Asymptotic structure of Carrollian limits of Einstein-Yang-Mills theory

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State of the art

• The Carroll algebra is the "ultrarelativistic limit" ($c \to 0$) of the Poincaré algebra (Lévy-Leblond '65, Bacry and Lévy-Leblond '68).

• Carrollian limit of Einstein gravity arose as "the strong coupling limit" (Isham '76) or "the zero signature limit" (Teitelboim '78). It is also relevant to description of the gravitational field near to a spacelike singularity (BKL analysis).

• The geometry of the Carrollian limit (manifestly diffeomorphism invariant formulation) has been studied as well (Henneaux '79). Since then, many exciting applications and properties of Carrollian structures has been discovered connecting the geometry of null hypersurfaces and the BMS symmetry! (Duval, Gibbons, Horvathy 2014)

• It was shown very recently at least 2 different Carrollian limits of GR: the **electric** limit and the **magnetic** limit (Henneaux and Salgado-Rebolledo 2109.06708)...as explained in Marc's talk

State of the art

• The asymptotic structure of both limits has been studied by using the strict set of parity conditions of *Regge-Teitelboim* (Poincaré algebra) and *Henneaux-Troessaert* (BMS algebra) in the recent interesting work by Alfredo Pérez 2110.15834 (see Alfredo's talk).

• But, there is a 3rd set of BMS invariant boundary conditions: the set twisted of parity conditions (Regge-Teitelboim + improper diffeomorphism) by Henneaux and Troessaert 2019.

• In the following:
i) Completing the asymptotic analysis of the Carrollian limits of Einstein gravity by considering the twisted set of parity conditions (spoiler: boosts remain improper in the electric limit) and
ii) Including the Yang-Mills field (spoiler: infinite-dimensional color group in the electric limit)
Moral: Electric Carrollian limit of EYM has a rich asymptotic structure!
work in collaboration with Marc Henneaux, Patricio Salgado-Rebolledo and Jakob Salzer. To appear soon!

Ideals of the Carroll algebra

- The Carroll algebra $\mathcal{C}$ reads

$$[M_i, M_j] = \epsilon_{ijk} M_k \qquad [M_i, B_j] = \epsilon_{ijk} B_k \qquad [B_i, B_j] = 0$$
$$[M_i, P_j] = \epsilon_{ijk} P_k \qquad [P_i, B_j] = \delta_{ij} E \qquad [M_i, E] = 0 \qquad [E, B_i] = 0$$
$$[P_i, P_j] = 0 \qquad [P_i, E] = 0$$

- P_i and M_i are kinematical transformations (defined within equal time hypersurfaces) forming a sublagebra isomorphic to $iso(3)$ (Euclidean displacements).

- E and B_i are dynamical transformations (involving time evolution) forming an Abelian ideal $\mathcal{D}$.

- The quotient of the Carroll algebra by the ideal $\mathcal{D}$ is isomorphic to $iso(3)$:

$$\frac{\mathcal{C}}{\mathcal{D}} \simeq iso(3)$$

Ideals of the Carroll algebra

• The energy E is also an ideal $\mathcal{I}$ (one-dimensional and so Abelian). Then:

$$\frac{\mathcal{C}}{\mathcal{I}} \simeq iso(3) \oplus_\sigma t_3 \equiv d_3.$$

with t_3 transforming in the vector representation of $iso(3)$.

• B_i and M_i form the subalgebra of homogeneous Carroll transformations, which is isomorphic to $so(3) \oplus_\sigma t_3$ (six-dimensional). This kind of transformation grows up at infinity linearly, while translations tend to constants (as we will see below).

• Both quotients (infinite-dimensional representations of the homogeneous Carroll group) will be relevant to our next discussion on asymptotic symmetries...but let us briefly discuss the sort of extensions we will find.

Supertranslations

- *Carroll-BMS algebra*: $\mathcal{C}$ is extended by Carroll supertranslations parametrized by one even function $T(\theta,\varphi)$ and one odd function $W(\theta,\varphi)$.

$$\begin{aligned}\hat{T} &= Y^A\partial_A T - 3bW - \partial_A b\overline{D}^A W - b\overline{\triangle}W \\ \hat{W} &= Y^A\partial_A W\end{aligned}$$

- *iso*(3)*-BMS algebra*: $\mathcal{C}/\mathcal{D} \simeq iso(3)$ is extended by Carroll supertranslations parametrized by one odd function $W(\theta,\varphi)$.

$$\hat{W} = Y^A\partial_A W$$

- d_3*-BMS algebra*: $\mathcal{C}/\mathcal{I} \simeq d_3$ is extended by Carroll supertranslations parametrized by one odd function $W(\theta,\varphi)$.

$$\hat{W} = Y^A\partial_A W$$

Asymptotic conditions for pure gravity

• Let us consider the fall-off of the fields (in Cartesian coordinates) for pure gravity:

$$g_{ij} = \delta_{ij} + \frac{\overline{h}_{ij}}{r} + \mathcal{O}(r^{-2}) \qquad \pi^{ij} = \frac{\overline{\pi}^{ij}}{r^2} + \mathcal{O}(r^{-3})$$

• Boundary conditions (parity conditions) must ensure that the symplectic structure is finite and the symplectic form is truly invariant (one can define canonical generators) under the transformations that preserve the fall-off of the fields (a true symmetry of the theory).

• Improper gauge transformations ($Q \neq 0$) strongly depends on this set of boundary conditions.

Summary of Carrollian limits for strict parity conditions

- Regge-Teitelboim parity conditions: $\overline{h}_{rr} \sim \overline{\pi}^{rA} \sim \overline{h}_{AB} = \text{even}$, $\overline{\pi}^{rr} \sim \overline{\pi}^{AB} = \text{odd}$.

Asymptotic symmetry algebra is the Poincaré algebra. Under the magnetic Carrollian limit of the theory is the Carroll algebra, while in the electric limit is $\mathcal{C}/\mathcal{D} \simeq iso(3)$ (Pérez 2110.15834).

- Henneaux-Troessaert strict parity conditions: $\overline{h}_{rr} \sim \overline{\pi}^{AB} = \text{even}$, $\overline{p} \sim \overline{k}_{AB} \sim \overline{\pi}^{rA} = \text{odd}$ with

$$\overline{p} = 2(\overline{\pi}^{rr} - \overline{\pi}^{A}_{A}) \qquad \overline{k}_{AB} = \frac{1}{2}(\overline{h}_{AB} + \overline{h}_{rr}\overline{g}_{AB})$$

Asymptotic symmetry algebra is the BMS algebra. Under the magnetic Carrollian limit of the theory is the Carroll-BMS algebra, while in the electric limit is the $iso(3)$-BMS algebra (Pérez 2110.15834). In this case, Carroll boosts are proper gauge transformations ($Q_b = 0$).

Carrollian limits for twisted parity conditions

- Henneaux-Troessaert twisted parity conditions: Regge-Teitelboim + improper gauge transformation (include Taub-NUT).

$$\bar{h}_{rr} = \text{even} \qquad \bar{\lambda}_A = (\bar{\lambda}_A)^{\text{odd}} + \bar{D}_A \zeta_r - \bar{\zeta}_A$$

$$\bar{h}_{AB} = (\bar{h}_{AB})^{\text{even}} + \bar{D}_A \bar{\zeta}_B + \bar{D}_B \bar{\zeta}_A + 2\bar{g}_{AB}\zeta_r$$

$$\bar{\pi}^{rr} = (\bar{\pi}^{rr})^{\text{odd}} - \sqrt{\bar{g}}\,\bar{\triangle} V \qquad \bar{\pi}^{rA} = (\bar{\pi}^{rA})^{\text{even}} - \sqrt{\bar{g}}\,\bar{D}^A V$$

$$\bar{\pi}^{AB} = (\bar{\pi}^{AB})^{\text{odd}} + \sqrt{\bar{g}}(\bar{D}^A \bar{D}^B V - \bar{g}^{AB}\bar{\triangle} V)$$

The function ζ_r is odd, while the angular component ζ_A and the function V are even under parity transformations.

The canonical realization of the asymptotic symmetries is the BMS algebra. In the magnetic limit one can obtain the Carroll-BMS algebra (similarly to the strict set of parity conditions).

But in the electric limit, one obtains instead the d_3-BMS algebra **(Carrollian boosts become improper, i.e., $Q_b \neq 0$).**

Action principle

The results in the magnetic limit for this set of twisted parity conditions are similar to the ones reported by Pérez (Carroll and Caroll-BMS at spatial infinity). Then, we focus in the electric Carrollian limit of Einstein gravity (Henneaux and Salgado-Rebolledo 2021):

$$S^E[g_{ij},\pi^{ij},N,N^i]=\int dt\left[\int d^3x\left(\pi^{ij}\dot{g}_{ij}-N\mathcal{H}^E-N^i\mathcal{H}^E_i\right)-B^E_\infty\right]$$

The Hamiltonian constraints

$$\mathcal{H}^E=\frac{1}{\sqrt{g}}\left(\pi^{ij}\pi_{ij}-\frac{\pi^2}{2}\right)\approx 0$$

$$\mathcal{H}^E_i=-2\nabla^j\pi_{ij}\approx 0$$

which in particular satisfy the “zero signature” deformation algebra

$$\{\mathcal{H}^E(x),\mathcal{H}^E(x')\}=0$$

This guarantees the invariance of the theory under Carroll transformations (Henneaux ’79).

Asymptotic symmetries

It is useful to write the fall-off in spherical polar coordinates:

$$g_{rr} = 1 + \frac{\overline{h}_{rr}}{r} + \frac{h^{(2)}_{rr}}{r^2} + \mathcal{O}\left(r^{-3}\right) \qquad \pi^{rr} = \overline{\pi}^{rr} + \frac{\pi^{(2)}_{rr}}{r} + \mathcal{O}\left(r^{-2}\right)$$

$$g_{rA} = \overline{\lambda}_A + \frac{h^{(2)}_{rA}}{r} + \mathcal{O}\left(r^{-2}\right) \qquad \pi^{rA} = \frac{\overline{\pi}^{rA}}{r} + \frac{\pi^{(2)rA}}{r^2} + \mathcal{O}\left(r^{-3}\right)$$

$$g_{AB} = r^2 \overline{g}_{AB} + r\,\overline{h}_{AB} + h^{(2)}_{AB} + \mathcal{O}\left(r^{-1}\right) \qquad \pi^{AB} = \frac{\overline{\pi}^{AB}}{r^2} + \frac{\pi^{(2)AB}}{r^3} + \mathcal{O}\left(r^{-4}\right)$$

This is preserved under the following deformations

$$\xi = br + T + \mathcal{O}(r^{-1})$$
$$\xi^r = W + \mathcal{O}(r^{-1})$$
$$\xi^A = Y^A + \frac{1}{r} I^A + \mathcal{O}(r^{-2})$$

→ In the magnetic limit, the Carrollian boosts are canonical provided $g_{rA} \sim r^{-1}$ or $\overline{\lambda}_A = 0$ ($I^A = \overline{D}^A W$) (Henneaux and Troessaert 2018, Pérez 2021).

Asymptotic symmetries

However, in the electric limit, there is no need to impose this requirement! **Carrollian boosts are immediately canonical** (absence of spatial derivatives in $\mathcal{H}^E \sim \pi^2$).

• Let us assume first that $\overline{\lambda}_A = 0$. In order to maintain this condition it is necessary to perform the following correcting gauge transformation

$$\xi^r = \mathcal{O}\left(r^{-2}\right) \qquad \xi^A = \frac{1}{r} I^A_{(b)} + \mathcal{O}\left(r^{-2}\right)$$

where

$$I^A_{(b)} = \frac{2b}{\sqrt{\overline{g}}} \overline{\pi}^{rA}$$

The charge is then given by

$$Q^E_\xi[g_{ij}, \pi^{ij}] = \oint d^2x \left[2Y^A\left(\pi^{(2)r}_A + \overline{h}_{AB}\overline{\pi}^{rB}\right) + \frac{2b}{\sqrt{\overline{g}}}\overline{\pi}^{rA}\overline{\pi}^r_A + 2W(\overline{\pi}^{rr} - \overline{\pi}^A_A)\right]$$

It is easy to check that for any set of strict parity conditions (Regge-Teitelboim or Henneaux-Troessaert), the boost term vanishes (recall that b is odd). Giving $iso(3)$ and $iso(3)$-BMS as asymptotic symmetry algebras, respectively.

Asymptotic symmetries

For the twisted set of parity conditions, $\overline{\pi}^{rA}$ does not have a definite-parity:

$$\overline{\pi}^{rA} = (\overline{\pi}^{rA})^{\text{even}} - \sqrt{\overline{g}}\,\overline{D}^A V$$

then the boost term does not vanish!

$$Q^E_\xi[g_{ij}, \pi^{ij}] = \oint d^2x \left[2Y^A \left(\pi^{(2)r}_A + \overline{h}_{AB}\overline{\pi}^{rB} \right) + \frac{2b}{\sqrt{\overline{g}}}\overline{\pi}^{rA}\overline{\pi}^r_A + 2W(\overline{\pi}^{rr} - \overline{\pi}^A_A) \right]$$

Carrollian boosts are then improper symmetries ($Q_b \neq 0$). The canonical generators obey the d_3-BMS algebra:

$$\begin{aligned}
\{B^i, B^j\} &= 0 \\
\{B^i, M^{jk}\} &= \frac{1}{2}\left(\delta^{ij}B^k - \delta^{ik}B^j\right) \\
\{M^{ij}, M^{kl}\} &= \frac{1}{2}\left(\delta^{jk}M^{il} - \delta^{ik}M^{jl} + \delta^{li}M^{jk} - \delta^{lj}M^{ik}\right) \\
\{B^i, \mathcal{W}(\theta,\varphi)\} &= 0 \\
\{M^{ij}, \mathcal{W}(\theta,\varphi)\} &= -\overline{D}_B(x^{[i}e^{j]B}\mathcal{W}(\theta,\varphi))
\end{aligned}$$

Asymptotic symmetries

- Let us now assume that $\overline{\lambda}_A \neq 0$. In this case the canonical generator reads

$$Q_\xi^E[g_{ij}, \pi^{ij}] = \oint d^2x \left[2Y^A \left(\pi_A^{(2)r} + \overline{h}_{AB}\overline{\pi}^{rB} + \overline{\pi}^{rr}\overline{\lambda}_A \right) + 2I^A \overline{\pi}_A^r + 2W\overline{\pi}^{rr} \right]$$

The algebra of the asymptotic symmetries is given by the semi-direct product of $iso(3)$-BMS and a generalized set of Carrollian supertranslations denoted by $\mathcal{W}_A$:

$$\{M^{ij}, M^{kl}\} = \frac{1}{2}\left(\delta^{jk}M^{il} - \delta^{ik}M^{jl} + \delta^{li}M^{jk} - \delta^{lj}M^{ik}\right)$$
$$\{M^{ij}, \mathcal{W}(\theta,\varphi)\} = -\overline{D}_B(x^{[i}e^{j]B}\mathcal{W}(\theta,\varphi))$$
$$\{M^{ij}, \mathcal{W}_A(\theta,\varphi)\} = \overline{D}_B(x^{[i}e^{j]B}\mathcal{W}_A(\theta,\varphi)) - x^{[i}\overline{D}_A e^{j]B}\mathcal{W}_B(\theta,\varphi)$$

Theory and asymptotic symmetries

(Higher-dimensional) electromagnetism: infinite-dimensional symmetries are also present in this context at spatial infinity (Henneaux and Troessaert 1803.10194, 1903.04437).
→ By introducing an appropriate set of parity conditions (preserved under Poincaré and gauge symmetries).
→ The resolution of the "boost problem" requires additional boundary fields. The results perfectly match with null-infinity (combining both functions with different parities...see 1803.10194).

Yang-Mills theory: It is not possible to find a consistent set of parity conditions, which allows to have improper gauge transformations consistent with Lorentz invariance (Tanzi and Giulini 2006.07268). Not even with the help of boundary degrees of freedom!

Natural question: Is it possible to find improper gauge transformations consistent with Carroll invariance of the theory?

Carrollian limits

Under the magnetic limit of Einstein-Yang-Mills, the "Carrollian boost problem" persists, and the asymptotic symmetry is trivial described by the Carroll algebra (no improper gauge symmetries).

Under the electric limit the "Carrollian boost problem" *disappears.* Non-integrability is not present since $\mathcal{H}^E_{\rm YM} \sim \pi^a_i \pi^i_a$. This allows to find non-trivial improper gauge symmetries by imposing an appropriate set of parity conditions.

The electric contraction is implemented by the rescalings (Henneaux and Salgado-Rebolledo 2021)

$$A^a_i \to c A^a_i \qquad \pi^i_a \to \frac{1}{c}\pi^i_a \qquad A^a_0 \to c A^a_0 \qquad \alpha \to \frac{1}{c}\alpha \tag{1}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \alpha[A_\mu, A_\nu]^a$ and $c \to 0$. The Hamiltonian action then reads

$$S^E_{\rm YM} = \int d^D x \left(\pi^i_a \dot{A}^a_i - \mathcal{H}^E_{\rm YM} + A^a_0 D_i \pi^i_a\right) \qquad \mathcal{H}^E_{\rm YM} = \frac{1}{2\sqrt{g}}\pi^a_i \pi^i_a$$

where $\{\mathcal{H}^E_{\rm YM}(x), \mathcal{H}^E_{\rm YM}(x')\} = 0$.

Fall-off and parity conditions

We take the same fall-off of the fields as the YM theory:

$$A_r = \frac{1}{r}\bar{A}_r(x^A) + O\left(\frac{1}{r^2}\right) \qquad \pi^r = \bar{\pi}_r(x^A) + O\left(\frac{1}{r}\right)$$

$$A_A = \bar{A}_A(x^A) + O\left(\frac{1}{r}\right) \qquad \pi^A = \frac{1}{r}\bar{\pi}^A(x^A) + O\left(\frac{1}{r^2}\right)$$

whereas for the gauge parameters we consider

$$\varepsilon^A = \bar{\varepsilon}^A(x^A) + O\left(\frac{1}{r}\right)$$

We make use of the parity conditions

$$\bar{A}_r = \bar{U}^{-1}\bar{A}_r^{\text{even}}\bar{U} \qquad \bar{\pi}^r = \bar{U}^{-1}\bar{\pi}^r_{\text{odd}}\bar{U}$$

$$\bar{A}_A = \bar{U}^{-1}\bar{A}_A^{\text{odd}}\bar{U} + \bar{U}^{-1}\partial_A\bar{U} \quad \bar{\pi}^A = \bar{U}^{-1}\bar{\pi}^A_{\text{even}}\bar{U}$$

where $\bar{U} = \exp(-\phi_{\text{odd}})$ (analog of the twisted parity conditions for EM!). The symplectic form is finite under this set of boundary conditions provided the asymptotic form of the Gauss constraint is imposed (Tanzi and Giulini 2020). **This is also the case for Carrollian limits.**

Asymptotic symmetries: Kac-moody algebra

Color charge

$$Q_{\epsilon}^{\text{YM}} = -\oint d^2x \bar{\epsilon}^a \bar{\pi}_a^r$$

The leading order of the momentum π_a^r has not a definite-parity then the color charges become improper ($Q_{\epsilon}^{\text{YM}} \neq 0$), being compatible with the rest of the symmetries.

In this case, improper symmetries are given by spatial rotations and the infinite-dimensional color charges

$$\begin{aligned} Q_{\xi,\epsilon}^{\text{YM}}[A_i, \pi^i] &= \oint d^2x \left(-\bar{\epsilon}^a \bar{\pi}_a^r + Y^A \overline{A}_A^a \bar{\pi}_a^r \right) \\ &= \frac{1}{2}\omega_{ij} M^{ij} + \oint d^2x \bar{\epsilon}^a \mathcal{T}_a \end{aligned}$$

Then, the algebra of the asymptotic symmetries is given by

$$\begin{aligned} \{M^{ij}, M^{kl}\} &= \frac{1}{2}\left(\delta^{jk} M^{il} - \delta^{ik} M^{jl} + \delta^{li} M^{jk} - \delta^{lj} M^{ik}\right) \\ \{M^{ij}, \mathcal{T}_a(x^A)\} &= -\overline{D}_B(x^{[i} e^{j]B} \mathcal{T}_a(x^A)) \\ \{\mathcal{T}_a(x_1^A), \mathcal{T}_b(x_2^A)\} &= f^c{}_{ab} \mathcal{T}_c(x_1^A) \delta^{(2)}(x_1^A - x_2^A) \end{aligned}$$

Conclusions

• We have completed the asymptotic analysis of the Carrollian limits of Einstein gravity by considering the twisted set of parity conditions. We have found other infinite-dimensional extensions of the possible quotient of the Carroll algebra by one the possible ideals: d_3-BMS (Carrollian boosts are included) and generalized $iso(3)$-BMS.

• We have included the Yang-Mills field, finding that the electric Carrollian limit possesses an infinite-dimensional set of color charges.

• Connection with null infinity through a hyperbolic slicing?

• Relation with the results at null infinity for Yang-Mills theory by He, Mitra, Strominger?...to be explored.

• Carrollian limits in higher dimensions? We have access to the asymptotic structure of all higher dimensions at spatial infinity through the Hamiltonian tools. Nonlinear deformations of the found infinite-dimensional algebras presented here (Fuentealba and Henneaux to appear).