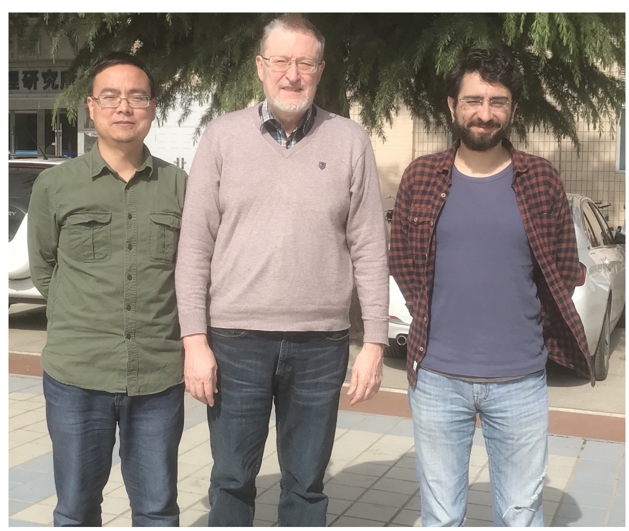
# Sleeping Beauty Memories :
# Carroll symmetry

Ussé castle – (Ch. Perrault 1697)

C. Duval, G. Gibbons, M. Cariglia    PM Zhang, P. Horvathy,
M. Elbistan



*Carroll Workshop, TU Wien. 17 Febr. 2022*

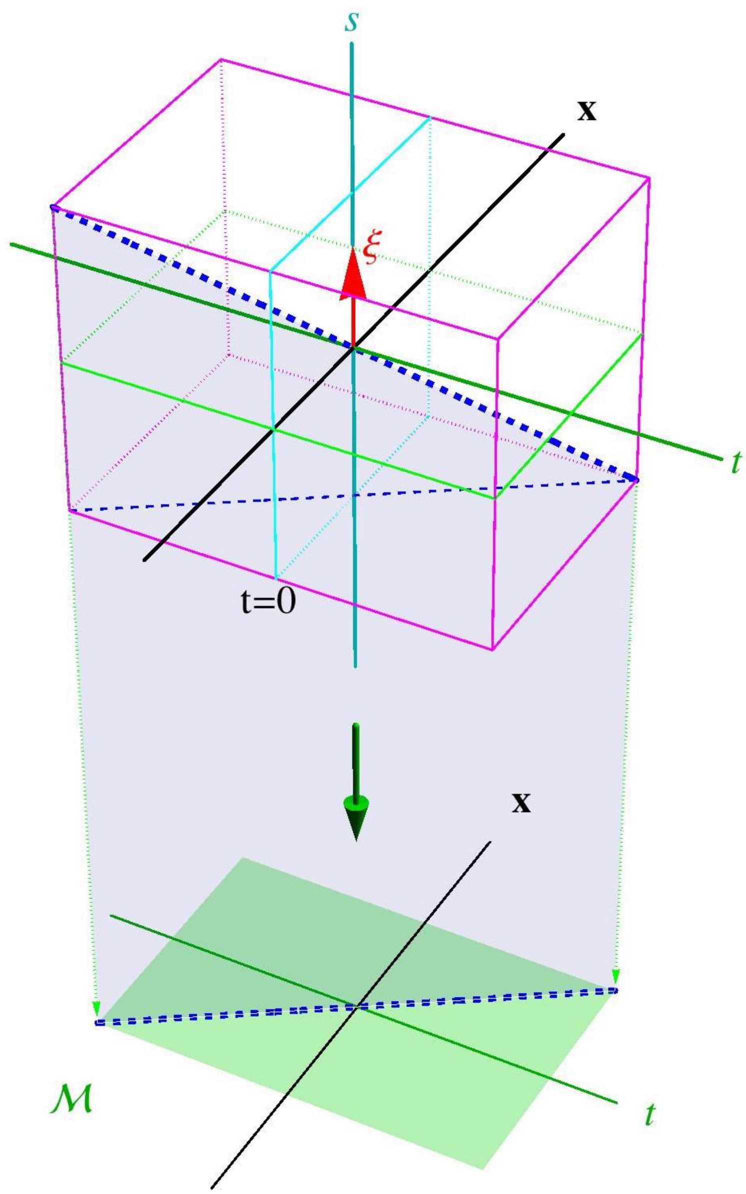
# Abstract

The $c \to \infty$ ( Galilei) and $c \to 0$ ("Carroll") contractions are unified in the "Bargmann" (B) framework : lightlike projection $(\boldsymbol{x}, t, s) \to (\boldsymbol{x}, t)$ yields non-relativistic spacetime & restriction to lightlike hypersurface $t = \text{const}$ has a Carroll structure.

The Bargmann subgroup of $d+2$ dim Poincaré group acts on the underlying $d+1$ dim non-relativistic spacetime, whereas its restriction to $t = \text{const}$ is the $d+1$ dim "Carroll" group.

The isometry group of a plane gravitational wave is isomorphic to Carroll with no rotations, implemented on the $t = \text{const}$ "Carroll" hypersurfaces through the Souriau matrix $H$. The geodesics are determined by Carrollian conserved quantities.

The unified framework refines the "no motion for Carroll" statement and conciliates it with complicated geodesic motion in a plane gravitational wave.




Bargmann space is $(d+1,1)$ dim manifold endowed with Lorentz signature metric, plus covariantly constant null vector $\boldsymbol{\xi} = \partial_s$. Lightlike projection $(\boldsymbol{x}, t, s) \to (\boldsymbol{x}, t)$ is **Galilei (Newton-Cartan)** spacetime. Null geodesics project to NR motions. Restriction to $t =$ const is **Carroll** spacetime with coords $(\boldsymbol{x}, s)$.

Plan:

1. Galilean / Carroll structures
2. Eisenhart - Duval unification
3. Plane gravitational waves (in Brinkmann)
4. Isometries of plane GWs
5. Carroll symmetry of GWs
6. Memory Effect
7. Geodesic Motion (in Brinkmann)
8. Motion from Carroll symmetry (in BJR)
9. Motions in Brinkmann
10. Outlook

# I. Galilean / Carroll structures

C. Duval, G. W. Gibbons and P. Horvathy, *"Celestial mechanics, conformal structures and gravitational waves,"* Phys. Rev. D **43** (1991), 3907 doi:10.1103/PhysRevD.43.3907 [arXiv:hep-th/0512188 [hep-th]].

C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016. [arXiv:1402.0657 [gr-qc]].

• **GR**: Space and time unified into $D = d{+}1$ manifold with coordinates $x^\mu$, $\mu = 0, \ldots, d$, endowed with metric $g_{\mu\nu}x^\mu dx^\nu$ of Lorentz signature $(d, 1)$.

isometries (for $d = 3$) : 10-parameter Poincaré

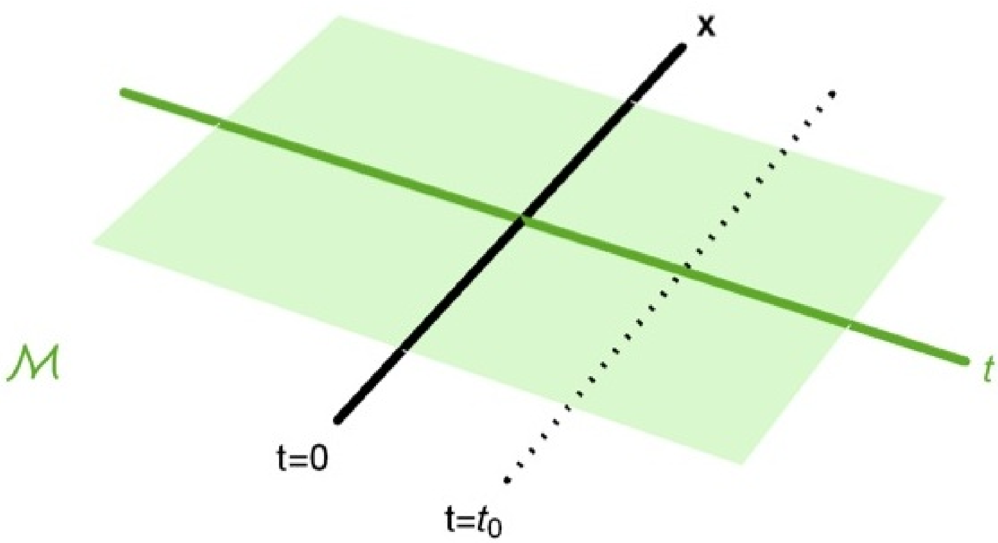
Motion of spinless particle: geodesics.

(textbooks)

• **Galilean (Newton-Cartan) spacetime**:

$(N, \gamma, \theta, \nabla)$ where $N$ $(d+1)$-dim manifold, $(\gamma^{ij})$ twice-symmetric *contravariant*, positive tensor whose kernel is generated by nowhere vanishing closed 1-form ("clock") $\theta$. $\nabla$ symmetric affine connection that parallel-transports both $\gamma$ and $\theta$.

Coordinates $(x^i, t)$, where $\theta = dt$. "metric" of signature $(d, 0)$.




**Galilean space-time** + connection $\nabla \rightsquigarrow$ Newton-Cartan str.

Projects onto absolute time axis.

Space & time (loosely) related through connection $\nabla$.

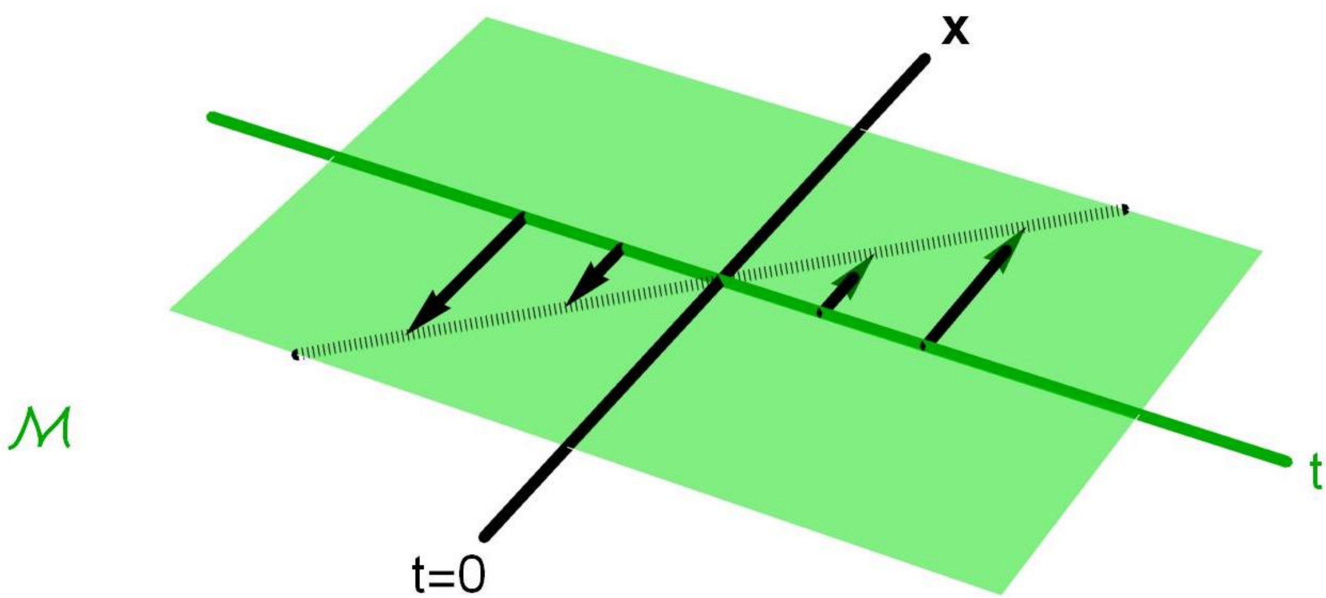
Automorphismes: transf which preserve $\gamma$, $\theta$, $\nabla$.
In $d+1$-dim flat spacetime: Galilei group

$$\begin{pmatrix} R & \mathbf{b} & \mathbf{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Gal}(d+1), \tag{1}$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $e \in \mathbb{R}$ represent orthogonal transfs, boosts, space & time transls.
Action generated by

$$(\omega^A_B x^B + \boxed{\beta^A t} + \gamma^A)\frac{\partial}{\partial x^A} + \varepsilon\frac{\partial}{\partial t} \in \mathfrak{gal}(d+1) \tag{2}$$

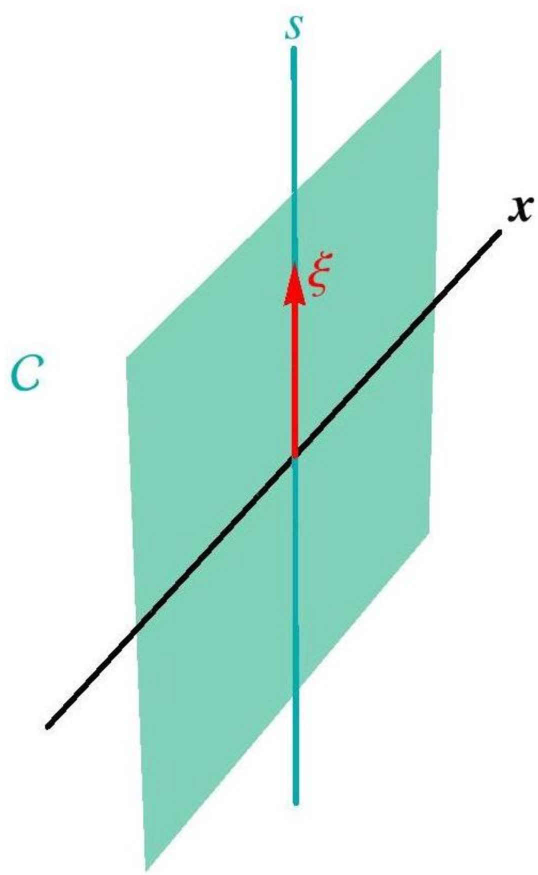
$\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$, $\varepsilon \in \mathbb{R}$.




**Galilei boost** acts on Galilei space but NOT on time.

• **Carroll structure**:

$(\mathcal{C}, \mathrm{g}, \xi, \nabla)$ where $C$ $(d+1)$-dim manif endowed with twice-symmetric *covariant*, positive tensor $\mathrm{g} = g_{ij}dx^i dx^j$ whose kernel generated by nowhere vanishing complete vector field $\boldsymbol{\xi}$. $\nabla$ symmetric affine connection that parallel-transports both g and $\xi$. Coordinates: $(x^i, s)$, $\xi = \partial_s$ generates "Carrollian time". $s$ has dim *action / mass*.




Carroll space-time $\mathcal{C}$ described by $\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix}$ endowed with "vertical" vector $\boldsymbol{\xi}$ which generates kernel of (singular) "metric".

Flat Carroll structure :

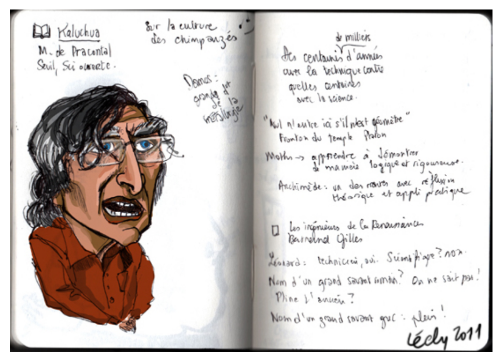
$$C^{d+1} = \mathbb{R}^d \times \mathbb{R}\,, \quad \mathsf{g} = \delta_{AB}\, dx^A dx^B, \quad \xi = \frac{\partial}{\partial s}, \quad \Gamma^k_{ij} = 0\,. \tag{3}$$

Carroll group $\mathrm{Carr}(d+1)$

$$\begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $f \in \mathbb{R}$. Implemented

$$\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix} \mapsto \begin{pmatrix} R\,\boldsymbol{x} + \mathbf{c} \\ s - \boxed{\mathbf{b} \cdot (R\,\boldsymbol{x})} + f \end{pmatrix} \tag{5}$$



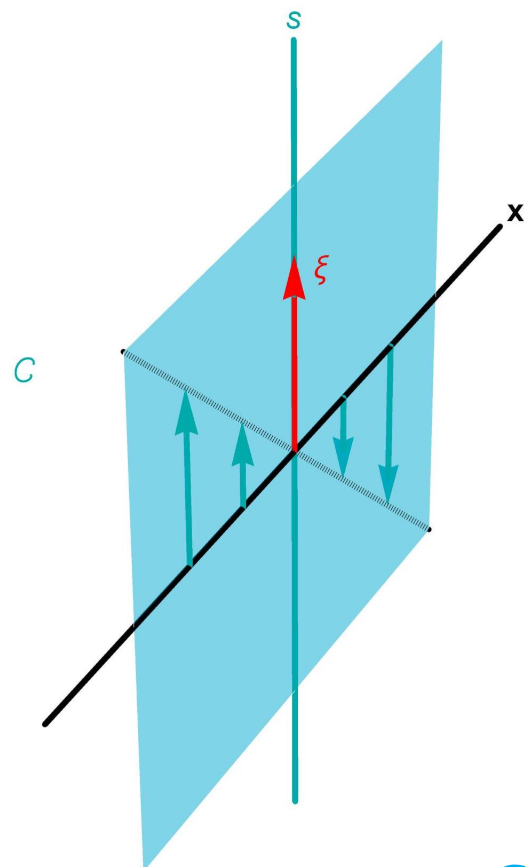
$\boldsymbol{x} \in \mathbb{R}^d$, and $s \in \mathbb{R}$. Lévy-Leblond 1965 .

**Carroll boosts** act "vertically" on "Carrollian time", NOT on space, $\boldsymbol{x}' = \boldsymbol{x}$, $s' = s - \boldsymbol{b} \cdot \boldsymbol{x}$.

Carroll algebra $\mathfrak{carr}(d+1) \sim$ vector fields

$$(\omega^A_B\, x^B + \gamma^A)\frac{\partial}{\partial x^A} + \Big(\varphi \boxed{-\beta_A\, x^A)}\Big)\frac{\partial}{\partial s}, \tag{6}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$, and $\varphi \in \mathbb{R}$.




**Carroll boosts** $\boldsymbol{x}' = \boldsymbol{x},\ s' = s - \boldsymbol{b} \cdot \boldsymbol{x}$

acts on flat Carroll space-time J. M. Lévy-Leblond, *"Une nouvelle limite non-relativiste du group de Poincaré,"* Ann. Inst. H. Poincaré **3** (1965) 1

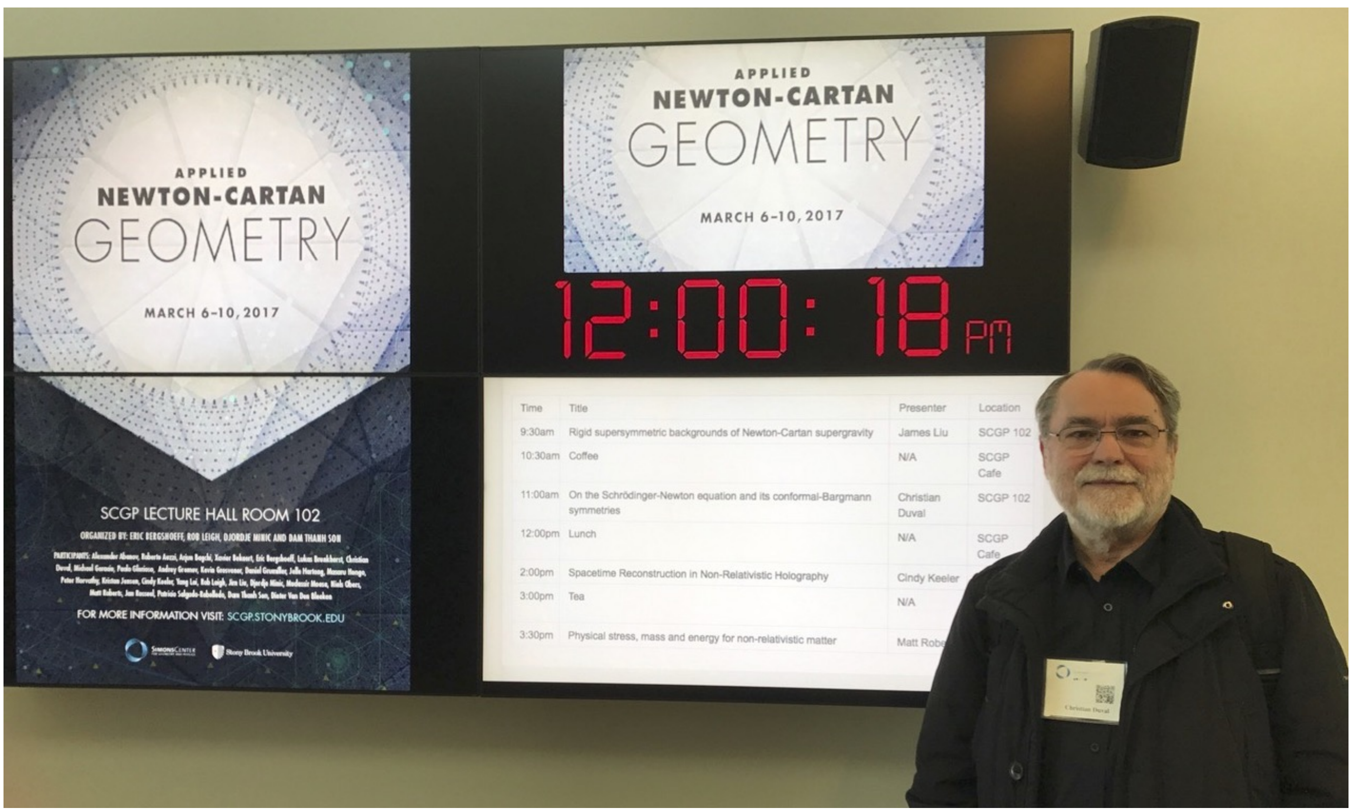
Fundamental flaw : Carroll - invariant particle

can not move why ? $\rightsquigarrow$ 50 years of sleep


($\sim$ 1/2 of *La Belle's* . . . )

# II. Eisenhart - Duval unification

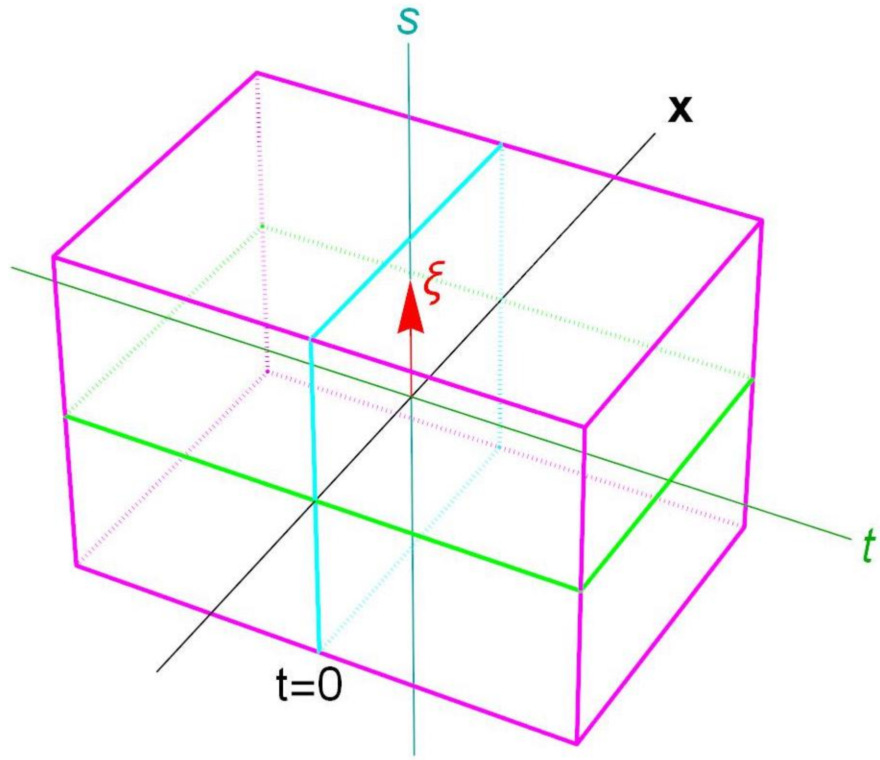
C. Duval, G. Burdet, H. P. Kunzle and M. Perrin,
*"Bargmann Structures and Newton-Cartan Theory,"*
Phys. Rev. D **31** (1985) 1841.



Bargmann manifold: relativistic framework for non-relativistic physics

(i) a $(d+2)$-dim manif $\mathcal{B}$ endowed with metric of Lorentz signature $(d+1,1)$.

(ii) carries nowhere vanishing, complete, null "vertical" vector $\boldsymbol{\xi}$, parallel-transported by Levi-Civita




connection, $\boldsymbol{\nabla}$

- e.g. Minkowski in light-cone coordinates

$$ds_0^2 = d\boldsymbol{x}^2 + 2dtds, \quad \xi = \partial_s\,. \tag{7}$$

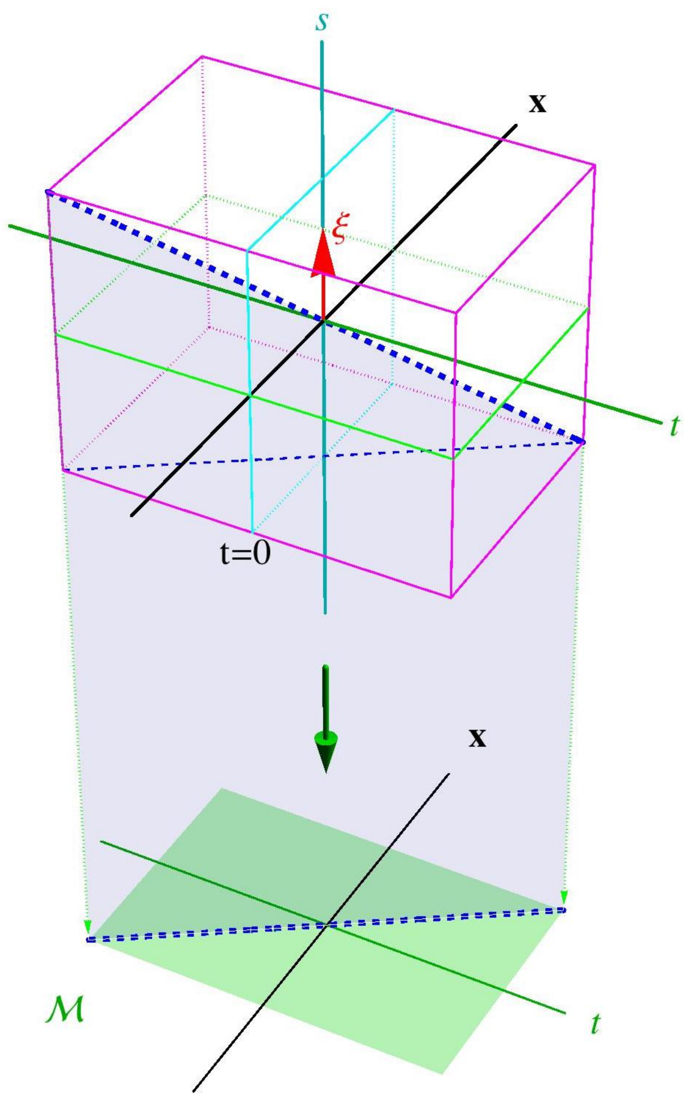
• $V(\boldsymbol{x},t)$ Newtonian potential

$$\boxed{ds_V^2 = d\boldsymbol{x}^2 + 2dtds - 2V(\boldsymbol{x},t)dt^2,\ \xi = \partial_s\,.} \qquad (8)$$

NULL GEODESICS UPSTAIRS

project to

NR MOTIONS DOWNSTAIRS




Factoring out Bargmann space by "vertical" translations generated by $\xi$, $(d+1)$-dim quotient acquires Newton-Cartan structure.

Stability subgroup of $\boldsymbol{\xi}$ in Poincaré: *Bargmann group* :

$$\begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & e \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Barg}(d+1), \qquad (9)$$

$R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $e, f \in \mathbb{R}$. Acts affinely.

Bargmann algebra $\mathfrak{barg}(d+1)$

$$(\omega^A_B x^B + \boxed{\boldsymbol{\beta}^A t} + \gamma^A)\frac{\partial}{\partial x^A} + \varepsilon\frac{\partial}{\partial t} + (\varphi \boxed{-\boldsymbol{\beta}_A x^A})\frac{\partial}{\partial s} \qquad (10)$$

$\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\varepsilon, \varphi \in \mathbb{R}$.

Framework proposed by

L. P. Eisenhart, *"Dynamical trajectories and geodesics"*, Annals. Math. **30** 591-606 (1928) (forgotten)

⋮

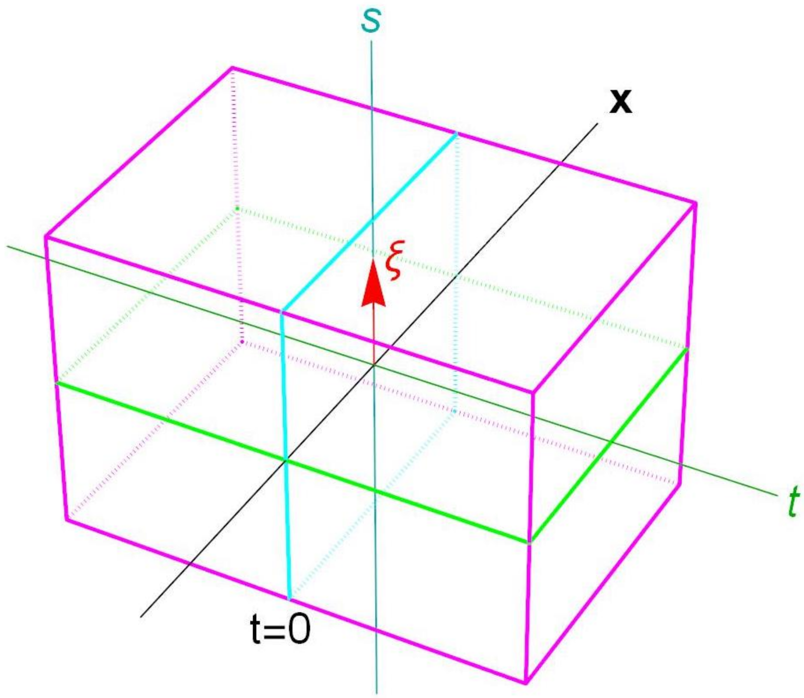
J. Gomis and J. M. Pons, *"Poincare Transformations and Galilei Transformations,"* Phys. Lett. A **66** (1978) 463.

C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, *"Bargmann Structures and Newton-Cartan Theory,"* Phys. Rev. D **31** (1985) 1841.

C. Duval, G. W. Gibbons, and P. A. Horvathy, *"Celestial Mechanics, Conformal Structures and Gravitational Waves,"* Phys. Rev. **D43**, 3907 (1991)

(... Künzle heard about Eisenhart around 1995 ...)

- **Carroll structure**




$(d{+}1)$-dimensional integrable distribution defined by orthogonal complement of "vertical" vector field $\boldsymbol{\xi}$, tangent to foliation. Embedding $\iota : \mathcal{C} \hookrightarrow \mathcal{B}$ i.e.,

$$\boxed{\mathcal{C} = \Big\{ \boldsymbol{x}, t = 0, s \Big\}} \tag{11}$$

of leaf above $t = 0$ endows $\mathcal{C}$ with Carroll structure $\mathrm{g}^{\mathcal{C}} = \iota^* G$, $\Rightarrow$ $(\mathcal{C}, \mathrm{g}^{\mathcal{C}}, \xi)$ is Carroll manifold.

N-C [Galilei] vs Carroll: "dual" structures.

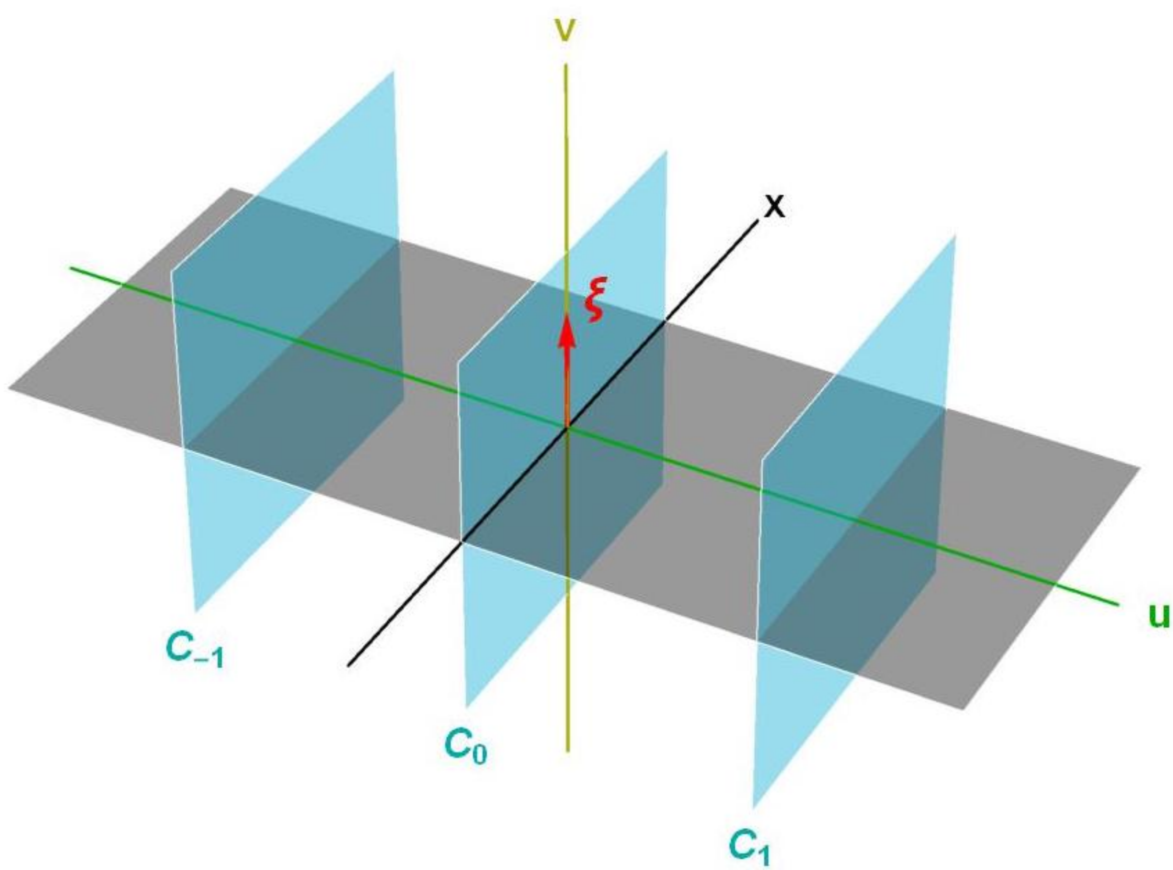
Restriction of Bargmann group "upstairs" to $\boxed{e = 0}$

$$\begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & \boxed{e = 0} \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (12)$$

Leaves $\mathcal{C} \sim t = 0$ hypersurface invariant $\rightsquigarrow$ Carroll group

NO GALILEAN TIME TRANSLATIONS

N.B. : 1-param fam of "Carroll slices" $\mathcal{C}_t$, param by $t =$ const.



## III. Plane GWs (in Brinkmann)*

$$\delta_{ij}\, dX^i dX^j + 2dU dV + K_{ij}(U) X^i X^j\, dU^2, \qquad (13)$$

where $K(U) = \big(K_{ij}(U)\big)$ symmetric and traceless matrix. $\xi = \partial_V$ covar. const.

Vacuum Einstein solutions : Ricci flat

$$R_{\mu\nu} = 0 \Leftrightarrow \mathrm{Tr}(K_{ij}) = 0. \qquad (14)$$

Bondi et al 1959: metric (13) has 5-dim isometry group - NO explicit form given why ?

$\Rightarrow$ NO Lie algebra structure identified.

*M. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94** (1925) 119–145.

# IV. Isometries of plane GWs

Torre *"Gravitational waves: Just plane symmetry,"* Gen. Rel. Grav. **38** (2006) 653 :

Killing vectors

$$S_i(U)\partial_i + \dot{S}_i(U)X^i\,\partial_V\,, \qquad \partial_V, \tag{15}$$

"dot" $= d/dU$. $S_i$, $i = 1, 2$ is solution of vector eqn

$$\boxed{\ddot{S}_i(U) = K_{ij}(U)S_j(U)\,.} \tag{16}$$

• In Minkowski $K_{ij} \equiv 0$, (16) solved by

$$S_i = \gamma_i + \beta_i U\,. \tag{17}$$

Transls in transverse plane + Galilei boosts. Lifted to Bargmann space,

$$Y = (\gamma_i + U\beta_i)\partial_i + \left(\delta + X^i\beta_i\right)\partial_V. \tag{18}$$

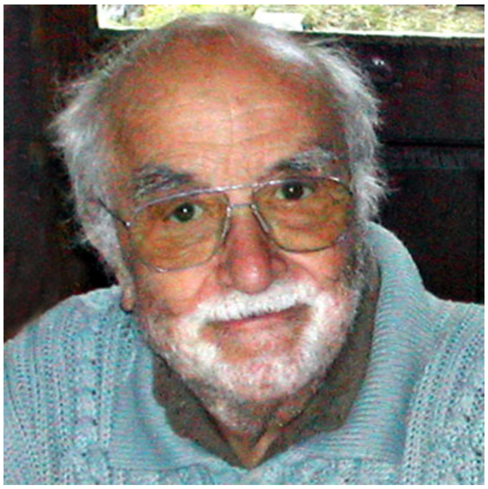
$i = 1, 2$, $\delta =$ const. (5th isometry = "vertical translation" generated by $\partial_V$).

• For $K_{ij} \neq 0$ :

3 translations + **2** WHAT ??? (eqn (16) Stum-Liouville . . . !!!)

# V. Carroll symmetry of plane GWs

1973 Souriau *"Ondes et radiations gravitationnelles,"* Colloques Internationaux du CNRS No 220, pp. 243-256. Paris (1973).



Jean-Marie Souriau

Metric written in "Rosen" (BJR $\equiv$ Baldwin-Jeffery-Rosen ) coordinates $(\boldsymbol{x}, u, v)$

$$\boxed{a_{ij}(u)}\, dx^i dx^j + 2du\, dv\,, \tag{19}$$

cf. Landau-Lifshitz. $a(u) \equiv (a_{ij}(u))$ is strictly positive $2 \times 2$ matrix.

BJR coords $(u, \boldsymbol{x}, v)$ typically <u>non global</u>; exhibit coordinate singularities [caustics] Bondi,Pirani.

Isometries implemented on space-time explicitly,

$$\begin{array}{rcl} u & \to & u, \\ \boldsymbol{x} & \to & \boldsymbol{x} + \boxed{H(u)}\, \mathbf{b} + \mathbf{c}, \\ v & \to & v - \mathbf{b}\cdot\boldsymbol{x} - \tfrac{1}{2}\mathbf{b}\cdot \boxed{H(u)}\, \mathbf{b} + \nu, \end{array} \tag{20}$$

$\mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ $\nu \in \mathbb{R}$, where

$$\boxed{H(u) = \int_0^u a(t)^{-1} dt} \tag{21}$$

symmetric $2 \times 2$ "Souriau" matrix.

• Minkowski: $a = Id \Rightarrow H(u) = u\, Id \Rightarrow$ Galilei.

Restriction to $u = 0 \Rightarrow H(u) = 0 \rightsquigarrow$ boost implemented by

$$\boxed{\left\{ \begin{array}{rcl} \boldsymbol{x}' & = & \boldsymbol{x} \\ v' & = & v - \boldsymbol{b}\cdot\boldsymbol{x} \end{array} \right.} \tag{22}$$

"Carroll" group of Lévy-Leblond 1965 !!!

M. Henneaux, *"Geometry of Zero Signature Space-times,"* Bull. Soc. Math. Belg. **31** (1979), 47-63 PRINT-79-0606 (PRINCETON).

C. Duval, G. W. Gibbons, and P. A. Horvathy, *"Celestial Mechanics, Conformal Structures and Gravitational Waves,"* Phys. Rev. **D43**, 3907 (1991)

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J. Gomis and F. Passerini, *"Super Carroll space, Carrollian super-particle and Carrollian super-string,"* (unpublished notes, 2005)

C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016. [arXiv:1402.0657 [gr-qc]].

E. Bergshoeff, J. Gomis and G. Longhi, *"Dynamics of Carroll Particles,"* Class. Quant. Grav. **31** (2014) no.20, 205009 doi:10.1088/0264-9381/31/20/205009 [arXiv:1405.2264 [hep-th]].

etc $\rightsquigarrow$ rapidly growing interest (once awaken . . . )

Carroll particle: To Move or Not to Move ?

?

Brinkmann $\Leftrightarrow$ BJR : $(\boldsymbol{X}, U, V) \to (\boldsymbol{x}, u, v)$ "potential" in (13),

$$\delta_{ij}\, dX^i dX^j + 2dUdV + \boxed{K_{ij}(U)X^iX^j}\, dU^2,$$

traded for transverse metric $\boxed{a_{ij}(u)dx^idx^j}$ :

$$U = u\,, \quad \boldsymbol{X} = P(u)\boldsymbol{x}\,, \quad V = v - \frac{1}{4}\boldsymbol{x}\cdot\dot{a}(u)\boldsymbol{x} \tag{23}$$

where $a_{ij} = \left(P^\dagger P\right)_{ij}$ with $2\times 2$ matrix $P = (P_{ij})$ is solution of matrix Sturm-Liouville pb :

$$\boxed{\ddot{P} = K(u)P\,, \quad P^\dagger\dot{P} = \dot{P}^\dagger P} \tag{24}$$

**Q**: SL $\rightsquigarrow$ SL DOES IT HELP ?

**A**: allows to clarify symmetry structure $\rightsquigarrow$ explicit integration of geodesic eqns in BJR (later)

Why is it important to know ?

# VI. Memory Effect

Ya. B. Zel'dovich and A. G. Polnarev, *"Radiation of gravitational waves by a cluster of superdense stars,"* Astron. Zh. **51**, 30 (1974)

> *. . . another,* **nonresonance**, *type of detector, consisting of two noninteracting bodies (such as satellites). [ . . . ] the distance between a pair of free bodies should change, and this effect might possibly serve as a nonresonance detector [ . . . ]*


(atomic + H bomb)

V.B. Braginsky & L. P. Grishchuk *"Kinematic resonance and the memory effect in free mass gravitational antennas," Zh. Eksp. Teor. Fiz.* **89** *744-750 (1985)* introduce

$$\boxed{\text{“memory effect”}}$$

> *"distance between a pair of bodies is different from the initial distance in the presence of a gravitational radiation pulse. . . . possible application to detect gravitational radiation . . ."*

Our assumptions :

- for very large distances GW approximated with exact plane GW

- spinless particles move on geodesics. (spinning ???)

# VII. Geodesic Motion (in Brinkmann)

Symmetry $\rightsquigarrow$ conserved quantities $\rightsquigarrow$ explicit integration of geodesic eqns (in BJR coords).

Plane GWs in Brinkmann coords (13)

$$\delta_{ij}dX^i dX^j + 2dUdV + K_{ij}(U)X^i X^j dU^2 \tag{25}$$

$\boldsymbol{X} = (X^i)$ transverse, $U$, $V$ light-cone coords.

Sandwich wave: profile $K(U) \neq 0$ only in "wave zone" $U_i < U < U_f$. Assumption : metric Minkowski in "before-zone" $U < U_i$ and flat in "after-zone" $U_f < U$.

Geodesics : solution of system

$$\frac{d^2X^1}{dU^2} - \frac{1}{2}\mathcal{A}_+ X^1 = 0, \tag{26a}$$

$$\frac{d^2X^2}{dU^2} + \frac{1}{2}\mathcal{A}_+ X^2 = 0, \tag{26b}$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}_+}{dU}\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_+\Big(X^1\frac{dX^1}{dU} - X^2\frac{dX^2}{dU}\Big) = 0\,. \tag{26c}$$

$X^{1,2}$-components decoupled. Projection of $4D$ worldline to transverse ($X^1 - X^2$) plane independent of $V(U_0)$ & $\dot{V}(U_0)$.

N.B. For affine parameter ($\sim$ "dot")

$$-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\mu} = m^2$$

const of the motion. For $m = 0$ : null lift.

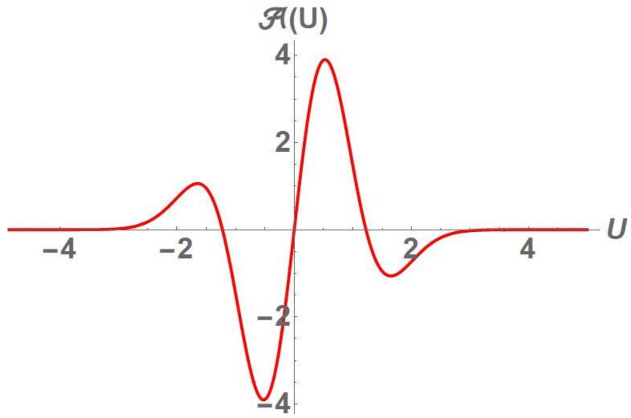
For $m^2 \neq 0$ vertical ($\sim V$) shift $m^2 U \Rightarrow$ restrict to $m = 0 \Rightarrow$ Enough to solve transv eqn (26a)-(26b).

Assumption: particle initially at rest in (approx) "before zone":

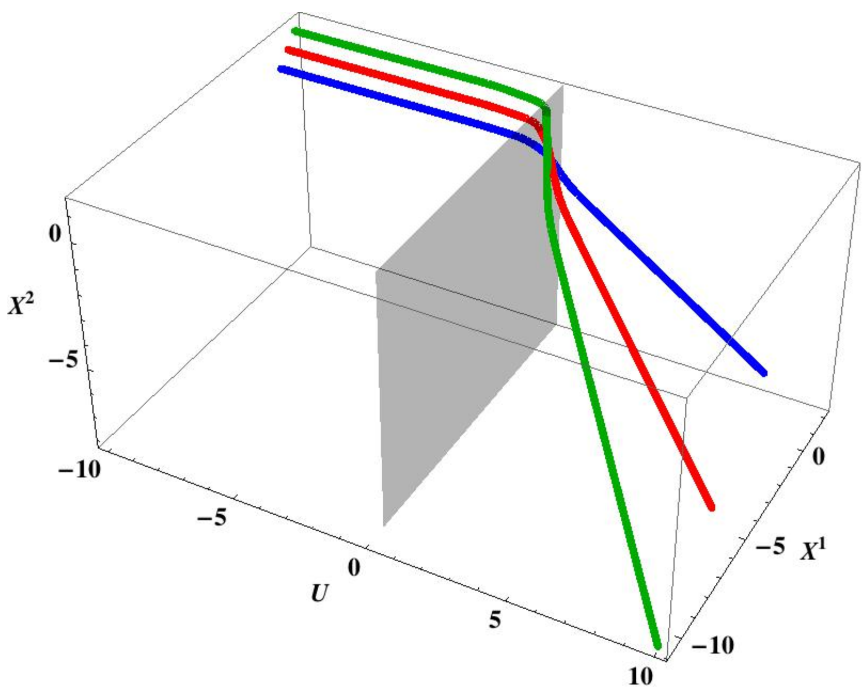
$$\boxed{\boldsymbol{X}(U) = \boldsymbol{X}_0, \quad \dot{\boldsymbol{X}}(U) = 0 \quad U \leq U_i.} \qquad (27)$$

For gravitational collapse: "sandwich" profile $\sim$ potential

$$\boxed{\frac{1}{2}\frac{d^3(e^{-U^2})}{dU^3}\,\mathrm{diag}(1,-1)} \tag{28}$$




Geodesics found numerically




N.B. After-zone motion $\approx$ diverging straight lines with const velocity $\equiv$ Newton's 1st law !!!

# VIII. Motions from Carromm symmetry (in BJR)

Return to GW $[t \to u, s \to v]$ in BJR $u$ can be choosen as affine coord : Noether's thm $\Rightarrow$ 5 isometries $\Rightarrow$ 5 conserved quantities.

$$\mathbf{p} = a(u)\,\dot{\boldsymbol{x}}\,, \qquad \mathbf{k} = \boldsymbol{x}(u) - H(u)\,\mathbf{p}\,, \tag{29}$$

where $\boxed{H(u)}$ Souriau matrix (21) conserved linear & boost-momentum, + $m = \dot{v} = 1$.

+ const of the motion $\boxed{e = \frac{1}{2}\mathrm{g}_{\mu\nu}\,\dot{x}^{\mu}\dot{x}^{\nu}}$ Timelike / lightlike / spacelike if $e$ negative/zero/positive.

Geodesics determined by Noether quantities $\mathbf{p}$, $\mathbf{k}$,

$$\boldsymbol{x}(u) = H(u)\,\mathbf{p} + \mathbf{k}, \tag{30a}$$

$$v(u) = -\tfrac{1}{2}\mathbf{p}\cdot H(u)\,\mathbf{p} + e\,u + d, \tag{30b}$$

• In flat Minkowski space $a = \mathbf{1} \Rightarrow H(u) = u\,\mathbf{1}$, yielding free motion

$$\boldsymbol{x}(u) = u\,\mathbf{p} + \mathbf{k}, \tag{31a}$$

$$v(u) = \Big(-\tfrac{1}{2}|\mathbf{p}|^2 + e\Big)\,u + v_0. \tag{31b}$$

For sandwich wave $K \equiv 0$ in "before-zone" $U < U_i \Rightarrow$ SL eqn. solved by $P(u) = \mathbf{1} \Rightarrow$ Brinkmann and BJR coords coincide.

"Memory init cond" (27) : particle at rest for $u \leq U_i \;\Rightarrow$ momentum vanishes by (29), $\mathbf{p} = 0$. But $\mathbf{p}$ conserved $\Rightarrow$

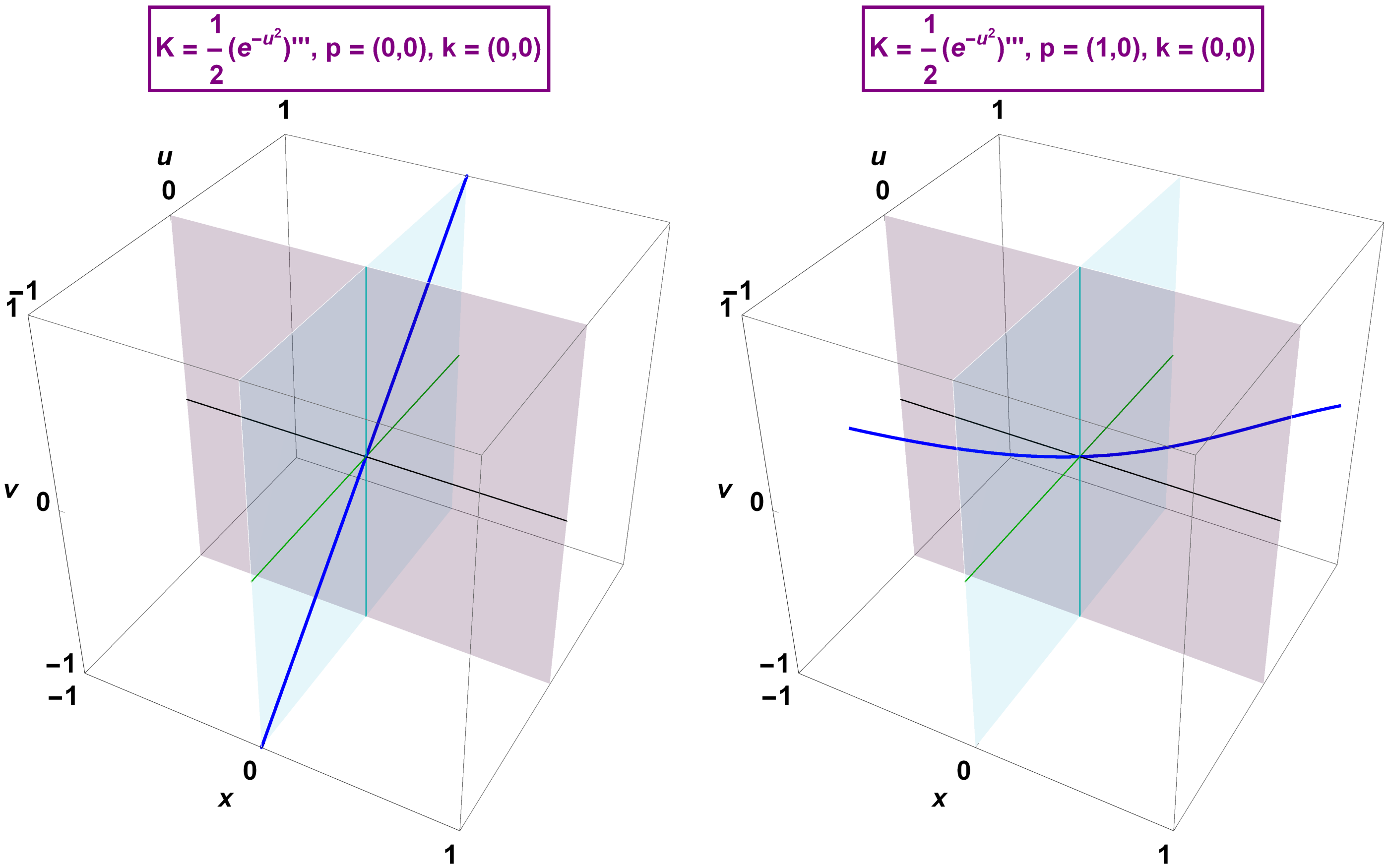
$$\boxed{\mathbf{p} = 0} \qquad \text{for all } u\,. \tag{32}$$

In BJR : $\boldsymbol{x}(u) = H(u)\,\mathbf{p} + \mathbf{k} \Rightarrow$ for any metric

$$\boxed{\boldsymbol{x}(u) = \boldsymbol{x}_0 = \mathbf{k}\,, \qquad v(u) = e\,u + v_0\,.} \tag{33}$$

NO transverse motion in BJR coords !!! *particles initially at rest remain at rest during and after passage of wave* !!

Ex: Collapse profile (in BJR) :




(a) (b)

Fig. *If* $p_2 = 0 \rightsquigarrow 3D$ *pictures for* $\big(x^1(u), u, v(u)\big)$.

*(a) "rest in before zone"* $\Rightarrow \mathbf{p} = 0 \Rightarrow$ $x^1(u) = 0$ :
*particle does NOT move*.

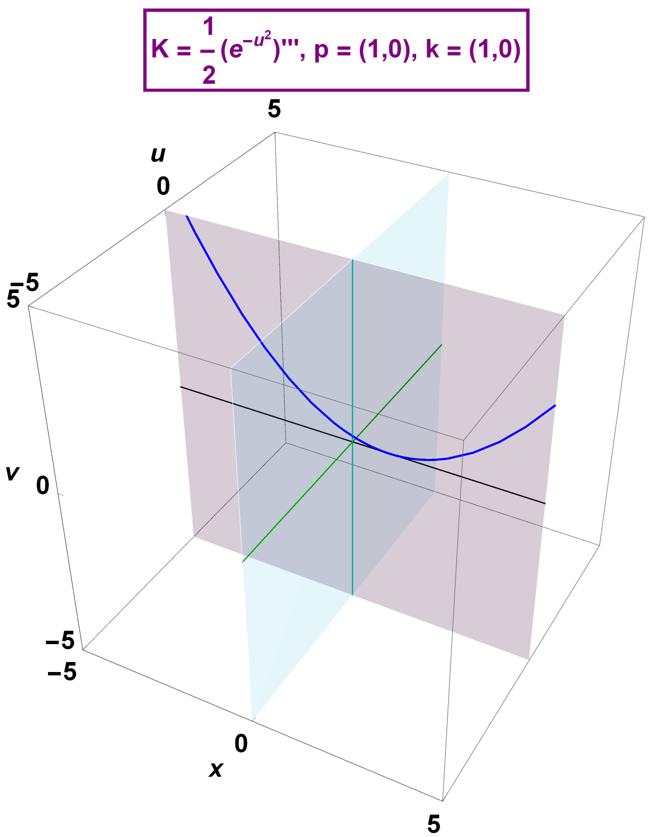
*(b) For "non-memory init cond"* $\mathbf{p} \neq 0 \Rightarrow x^1(u) \neq \text{const.}$

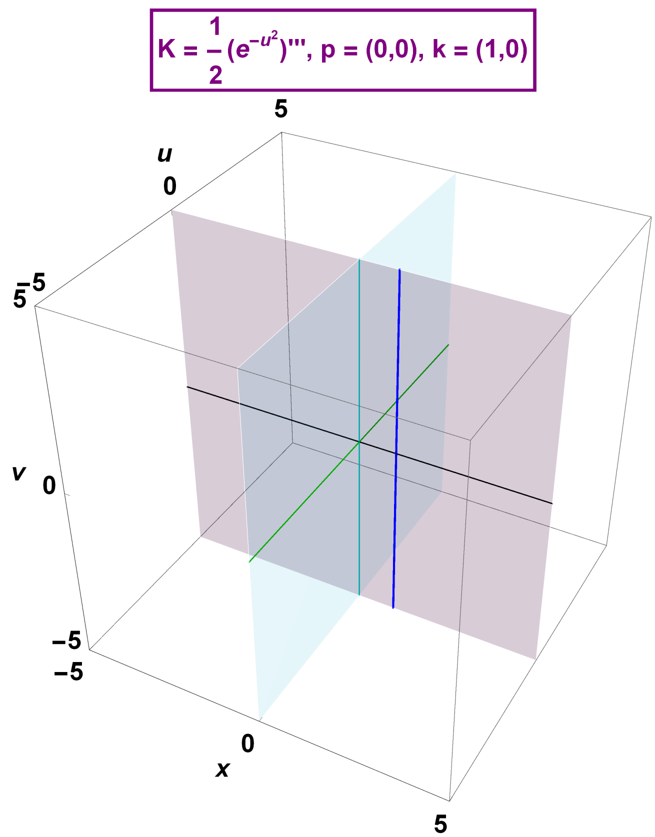
"eppur si muove"

On “Carroll slice”: $\mathcal{C} \equiv \mathcal{C}_0 = \left\{x, \boxed{u = 0}, v\right\}$

$u$ can not be used as coord. Affine coord $\lambda \rightsquigarrow$

particle moves on $\mathcal{C}$ (stable under dynamics) :



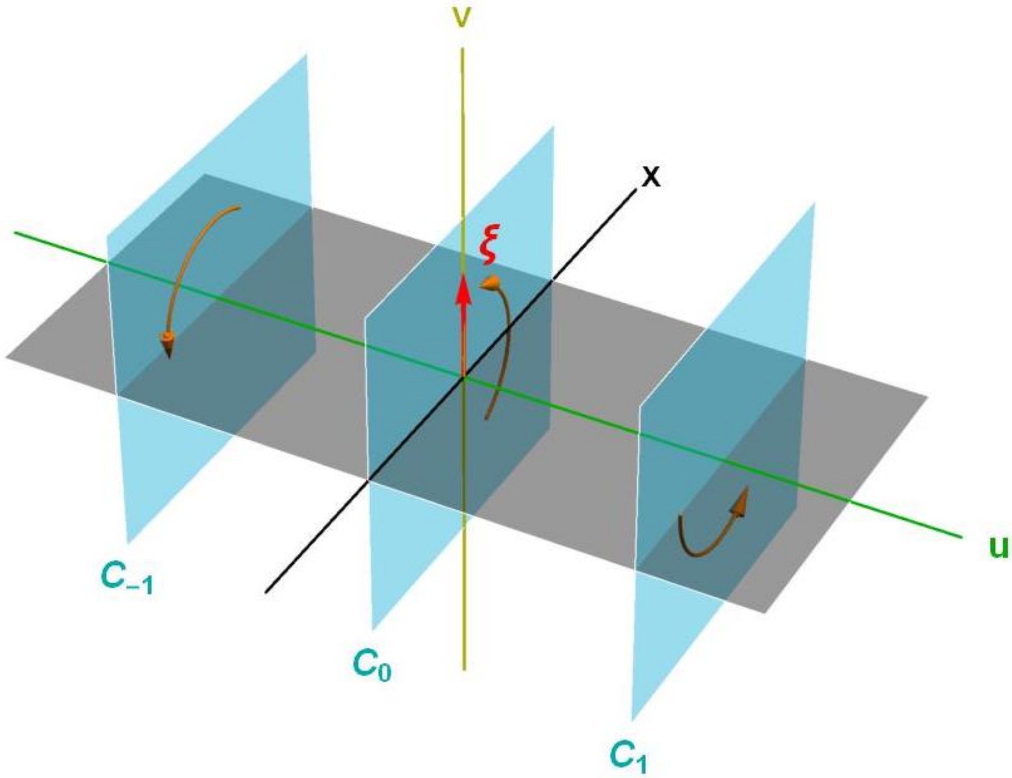




- For init cond $x^2 = p_2 = 0$ $x^2$ dropped. $p_1 \neq 0$ : trajectory: parabola.

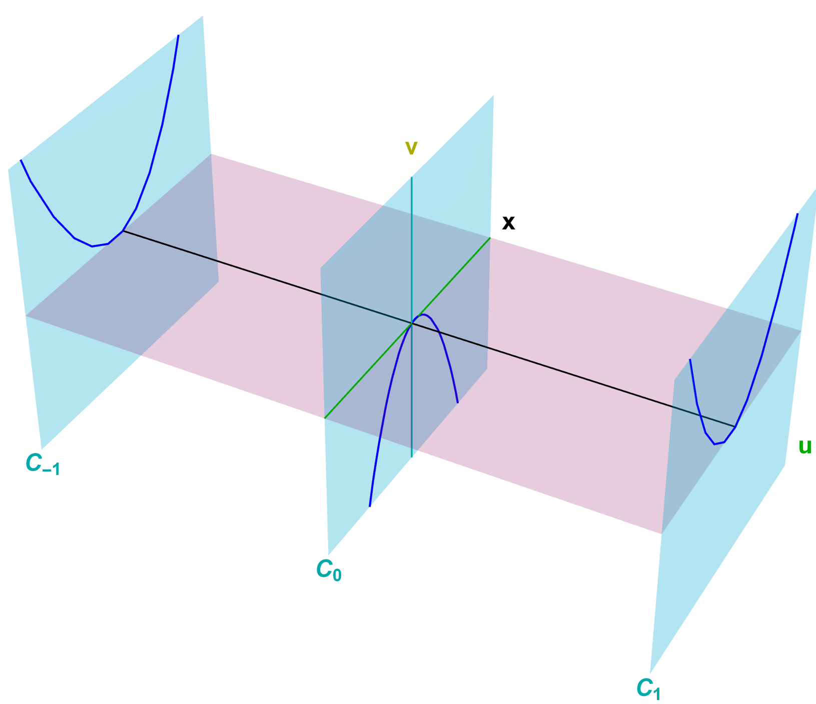
- for “memory” init cond $\mathbf{p} = 0$ trajectory degenerates: particle does NOT move in transverse space

# On “Carroll slices” $\mathcal{C}_u$

Carroll group implemented through Souriau matrix $H(u)$ which links slices



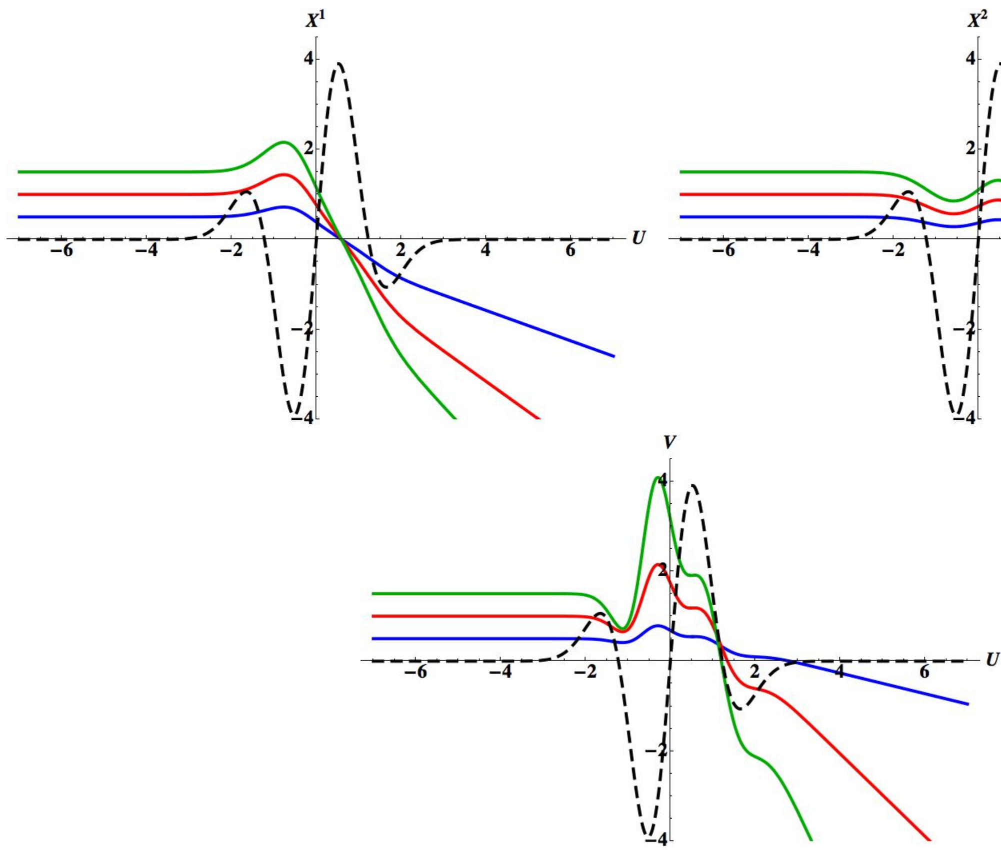

Motions : parabolæ, turned by $H(u)$



# IX. Motions in Brinkmann

- for $u \neq \text{const}$: pull back to Brinkmann by $P(u)$ -matrix $\Rightarrow$

$$\boxed{\boldsymbol{X}(U) = P(u)\,\boldsymbol{x}^0, \quad \boldsymbol{x}^0 = \text{const}} \tag{34}$$




*Geodesics of particles initially at rest for collapse profile.*

"no-motion" of Carroll particle in BJR
&
complicated motion in Brinkmann conciliated

Everything comes from $\mathbf{P(u)}$ !!!

• Motion on Carroll slice $\mathcal{C}_u$ : $\lambda$ aff param. Geodesic:

$$x(\lambda) = x_0 + b_1\lambda\,, \tag{35a}$$

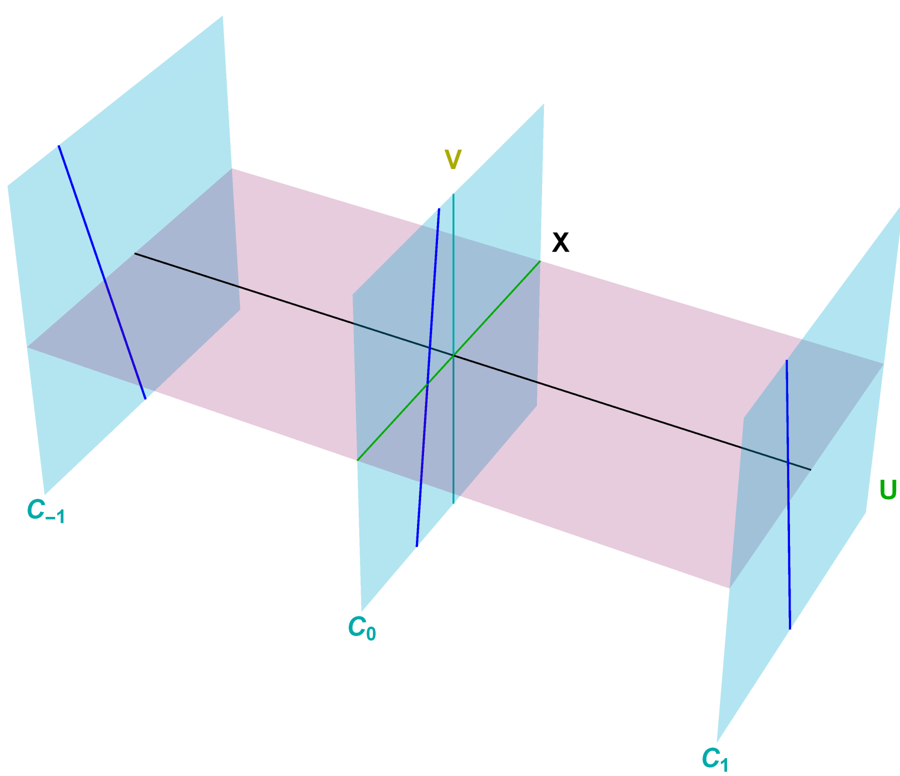
$$v(\lambda) = v_0 + \frac{b_3}{b_1}(x - x_0) + \frac{1}{4}a'_{11}(u_0)(\boldsymbol{x} - \boldsymbol{x}_0)^2 \tag{35b}$$

But BJR $\to$ B (cf. (23))

$$\boldsymbol{X} = P(u)\boldsymbol{x}\,, \;\; V = v - \frac{1}{4}\boldsymbol{x}\cdot\dot{a}(u)\boldsymbol{x}$$

Quadratic terms in $V$ cancel, yielding straight (but displaced) trajectories.

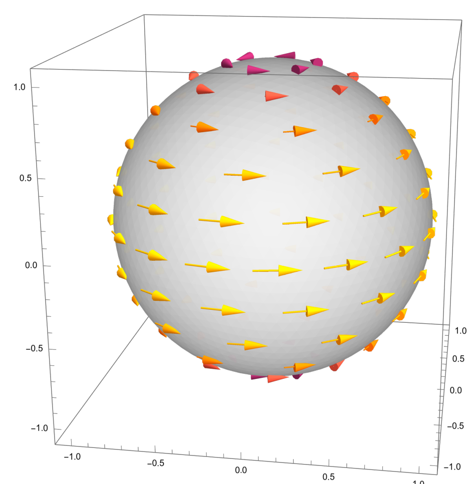
$$\boldsymbol{X} = \boldsymbol{X}_0 + \mathbf{k}\lambda \qquad V = V_0 + k_v\lambda. \tag{36}$$



## X. Outlook:

fairy tale continues:

- (Marsot, Donnay & Marteau)




Carroll on BH horizon

• (Harte and Oancea - unpublished) spinning memory. (caveat: prick from spindle may make you asleep & loose your memory !)