\right)' = \left(K - \frac{\Lambda}{3} - i\Omega\right)$$

$$\bar{m}^A \bar{m}^B\, {}^{(2)}\nabla_A\, {}^{(2)}\nabla_B \left(K - \frac{\Lambda}{3} + i\Omega\right)^{-\frac{1}{3}} = 0$$

$$\bar{m}^A \bar{m}^B\, {}^{(2)}\nabla_A\, {}^{(2)}\nabla_B \left(K - \frac{\Lambda}{3} - i\Omega\right)^{-\frac{1}{3}} = 0$$

$\Rightarrow$ g_{AB}, ω_A **axially symmetric**

JL, Szereszeski 2018

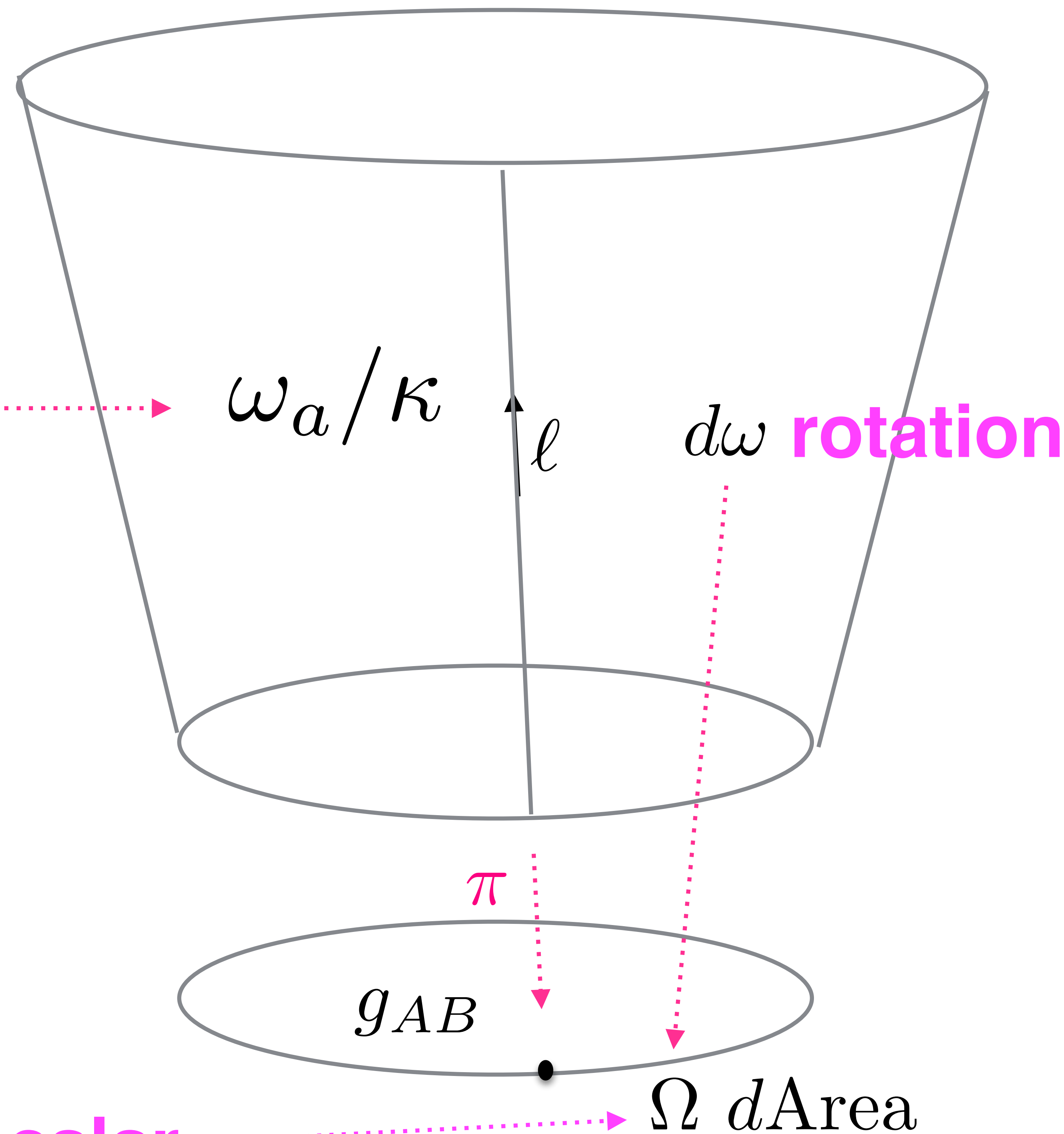
Type horizon as the Hopf bundle structure

$$\omega_a \ell^a = \kappa \neq 0$$

a connection

a principal fiber bundle:

$$\begin{array}{ccc} & & H \\ U(1),\ \mathbb{R}^+ = & G & \downarrow \\ & & S \end{array}$$



rotation pseudo scalar

The results on the Λ-vacuum Petrov type D horizons of the non-trivial bundle structure over S_2

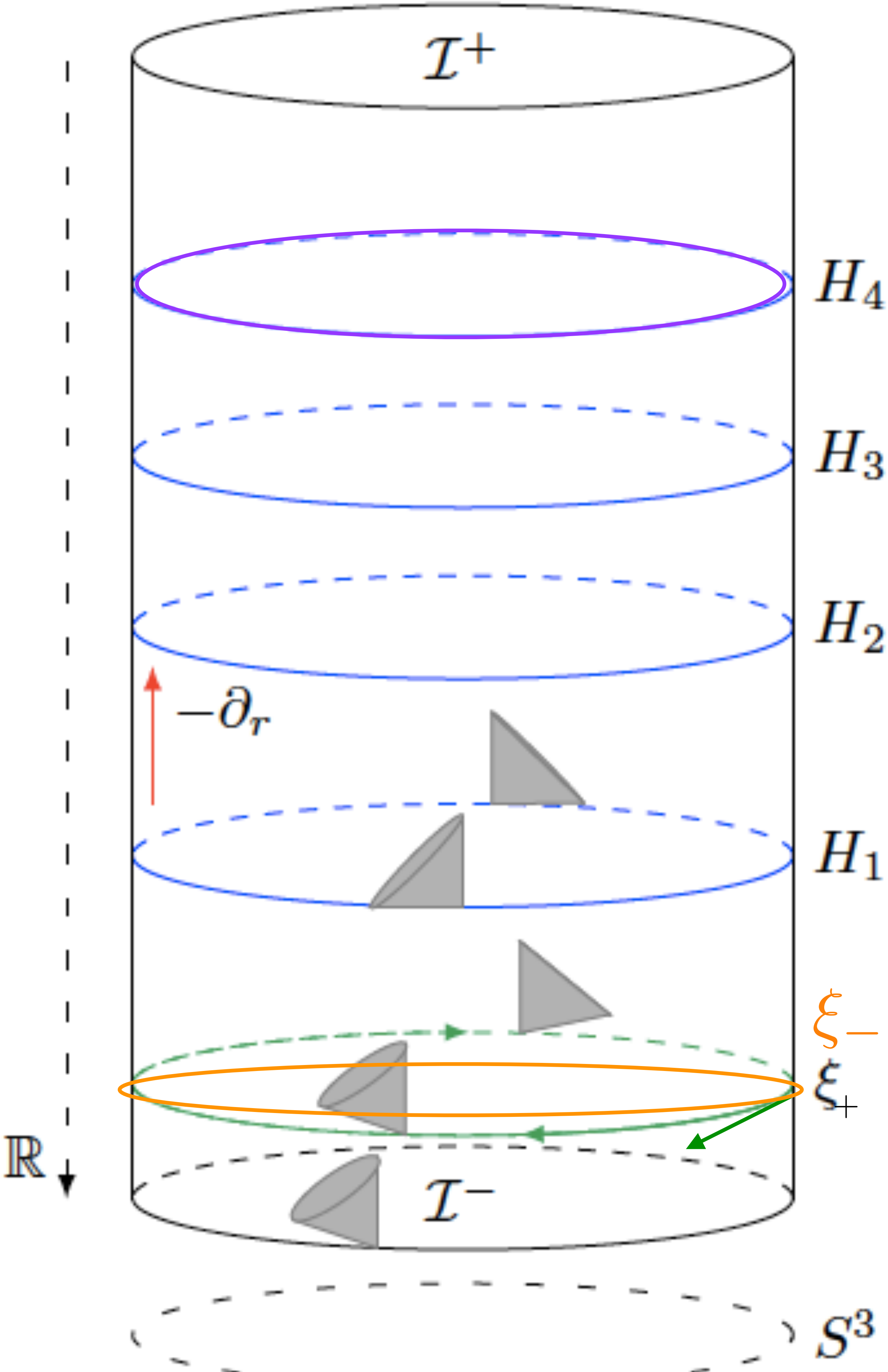
Dobkowski-Ryłko, JL, Rácz 2019

$$\int_{\mathcal{S}} \Omega \, d\mathrm{Area} = 2\pi\kappa\, m =: n \neq 0$$

We found all the axisymmetric solutions. For every value of the topological charge m and the cosmological constant Λ they set a 3d family that can be parametrized by:

- **the area radius R**
- **surface gravity times m denoted by n**
- **and one more parameter η corresponding to the rotation**

All together, there is a 4-dimensional family of solutions.



Embedded in NUT spacetimes

$$M = S_3 \times \mathbb{R}$$

the Hopf bundle

however, generically the null generators of the Killing horizons are not the fibres of the bundle.

unless:

$$\Lambda = \frac{3}{a^2 + 2l^2 + 2r_0^2}.$$

$$m = \frac{a^4 - 2a^2l^2 + l^4 + 2a^2r_0^2 - 6l^2r_0^2 + r_0^4}{2a^2r_0 + 4l^2r_0 + 4r_0^3}$$