Carrollian geometry of null infinity

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Introduction and motivations

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- in the quantum theory the related ward identities are just “Weinberg soft-theorems” (Strominger et al 2016, ...)
- gravitational memory effect is tightly related to the transition between gravity vacua (Ashtekar 2016, Strominger et al 2016, ...)
- Hawking’s initial derivation of the “information paradox” implicitly relied on the uniqueness of the gravity vacua (Hawking, Perry, Strominger 2016 and 2017)
- There has also been a surge of interest for the possibility of realising “flat holography” on the celestial sphere (Strominger et al 2019, Donnay et al 2022 ...)

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Many of these results rely on subtle geometrical features of asymptotically flat-space-times, such as “degeneracy of gravity vacua”, which are already present at classical level.

These deserve full intrinsic (i.e. Carrollian) realisations and conceptual clarity.

Introduction and motivations

Asymptotically flat space-times are essential tools in our physical understanding of General Relativity (e.g. model gravitational radiations as seen in LIGO), as such they have a venerable history:

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Accordingly, the geometry of null infinity has a history just as long:

- Penrose (1963), Geroch (1977), Newman (1981), Ashtekar (1981) ...

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Accordingly, the geometry of null infinity has a history just as long:

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However not as intrinsic/Carrollian nor as geometrical/conceptually clear as one might hope for a subject which is 60 years old.

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There are of course classical (technical) answers to these questions, they are however not particularly illuminating :

This is especially true if one takes the intrinsic/Carrollian perspective.

Highlight

Building up on relatively recent results from Gover et al (2010) – (2018) on tractor calculus (see also Penrose–MacCallum (1973) “asymptotic local twistors”), one can shed a fully Carrollian new light on the subject (YH (2010), YH (2021), YH (2021)):

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- *Radiative characteristic data of general relativity are equivalent to strongly conformally Carrollian geometry*
- *These are in turn equivalent with genuine Cartan connections with completely intrinsic meaning*
- *Gravitational radiation correspond to the presence of curvature*
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What is more...

This perspective suggests generalisations, e.g. to higher-spin geometry (*see I. Lovrekovic’s talk*), and is useful to construct intrinsic functional (*see J. Salzer’s talk*).

1. Carroll manifolds: A view from Cartan geometry
2. Carroll manifolds and the geometry of null infinity
3. Null-infinity as a conformal boundary of spacetimes

Carroll manifolds: A view from Cartan geometry

Carrollian manifold and BMS

Levy-Leblond (1965) *"Une nouvelle limite non-relativiste du groupe de Poincaré"*
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Levy-Leblond (1965) *"Une nouvelle limite non-relativiste du groupe de Poincaré"*
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A *Carrollian manifold* $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, h_{\mu\nu}, n^\mu\right)$ is the data of

- a nowhere vanishing vector field $n^\mu \in \Gamma\left[T\mathscr{I}\right]$
- whose integral lines form the fibres of a (trivial) bundle $\mathscr{I} \xrightarrow{\pi} \Sigma$

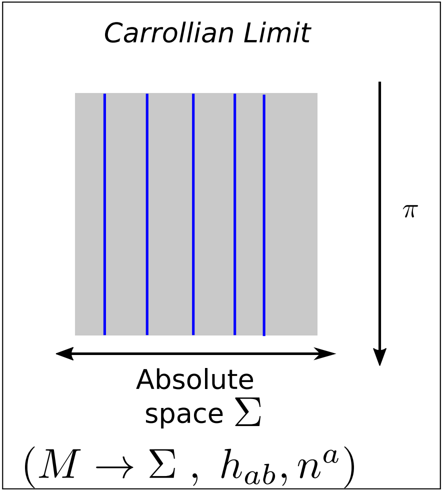
$$\mathscr{I}^{(n)} \simeq \mathbb{R} \times \Sigma^{(n-1)}$$

- a symmetric tensor $h_{\mu\nu} \in \Gamma\left[S^2 T^* \mathscr{I}\right]$ of constant rank $(n-1)$
- satisfying

$$h_{\mu\nu} n^\nu = 0, \qquad \mathcal{L}_n h_{\mu\nu} = 0.$$

In particular the base Σ is a Riemannian manifold (Σ, h_{AB}) with

$$h_{\mu\nu} := \pi^* h_{AB}.$$



"since absence of causality as well as arbitrarinesses in the length of time intervals is especially clear in Alice's adventures (in particular in the Mad Tea-Party) this did not seem out of place to associate Lewis Carroll's name"

Levy-Leblond (1965)

Carrollian manifold and BMS

Duval–Gibbons–Horvathy (2014) *“Conformal Carroll groups”*

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A *conformal Carrollian manifold* $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, [h_{\mu\nu}, n^{\mu}]\right)$ is the data of

- an equivalence class

$$(h_{\mu\nu}, n^{\mu}) \sim \left(\lambda^2 h_{\mu\nu}, \lambda^{-1} n^{\mu}\right) \qquad 0 < \lambda \in \mathcal{C}^{\infty}\left(\mathscr{I}\right),$$

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In particular the base Σ is a conformal manifold $(\Sigma, [h_{AB}])$ with $h_{\mu\nu} := \pi^* h_{AB}$.

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This coincides with Geroch’s universal structure (1977) for $\mathscr{I}$!

Carrollian manifold and BMS

Geroch (1977) *“Asymptotic Structure of Space-Time”*

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The group of automorphisms of $(\mathscr{I} \to \Sigma, [h_{ab}, n^a])$, i.e diffeomorphisms $\phi \colon \mathscr{I} \to \mathscr{I}$ satisfying

$$\phi^* h_{\mu\nu} = \lambda^2 h_{\mu\nu}, \qquad \phi_* n^\mu = \lambda^{-1} n^\mu,$$

is

$$Aut\,(\mathscr{I}, [h_{\mu\nu}, n^\mu]) \simeq \mathcal{C}^\infty\,(\Sigma) \rtimes Conf\,(\Sigma, [h_{AB}])$$

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If $(\Sigma, [h_{AB}])$ is the conformal sphere $\left(S^{n-2}, [h^{(S)}_{AB}]\right)$ then this is

$$\boxed{BMS_{n+1} \simeq \mathcal{C}^\infty\left(\Sigma\right) \rtimes \mathrm{SO}(n,1)}$$

the BMS group of Bondi–van der Burg–Metzner–Sachs (1962)

Carrollian manifold and gravitational radiation

Nice picture but ...

- where is Levy-Leblond's Carroll group in this picture ?
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- where are gravitational radiation ?

In a nutshell, the respective answers are

1. One needs "strong" Carroll structures
   (Duval–Gibbons–Horvathy–Zhang (2014)).
2. They are captured by the "radiative" structure at null infinity
   (Ashtekar (1981)).

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- where is Levy-Leblond's Carroll group in this picture ?
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In a nutshell, the respective answers are

1. One needs "strong" Carroll structures (Duval–Gibbons–Horvathy–Zhang (2014)).
2. They are captured by the "radiative" structure at null infinity (Ashtekar (1981)).

The interplay between these two pictures is subtle and will be most transparent from the perspective of Cartan geometry (YH (2020), YH (2021)).

Strongly Carrollian Manifold

Duval–Gibbons–Horvathy–Zhang (2014) *“Carroll versus Newton and Galilei”*

Bekaert–Morand (2015) *(“Through the looking class” Appendix)*

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A *strongly Carrollian manifold* $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, h_{\mu\nu}, n^\mu, \nabla\right)$ is the data of

- a Carrollian manifold $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, h_{\mu\nu}, n^\mu\right)$
- together with a torsion free connection ∇
- satisfying the compatibility conditions

$$\nabla_\rho n^\mu = 0, \qquad \nabla_\rho h_{\mu\nu} = 0.$$

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$$\nabla_{\rho} n^{\mu} = 0, \qquad \nabla_{\rho} h_{\mu\nu} = 0.$$

Let ∇ and $\hat{\nabla}$ be two compatible torsion free connections for $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, h_{\mu\nu}, n^{\mu}\right)$ then

$$\left(\nabla_{\rho} - \nabla_{\rho}\right) V^{\mu} = C_{\rho\nu} V^{\nu} n^{\mu}.$$

where $C_{\mu\nu} = (dy^{A})_{\mu}(dy^{B})_{\nu} C_{(AB)}$ is a symmetric tensor s.t. $n^{\mu} C_{\mu\nu} = 0$.

Carroll groups and Model spaces

Bacry–Levy-Leblond (1968) *"Possible Kinematics"*
Figueroa-O'Farrill–Grassie–Prohazka (2019) and Figueroa-O'Farrill–Prohazka (2019)

The Carroll groups:

$$\begin{aligned} \mathrm{Carr}_{dS}(n) &:= \mathbb{R}^n \rtimes \mathrm{SO}(n) \\ \mathrm{Carr}(n) &:= \mathbb{R}^n \rtimes \mathrm{ISO}(n-1) \\ \mathrm{Carr}_{AdS}(n) &:= \mathbb{R}^n \rtimes \mathrm{SO}(n-1,1) \end{aligned}$$

$$\boxed{\mathrm{Carr}_{\Lambda} := \mathbb{R}^n \rtimes \mathrm{ISO}_{\Lambda}(n-1)}$$

The model Carroll space-times:

$$\Lambda > 0 \qquad Carr^{(n)}_{dS} = \mathrm{Carr}_{dS}(n) / \mathrm{ISO}(n-1) \simeq \mathbb{R} \times S^{n-1}$$

$$\Lambda = 0 \qquad Carr^{(n)} = \mathrm{Carr}(n) / \mathrm{ISO}(n-1) \simeq \mathbb{R} \times \mathbb{R}^{n-1}$$

$$\Lambda < 0 \qquad Carr^{(n)}_{AdS} = \mathrm{Carr}_{AdS}(n) / \mathrm{ISO}(n-1) \simeq \mathbb{R} \times H^{n-1}$$

Equivalence problem of Cartan geometry

Hartong (2015), YH (2021)

Strongly Carrollian geometries $\left(\mathscr{I} \xrightarrow{\Sigma}, h_{\mu\nu}, n^{\mu}, \nabla\right)$ are (locally) equivalent to normal Cartan geometries $(\mathcal{G} \to \mathscr{I}, \omega)$ modelled on

$$Carr_{\Lambda}^{(n)} = {}^{\mathrm{Carr}_{\Lambda}(n)}\!\big/_{\mathrm{ISO}(n-1)}.$$

In particular if the geometry is "flat"

$$F^{A}{}_{B} - 2\Lambda \theta^{A} \wedge \theta_{B} = 0, \qquad F^{0}{}_{B} - 2\Lambda l \wedge \theta_{B} = 0$$

then its algebra of infinitesimal symmetry is $\mathfrak{carr}_{\Lambda}(n)$.

Proof (0) : Cartan Geometry

A Cartan geometry $(\mathcal{G} \to M, \omega)$ modelled on G/H is the data of

- a H-principal bundle $\mathcal{G} \to M$
- a "$\mathfrak{g}$-valued Cartan connection" ω, i.e a section ω of $\Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying
  1. $\omega\left(X^{\#}\right) = X$ for all $X \in \mathfrak{h}$
  2. $R_h^* \omega = Ad_{h^{-1}}(\omega)$
  3. s.t. $\omega \colon T\mathcal{G} \to \mathfrak{g}$ is an isomorphism.

The main example is the flat model :

$(G \to G/H, \omega_G)$ whith ω_G the Maurer-Cartan form on G.

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Fundamental theorem (E. Cartan)

A Cartan geometry $(\mathcal{G} \to M, \omega)$ is locally isomorphic to the flat model $(G \to G/H, \omega_G)$ if and only if the curvature $F = d\omega + \frac{1}{2}[\omega, \omega]$ vanishes.

Proof (1) : $\mathcal{G} \to \mathscr{I}$

Geroch–Held–Penrose (1973)
Ciambelli–Leigh–Marteau–Petropoulos (2019) and Ciambelli–Marteau (2019)

Let $(\mathscr{I}, h_{\mu\nu}, n^\mu)$ be a Carrollian manifold, a Carrollian frame

$$\{e_A^\mu, n^\mu\}_{A\in 1,\dots,n-1}$$

at $x \in \mathscr{I}$ is a basis of $T_x\mathscr{I}$ such that

$$e_A{}^\mu e_B{}^\nu h_{\mu\nu} = \delta_{AB}.$$

Carrollian frames form a $\mathrm{ISO}(n-1)$-principal bundle $\mathcal{G} \to \mathscr{I}$:

$$\begin{array}{ccc} e_A{}^\mu & \mapsto & m_A{}^B\left(e_B{}^\mu - t_B n^\mu\right) \\ n^\mu & \mapsto & n^\mu \end{array} \qquad \begin{pmatrix} m^A{}_B & 0 \\ t_B & 1 \end{pmatrix} \in \mathrm{ISO}(n-1)\,.$$

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Let $(\mathscr{I}, h_{\mu\nu}, n^\mu)$ be a Carrollian manifold, a Carrollian co-frame

$$\{l_\mu, \theta_\mu^A\}_{A\in 1,\dots,n-1}$$

at $x \in \mathscr{I}$ is a basis of $T_x^*\mathscr{I}$ such that

$$e_B{}^\mu \theta_\mu{}^A = \delta_B{}^A, \qquad n^\mu l_\mu = 1.$$

(other contractions vanish)

Carrollian co-frames form a $\mathrm{ISO}(n-1)$-principal bundle

$$\begin{aligned} \theta_\mu{}^A &\mapsto m^A{}_B\, \theta_\mu{}^B \\ l_\mu &\mapsto l_\mu + t_C\, \theta_\mu{}^C \end{aligned} \qquad \begin{pmatrix} m^A{}_B & 0 \\ t_B & 1 \end{pmatrix} \in \mathrm{ISO}(n-1)\,.$$

Proof (2) : $\omega \in \Omega^1(\mathfrak{g})$

$$\omega = \begin{pmatrix} 0 & -\theta_B & 0 & 0 \\ \Lambda\,\theta^A & \omega^A{}_B & \theta^A & 0 \\ 0 & -\Lambda\,\theta_B & 0 & 0 \\ \Lambda\, l & -\frac{1}{2}C_B & l & 0 \end{pmatrix} \in \mathfrak{carr}_\Lambda(n)$$

Under $\mathrm{ISO}(n-1)$-gauge transformations ...

- (l, θ^A) transform like a coframe
- $\left(\omega^A{}_B, C_B\right)$ transform like the components of a connection

$$\begin{pmatrix} \theta_\mu{}^A \nabla e^\mu_B & \theta_\mu{}^A \nabla n^\mu \\ l_\mu \nabla e^\mu_B & l_\mu \nabla n^\mu \end{pmatrix} = \begin{pmatrix} \omega^A{}_B & 0 \\ -\frac{1}{2}C_B & 0 \end{pmatrix}$$

Proof (3) : normality

$$F^I{}_J \in \mathfrak{carr}_\Lambda(n)$$
$$= \begin{pmatrix} 0 & -d^\omega \theta_B & 0 & 0 \\ \Lambda\, d^\omega \theta^A & F^A{}_B - 2\Lambda\, \theta^A \wedge \theta_B & d^\omega \theta^A & 0 \\ 0 & -\Lambda\, d^\omega \theta_B & 0 & 0 \\ \Lambda\left(dl - \frac{1}{2} C_C \wedge \theta^C\right) & -\frac{1}{2} d^\omega C_B - 2\Lambda\, l \wedge \theta_B & dl - \frac{1}{2} C_C \wedge \theta^C & 0 \end{pmatrix}.$$

The Carrollian connection ∇ is torsion-free iif and only if $F^I{}_J \in \mathfrak{iso}(n-1)$

and the only remaining curvature components are

$$F^A{}_B - 2\Lambda \theta^A \wedge \theta_B, \qquad -\frac{1}{2} d^\omega C_B - 2\Lambda l \wedge \theta_B.$$

By Cartan's theorem these are the obstruction to having a local identification

$$\mathscr{I} \simeq Carr^{(n)}_\Lambda = {}^{\mathrm{Carr}_\Lambda(n)}\!\diagup_{\mathrm{ISO}(n-1)}$$

i.e. a preferred $\mathfrak{carr}_\Lambda(n)$ symmetry group.

Carroll manifolds and the geometry of null infinity

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Recall that if $(\mathscr{I}, h_{\mu\nu}, n^\mu)$ is a Carrollian manifold and ∇, $\hat{\nabla}$ are two compatible connections then,

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Radiative structure (Ashtekar (1981))

A radiative structure at null infinity $(\mathscr{I}, h_{\mu\nu}, n^\mu, [\nabla])$ is a Carrollian manifold together with an equivalence class of torsion-free compatible connections $[\nabla]$

$$\nabla \sim \hat{\nabla} \qquad \Leftrightarrow \qquad \left(\nabla_\rho - \hat{\nabla}_\rho\right) V^\mu \propto h_{\rho\nu} V^\nu n^\mu$$

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This is great and very inspiring but...

- not very satisfactory geometrically
- conformal invariance has been fixed (Bondi gauge)
- relationship to the Poincaré group is obscure
- suggestive of a simpler picture, in Ashtekar's words (2018) “radiative aspects of null infinity are close to those of non-abelian gauge theory”.

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We can fix all these problems by resorting to Cartan geometry !

Homogenous models

YH (2020, 2021), Figueroa-O'Farrill–Have–Prohazka–Salzer (2021)

Null infinity

$$\mathscr{I}^n = {}^{\mathbb{R}^{n+1} \rtimes \mathrm{SO}(n,1)} \big/ {}_{\mathbb{R}^n \rtimes (\mathbb{R} \times \mathrm{ISO}(n-1))}$$

compare with ...

The boundary of AdS

$$\partial\left(AdS_{n+1}\right) = {}^{\mathrm{SO}(n+1,1)} \big/ {}_{\mathbb{R}^n \rtimes (\mathbb{R} \times \mathrm{SO}(n))}$$

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Carroll dS

$$Carr^{(n)}_{dS} = {\mathbb{R}^n \rtimes \mathrm{SO}(n)}\Big/{\mathrm{ISO}(n-1)}$$

Rq: topologically these are $\mathbb{R} \times S^{n-1}$

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Null infinity in Bondi gauge

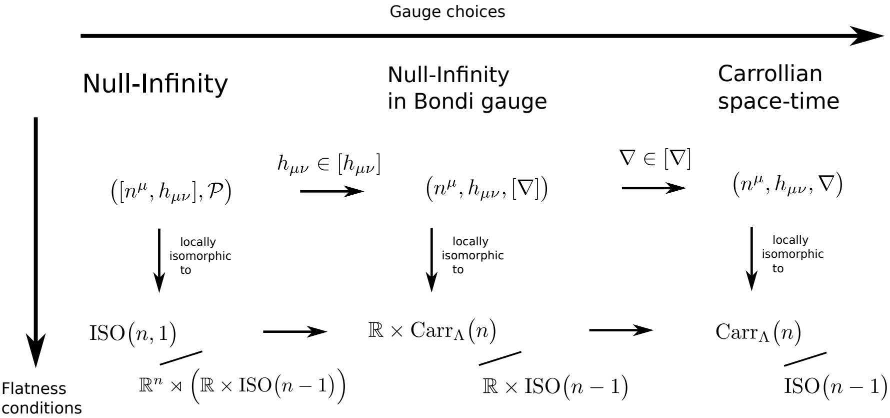
$$\mathscr{I}^n_B = {\mathbb{R}^{n+1} \rtimes \mathrm{SO}(n)} \big/ {\mathbb{R} \times \mathrm{ISO}(n-1)}$$

Carroll dS

$$Carr^{(n)}_{dS} = {\mathbb{R}^{n} \rtimes \mathrm{SO}(n)} \big/ {\mathrm{ISO}(n-1)}$$

Rq: topologically these are $\mathbb{R} \times S^{n-1}$

overview



Gauge choices
Null-Infinity
Null-Infinity
in Bondi gauge
Carrollian
space-time
$h_{\mu\nu} \in [h_{\mu\nu}]$
$\nabla \in [\nabla]$
$([n^\mu, h_{\mu\nu}], \mathcal{P})$
$(n^\mu, h_{\mu\nu}, [\nabla])$
$(n^\mu, h_{\mu\nu}, \nabla)$
locally
isomorphic
to
locally
isomorphic
to
locally
isomorphic
to
$\mathrm{ISO}(n,1)$
$\mathbb{R} \times \mathrm{Carr}_\Lambda(n)$
$\mathrm{Carr}_\Lambda(n)$
$\mathbb{R}^n \rtimes \left(\mathbb{R} \times \mathrm{ISO}(n-1)\right)$
$\mathbb{R} \times \mathrm{ISO}(n-1)$
$\mathrm{ISO}(n-1)$
Flatness
conditions

Strongly conformally Carrollian manifold

YH (2021), YH (2022)

Strongly conformally Carrollian manifold

YH (2021), YH (2022)

A Poincaré structure, a.k.a. *strongly conformally Carrollian manifold* $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, [h_{\mu\nu}, n^{\mu}], \mathcal{P}_{\mu\nu}\right)$ is the data of

- a conformally Carrollian manifold $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, [h_{\mu\nu}, n^{\mu}]\right)$
- together with a compatible Poincaré operator

$$\mathcal{P}_{\mu\nu} : \mathcal{E}\,[1] \to {}^{S^2 T^* \mathscr{I}}\!\big/_{h_{\mu\nu}} \otimes \mathcal{E}\,[1]$$

NB: $f \in \mathcal{E}\,[k]$ iff $f \mapsto \lambda^k f$ when $h_{\mu\nu} \mapsto \lambda^2 h_{\mu\nu}$.

Strongly conformally Carrollian manifold

YH (2021), YH (2022)

A Poincaré operator

$$\mathcal{P}_{\mu\nu} : \mathcal{E}[1] \to {}^{S^2 T^* \mathscr{I}}\diagup_{h_{\mu\nu}} \otimes \mathcal{E}[1]$$

is a second order linear differential operator of the form

$$\mathcal{P}(f)_{\mu\nu} =$$
$$\mathcal{P}_{00}(f)\,(du)_\mu\,(du)_\nu + \mathcal{P}_{A0}(f)\,2(du)_{(\mu}\,(dy^A)_{\nu)} + \mathcal{P}(f)_{AB}\,(dy^A)_\mu\,(dy^B)_\nu,$$

$$\mathcal{P}_{00}(f) = \nabla_0 \left(\dot{f} - \tfrac{h^{CD}\dot{h}_{CD}}{2(n-1)} f \right),$$
$$\mathcal{P}_{A0}(f) = \nabla_A \left(\dot{f} + \tfrac{h^{CD}\dot{h}_{CD}}{2(n-1)} f \right),$$
$$\mathcal{P}(f)_{AB} = \nabla_{(A}\nabla_{B)}\big|_{tf}\, f + \tfrac{1}{2} C_{AB}\dot{f} - \tfrac{1}{2}\dot{C}_{AB} f.$$

Claim: this transforms like a tensor under BMS transformations iif C_{AB} transforms like the asymptotic shear !

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Poincarré operators generalise to conformal Carroll geometries Möbius operators of 2D conformal geometry (Calderbank (2006)).

Equivalence problem of Cartan geometry $n \geq 3$

YH (2020)

Strongly conformal Carrollian geometries $\left(\mathscr{I} \xrightarrow{\Sigma}, [h_{\mu\nu}, n^\mu], \mathcal{P}_{\mu\nu}\right)$ are (locally) equivalent to normal Cartan geometries $(\mathcal{G} \to \mathscr{I}, \omega)$ modelled on

$$\mathscr{I}^{(n)} = {}^{\mathrm{ISO}(n,1)}\!\big/_{\mathbb{R}^n \rtimes (\mathbb{R} \times \mathrm{ISO}(n-1))}.$$

By Cartan's theorem, flat geometries will be equivalent to isomorphisms

$$\phi\colon \mathscr{I} \to {}^{ISO\,(3,1)}\!\big/_{(\mathbb{R} \times ISO\,(2)) \ltimes \mathbb{R}^3}$$

i.e. "gravity vacua" (Ashtekar (1981)).

Cartan Geometry of null infinity (YH (2020))

We now want to consider Cartan geometry $(\mathcal{G} \to \mathscr{I}, \omega)$ modelled on

$$ISO\,(n,1) \big/ (\mathbb{R} \times ISO\,(n-1)) \ltimes \mathbb{R}^n$$

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Null-tractors frames (YH (2020))

Let $(\mathscr{I}, [h_{ab}, n^a])$ be a n-dimensional conformal Carrollian manifold,
it is *canonically* equipped with a $(\mathbb{R} \times ISO\,(n-1)) \ltimes \mathbb{R}^n$-principal bundle

$$\mathcal{G} \to \mathscr{I}$$

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$$\mathcal{G} \to \mathscr{I}$$

Normal conformal Carrollian Cartan connection (YH (2020))

Let $(\mathscr{I}, [h_{ab}, n^a])$ be a n-dimensional conformal Carrollian manifold. Compatible normal Cartan connections

$$\omega \in \Omega^1(\mathcal{G}, \mathfrak{iso}(n,1))$$

form an affine space isomorphic to

- ($n \geq 4$) "zero modes for the asymptotic shear" $C_{AB}(u=0,y)$,
- ($n=3$) genuine choice of "asymptotic shear" $C_{AB}(u,y)$,
- ($n=2$) choices of "Mass M and Angular momentum N_A aspects".

The null-tractor bundle (YH (2020))

Let $(\mathscr{I} \to \Sigma, [h_{ab}, n^a])$ be a conformal Carrollian geometry.

The null-tractor bundle $\mathcal{T} \to \mathscr{I}$ is the canonical bundle obtained as the associated bundle to $\mathcal{G} \to \mathscr{I}$ for the fundamental representation.

Practically, a choice of trivialisation $u \in \mathcal{C}^\infty(\mathscr{I})$ gives an isomorphism

$$\mathcal{T} \overset{u}{\simeq} \mathbb{R} \oplus T\mathscr{I} \oplus \mathbb{R}$$

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In a trivialisation, a tractor $Y^I \in \mathcal{C}^\infty(\mathcal{T})$ can be written as:

$$Y^I \overset{u}{=} \begin{pmatrix} Y^+ \\ Y^A \\ Y^u \\ Y^- \end{pmatrix}$$

with $Y^+, Y^- \in \mathcal{C}^\infty(\mathscr{I})$
$Y^A\partial_A + Y^u\partial_u \in \mathcal{C}^\infty(T\mathscr{I})$

This is a 5-dimensional vector bundle, canonically defined from $([h_{ab}, n^a])$ and equipped with a degenerate metric :

$$Y^2 = 2Y^+Y^- + Y^AY^Bh_{AB}$$

and a preferred degenerate direction $I^I = (0, 0^A, 1, 0)$.

Tractor "transformation rules"

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Any other trivialisation $\hat{u}$ will give another isomorphism and we have the transformation rules

$$Y^I \overset{u}{=} \begin{pmatrix} Y^+ \\ Y^A \\ Y^u \\ Y^-. \end{pmatrix} \mapsto Y^I \overset{\hat{u}}{=}$$

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$$Y^I \overset{u}{=} \begin{pmatrix} Y^+ \\ Y^A \\ Y^u \\ Y^-. \end{pmatrix} \mapsto Y^I \overset{\hat{u}}{=} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda^{-1}U^A & \lambda^{-1}\delta^A{}_B & 0 & 0 \\ \beta & \lambda^{-1}\nabla_B u & 1 & 0 \\ -\frac{\lambda^{-1}}{2}U^2 & -\lambda^{-1}U_B & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} Y^+ \\ Y^A \\ Y^u \\ Y^-. \end{pmatrix}$$

here $\lambda := \mathcal{L}_n \hat{u}$,
$U_A := \lambda^{-1}\left(\nabla_A \lambda - \left(\frac{\dot{\lambda}}{\lambda} + \frac{1}{2(n-1)} h^{CD}\dot{h}_{CD}\right)\nabla_A \hat{u}\right)$,
and $\beta := -\frac{\lambda^{-1}}{d-2}\nabla^C\nabla_C \hat{u} + \frac{2\lambda^{-2}}{n-1}\nabla_C\lambda\nabla^C\hat{u} - \left(\frac{\nabla_C \hat{u}}{\lambda}\right)^2\left(\frac{\dot{\lambda}}{\lambda} + \frac{\Theta}{2}\right)$

Proposition (YH (2020))

Let $([h_{ab}, n^a])$ be a n-dimensional conformal Carrollian geometry. Compatible normal tractor connections form an affine space isomorphic to

- ($n \geq 4$) "zero modes for the asymptotic shear" $C_{AB}\,(u=0,y)$,
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Sketch of Proof

The coordinate of a "compatible tractor connection" are

$$A^I{}_J \overset{u}{=} \begin{pmatrix} 0 & -h_{BC}\,dy^C & 0 & 0 \\ -\xi^A & \omega^A{}_B & 0 & dy^A \\ 0 & \xi_B & 0 & 0 \\ -\psi & -\frac{1}{2}C_B & 0 & du \end{pmatrix}$$

- $F^I{}_J X^J = 0 \;\Rightarrow\; \omega^A{}_B = fct\,(h),\, \xi_B = \xi_{(AB)}dy^A,\, C_B = C_{(AB)}dy^A.$
- $F^a{}_{bcd}n^c = 0 \;\Rightarrow\; \xi_{AB}\big|_{TF} := \frac{1}{2}\dot{C}_{AB},\; (n-2)Tr\xi = -\frac{1}{2}R(h),\; \psi_b n^b \propto Tr\xi.$
- $h^{bc}F^a{}_{bcd} = 0 \;\Rightarrow\; (n-3)\frac{1}{2}\dot{C}_{AB} = -R_{AB}\big|_{TF},\, (n-2)\psi_A = -\frac{1}{2}\nabla^C C_{CA}.$

Null-infinity as a conformal boundary of spacetimes

Homogenous space perspective

YH (2020), Figueroa-O'Farrill–Have–Prohazka–Salzer (2021)

Let us define Minkowski space $M^{3,1}$ as the homogeneous space

$$M^{3,1} := {\color{red}ISO\,(3,1)} / SO\,(3,1)$$

then its conformal boundary $\mathscr{I}^3$ is also an homogeneous space (*see S. Prohazka's talk for a lot more examples of this type!*):

$$\mathscr{I}^3 := {\color{red}ISO\,(3,1)} / (\mathbb{R} \times ISO\,(2)) \ltimes \mathbb{R}^3.$$

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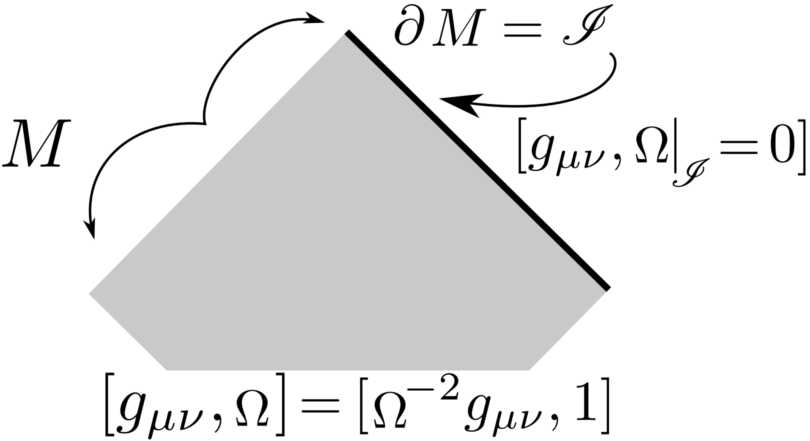
$$\mathscr{I}^3 := {ISO\,(3,1)}\diagup{(\mathbb{R}\times ISO\,(2))\ltimes\mathbb{R}^3}.$$

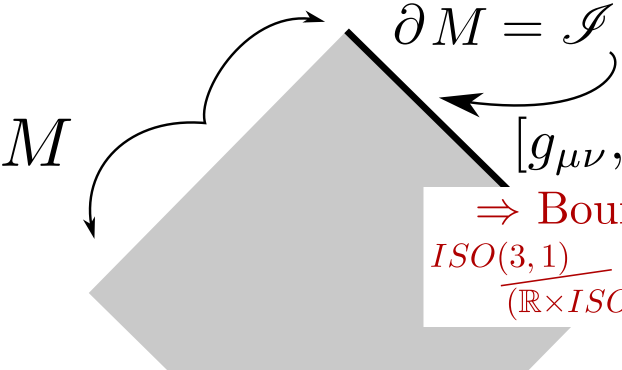
Compare with

$$AdS_4 := {SO\,(3,2)}\diagup{SO\,(3,1)}$$

with conformal boundary

$$S^1\times S^2 := {SO\,(3,2)}\diagup{(\mathbb{R}\times SO\,(2,1))\ltimes\mathbb{R}^3}$$


$\partial M = \mathscr{I}$
M
$[g_{\mu\nu}, \Omega|_{\mathscr{I}} = 0]$
$[g_{\mu\nu}, \Omega] = [\Omega^{-2} g_{\mu\nu}, 1]$

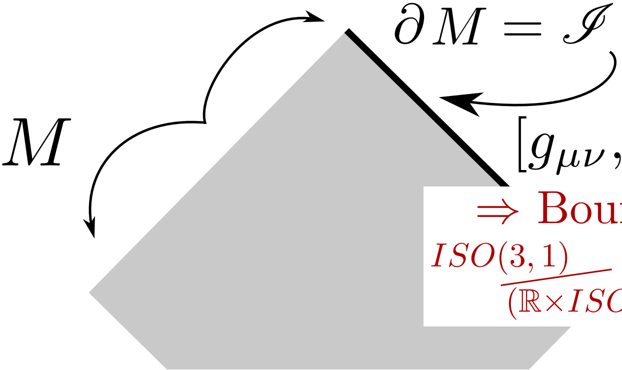


$$[g_{\mu\nu}, \Omega|_{\mathscr{I}} = 0]$$

$\Rightarrow$ Boundary $\frac{ISO(3,1)}{(\mathbb{R}\times ISO(2))\ltimes\mathbb{R}^3}$ CG

$$[g_{\mu\nu}, \Omega] = [\Omega^{-2} g_{\mu\nu}, 1]$$

$\Rightarrow \frac{ISO(3,1)}{SO(3,1)}$ CG (MacDowell-Mansouri)



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$$\Rightarrow \text{Boundary } \frac{ISO(3,1)}{(\mathbb{R}\times ISO(2))\ltimes\mathbb{R}^3} \text{ CG}$$

$$[g_{\mu\nu}, \Omega] = [\Omega^{-2} g_{\mu\nu}, 1]$$

$$\Rightarrow \frac{ISO(3,1)}{SO(3,1)} \text{ CG} \quad \text{(MacDowell-Mansouri)}$$

This follows from tractor methods (Gover 2010) and "orbit decomposition of Cartan geometry" from Cap-Gover-Hammerl (2014).

Induced Cartan geometry (YH (2021))

An asymptotically flat space-time induces at $\mathscr{I}$ more than a conformal Carrollian geometry $([h_{ab}, n^a])$:

It induces a full Cartan geometry modelled on $\mathrm{ISO}(3,1)\,/P$!

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Gravitational radiation = Cartan curvature

The curvature of the induced Cartan geometry coincides with the pull-back of the Weyl tensor

$$\iota^*\left(\Omega^{-1} n^\mu W_{\mu\nu\rho\sigma}\right)$$

equivalently with Ψ_4^0, Ψ_3^0, $Im\left(\Psi_2^0\right)$.

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Disclaimer: The spacetime connection itself is known to twistor experts since Penrose–MacCallum (1973) as “local twistor connection” (and needed to define asymptotic twistor space). New aspects here are: 1) Its relation to the intrinsic geometry of conformal Carrollian mfld, 2) The Cartan geometry interpretation.

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Conclusion

- Strongly (conformal) Carrollian geometries are equivalent to Cartan connections for Carroll (Poincaré) groups
- BMS symmetries are related to the non-uniqueness of “strong” geometries compatible with a fixed “weak”conformally Carroll geometry
- For asymptotically flat spacetimes, null infinity has a preferred Cartan connection induced from the spacetime tractor connection.
- In all dimensions but $d = 4$ these induced connections are flat and define a “gravity vacua”; i.e a map

$$\phi \colon \mathscr{I} \to {}^{\mathrm{ISO}(d-1,1)}\!\big/_{P}.$$

- In dimension $d = 4$ the tractor curvature invariantly encodes the presence of gravitational radiation : this *is the obstruction to finding such isomorphisms*.
- There is in fact a precise sense in which memory effect is related to the fact that “gravitational radiation induces transition between gravity vacua” (Ashtekar (2016), YH (2020)).

What's next?

- Can be used to write invariant functionals at null infinity e.g.

$$\int_{\mathscr{I}} CS\,(A)$$

(Nguyen-Salzer (2021)).

- Suggests further generalisations e.g. higher spin (Lovrekovic (2022))
- I did not touch on even more interesting invariants of null infinity which are *charges* (Barnich–Troessart (2011)) even thought they do have Carrollian interpretations (*See G. Barnich's, M. Petropoulos's and L. Ciambelli's talks !*).

$\Rightarrow$ Can we reproduce these results in this conformal framework?
(Penrose (1982), Dougan–Mason (1991), Cap–Gover (2021) suggest this is possible)

- Tractors are meant to "proliferate invariants". Can we use this machinery to produce new (physical!) invariants for asymptotically flat space-times (*see L. Freidel's talk for candidates*)?

Thank you for your attention ...

and see you in Mons in September !