Carrollian physics in cosmology

Béatrice Bonga – 22 Feb 2022 – Carroll Workshop
[BB+Prabhu, PRD, arXiv:2009.01243]

Radboud University

IN·DEI·NOMINE·FELICITER

Disclaimer

"An expert is someone with $\geq$ 2 paper on a given topic."

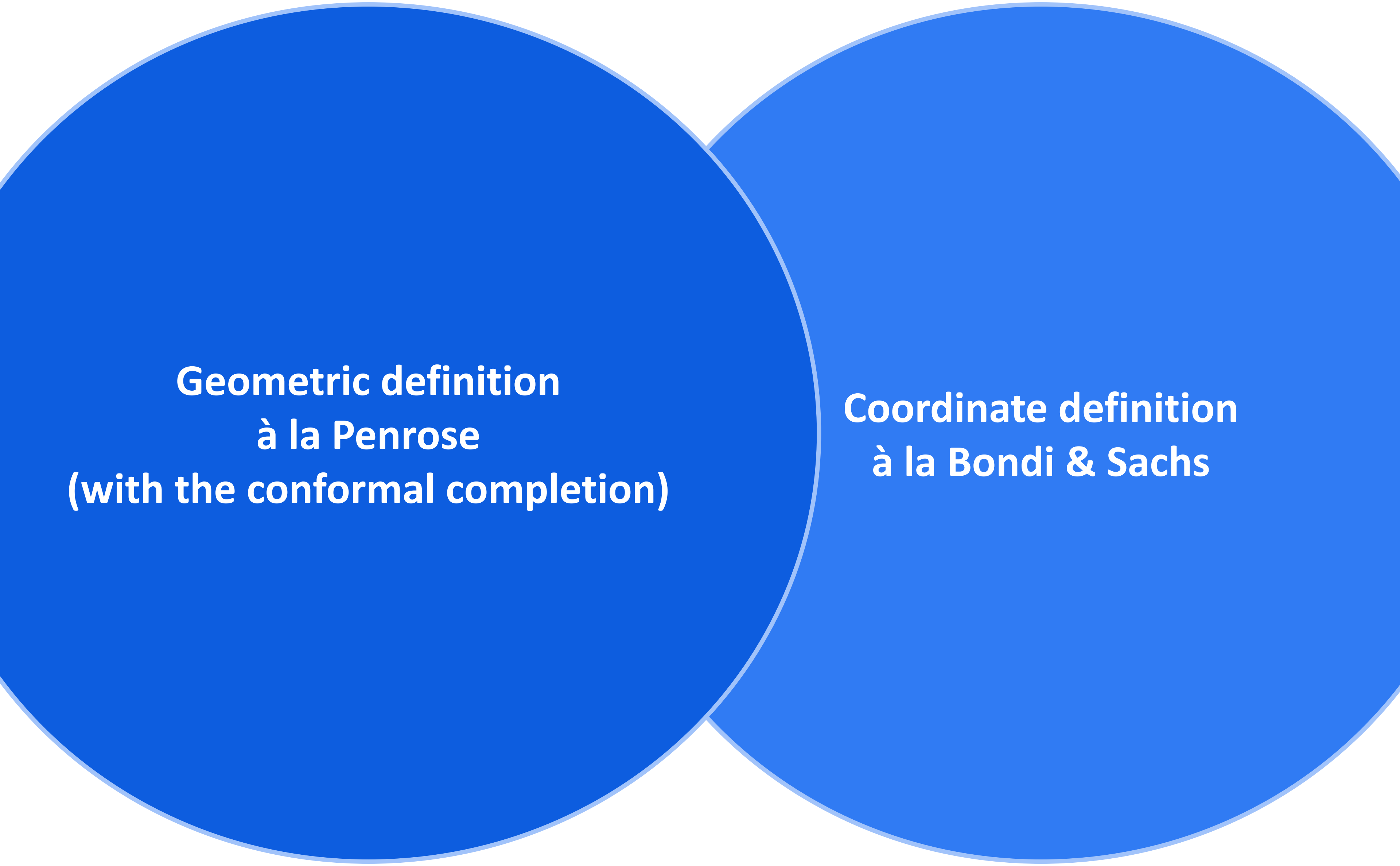
I have one remark on Carroll structures in one paper.

Outline

- Asymptotically flat spacetimes and the BMS algebra

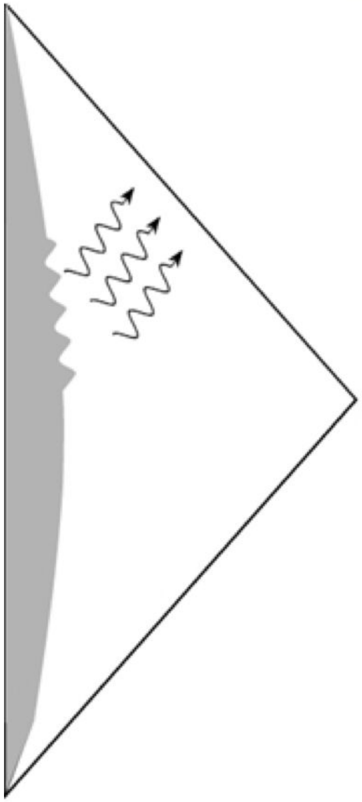
- Generalization to expanding spacetimes

Two definitions asymptotic flatness



Asymptotically flat spacetime

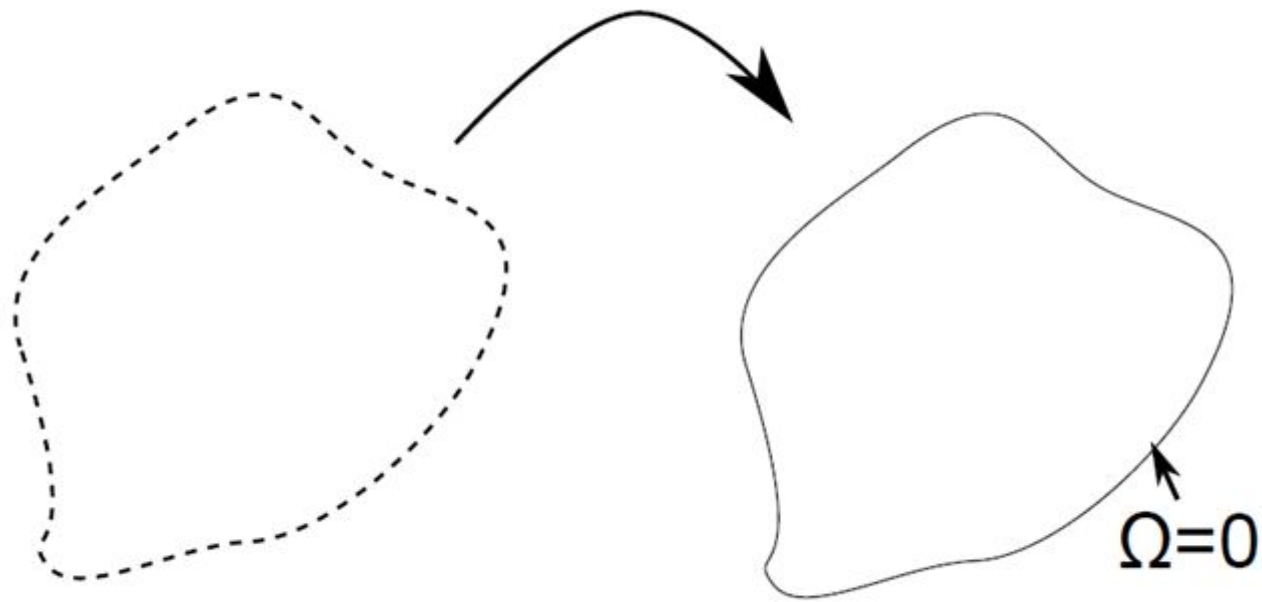
$$d\hat{s}^2 = -UV\,du^2 - 2U\,dudr + \gamma_{AB}(r\,d\theta^A + W^A\,du)(r\,d\theta^B + W^B\,du)$$



$$U = 1 + B/r^2 + O(r^{-3}),$$
$$V = 1 - 2M/r + N/r^2 + O(r^{-3}),$$
$$W^A = A^A/r + B^A/r^2 + O(r^{-3}),$$
$$\gamma_{AB} = \Omega_{AB} + f_{AB}/r + \tfrac{1}{4} f^2 \Omega_{AB}/r^2 + O(r^{-3})$$

Flat Space | Mass aspect | Radiative modes | Angular momentum aspect

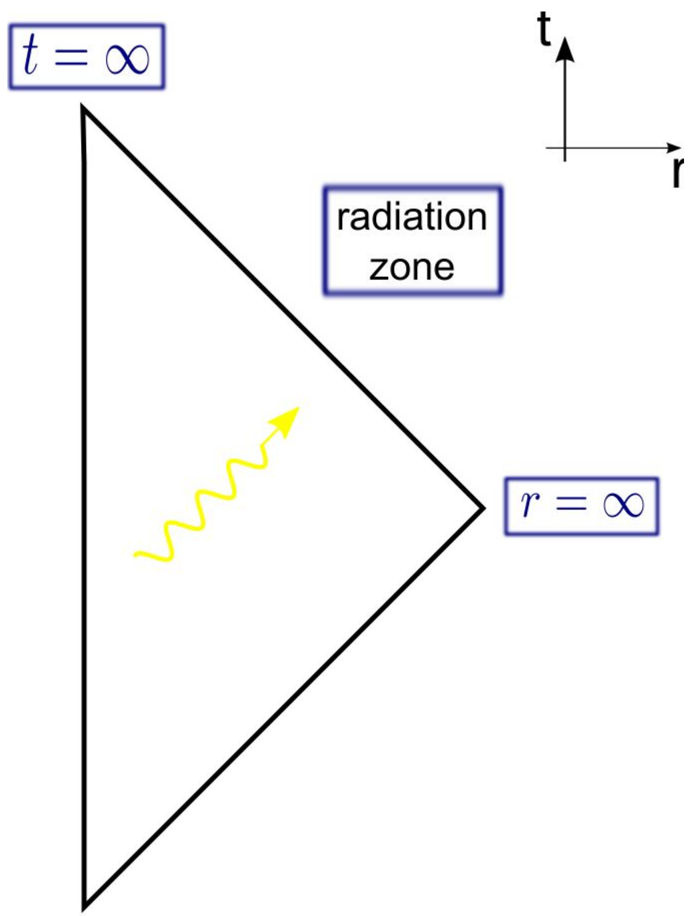
Key idea: bring infinity to a finite distance



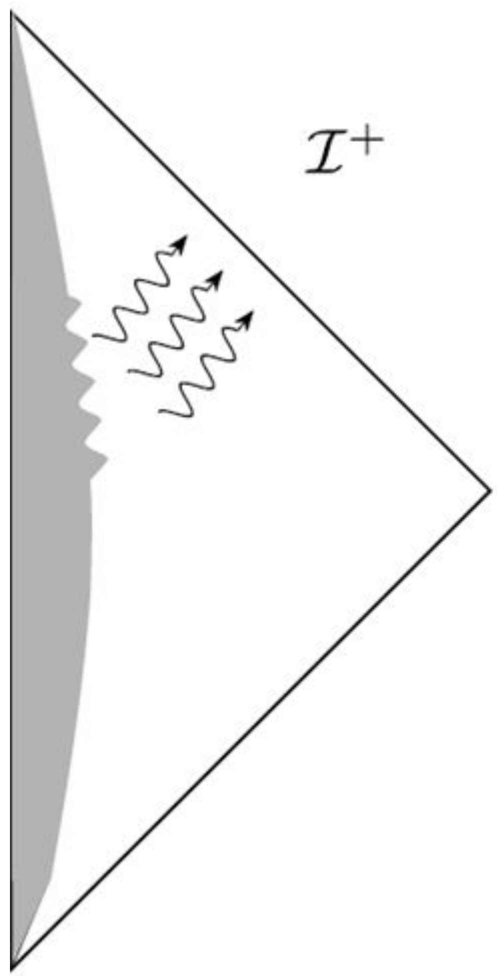
$(\hat{M}, \hat{g}_{ab})$ $(M, g_{ab} = \Omega^2 \hat{g}_{ab})$

Conformal completion

Conformal diagram Minkowski



Asymptotic flatness

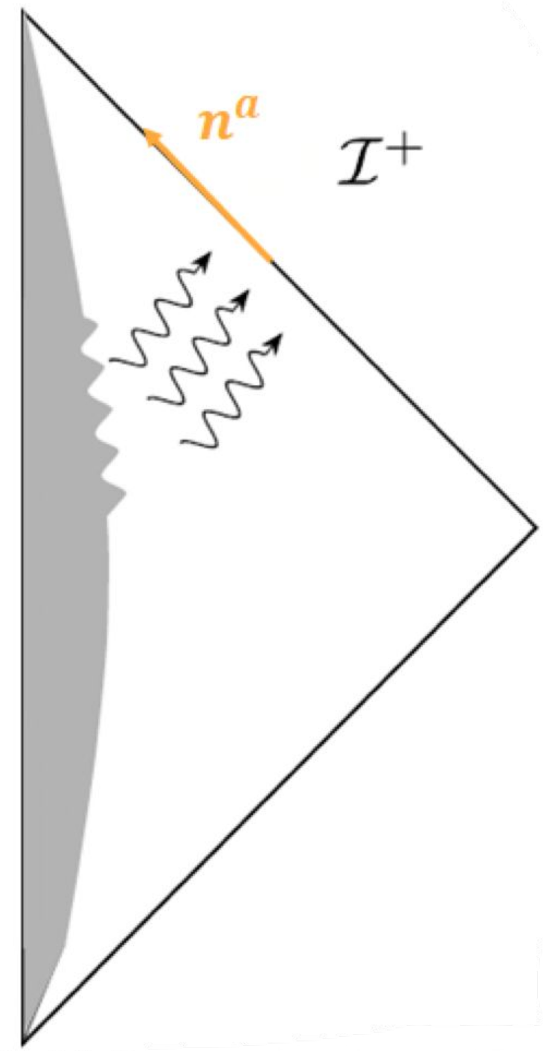


A physical spacetime $(\hat{M}, \hat{g}_{ab})$ is asymptotically flat if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{J} \cong \mathbb{R}\, x\, \mathbb{S}^2$ such that

1. Ω and $g_{ab} = \Omega^2\ \hat{g}_{ab}$ are smooth on M, $\Omega \mathrel{\hat{=}} 0$ and $n_a = \nabla_a \Omega$ is nowhere vanishing on $\mathcal{J}$
2. Einstein's equations are satisfied with $\hat{T}_{ab}$ such that $\Omega^{-2}\hat{T}_{ab}$ has a smooth limit to $\mathcal{J}$

Consequences

- Einstein's equation ⟹ n^a is null on $\mathcal{J}$
- q_{ab} = induced metric on $\mathcal{J}$ is degenerate: 0 + +



Universal structure

This is common to ***all*** asymptotically flat spacetimes

$$\{ q_{ab}, n^a \} = \{ \omega^2 q_{ab}, \omega^{-1} n^a \}$$

Gravitational radiation is encoded in the next-order structure and differs from spacetime to spacetime

Key points of asymptotics

Nowhere in this construction did we introduce a split of the background and "gravitational waves".

The split occurs naturally at null infinity:

- universal structure is like a background,
- first order structure contains gravitational radiation,

and it is fully non-linear!

Asymptotic symmetry algebra

Spacetime diffeomorphism that leave the universal structure at null infinity invariant

Universal structure

$$\mathcal{L}_\xi q_{ab} \widehat{=} 2\,\alpha\, q_{ab} \text{ with } \mathcal{L}_n \alpha \widehat{=} 0$$

$$\mathcal{L}_\xi n^a \widehat{=} -\,\alpha\, n^a$$

Coordinates

$$\Omega^2 \mathcal{L}_\xi \hat{g}_{ab} \widehat{=}\; 0$$

Bondi-Metzner-Sachs algebra (BMS)

- Asymptotic symmetry algebra is ***bigger*** than Poincaré
- BMS = supertranslations & rotations

$$\xi^a \partial_a = \left(f(\theta, \varphi) + \frac{1}{2} u D_A Y^A \right) \partial_u + Y^A \partial_A$$

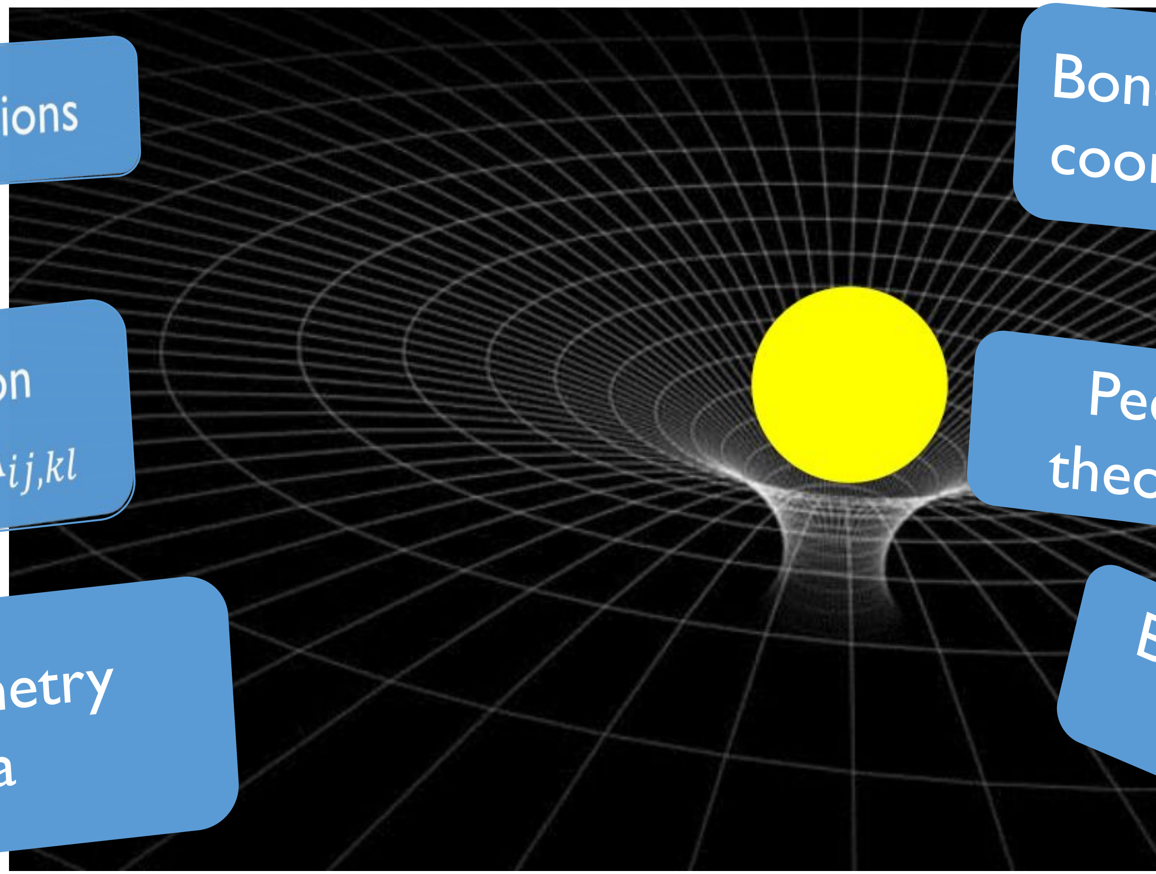
supertranslations

rotations

$$2D_{(A}Y_{B)} + q_{AB} D_C Y^C = 0$$

- Conformal Carroll algebra with N=2 (so that $g_{ab} n^c n^d$ is preserved)

Critical assumption



$\frac{1}{r}$ expansions

Projection operator $\Lambda_{ij,kl}$

BMS symmetry algebra

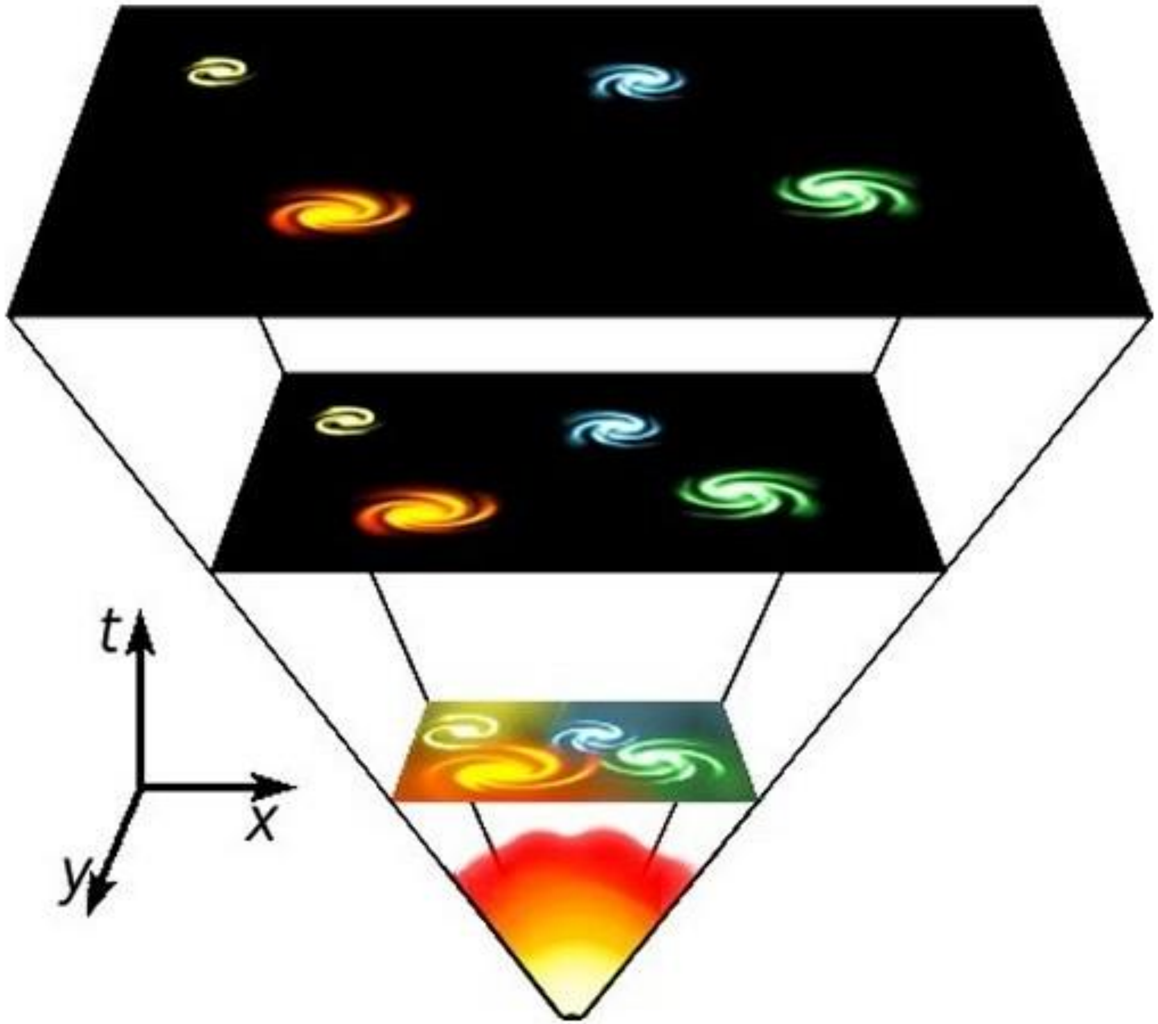
Bondi-Sachs coordinates

Peeling theorems

Bondi news tensor

Move far away from sources:
'spacetime becomes flat'

Expanding spacetimes are not asymptotically flat!



Why assume asymptotic flatness?

P. G. BERGMANN:

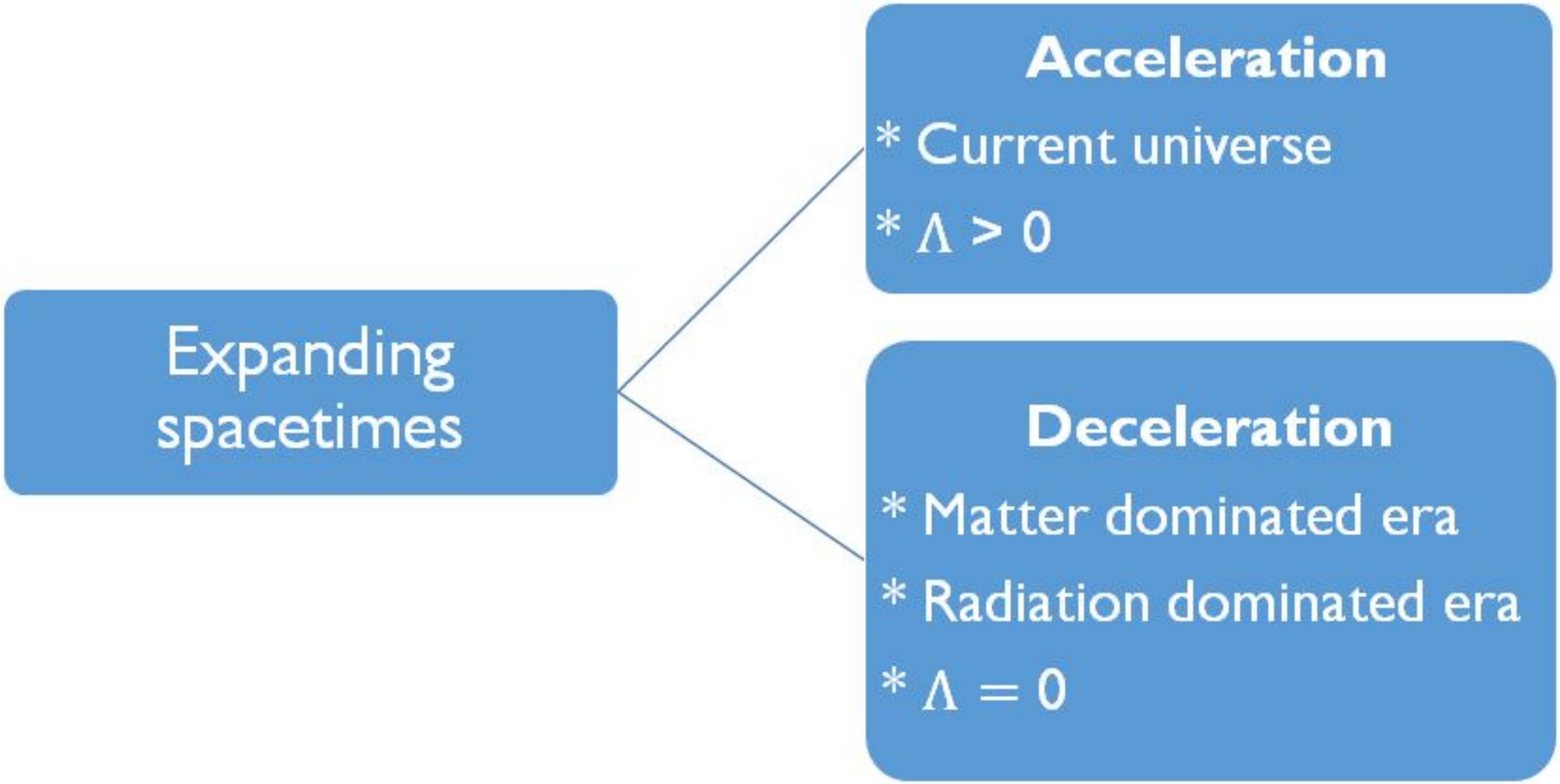
The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.

H. BONDI:

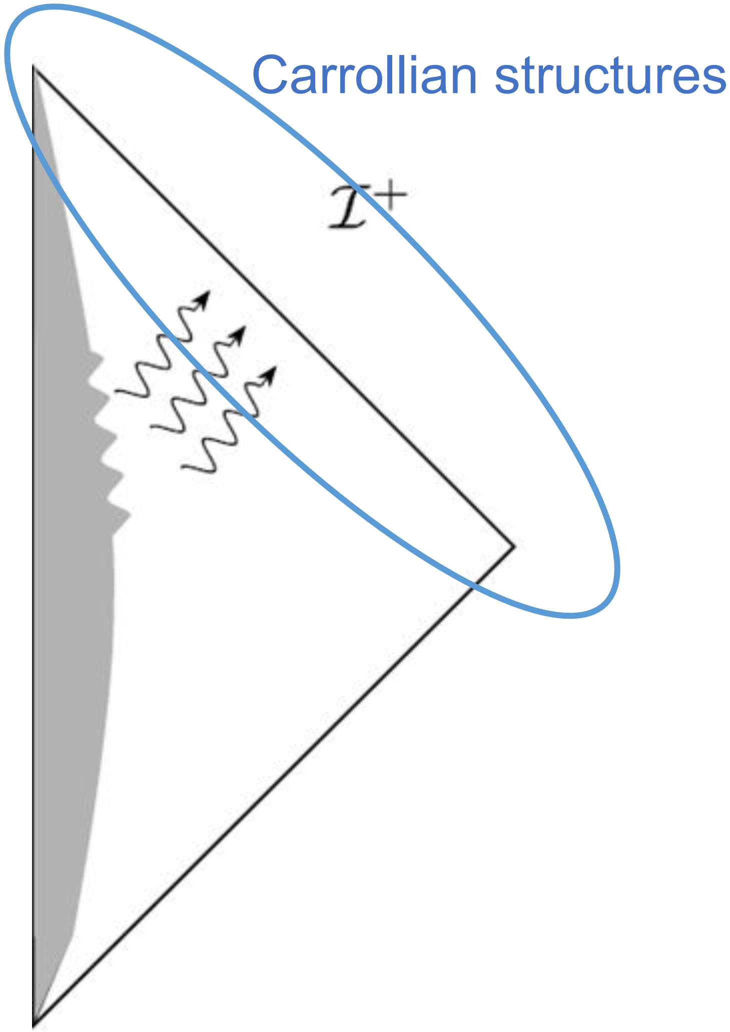
I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.

Conference Warsaw 1962

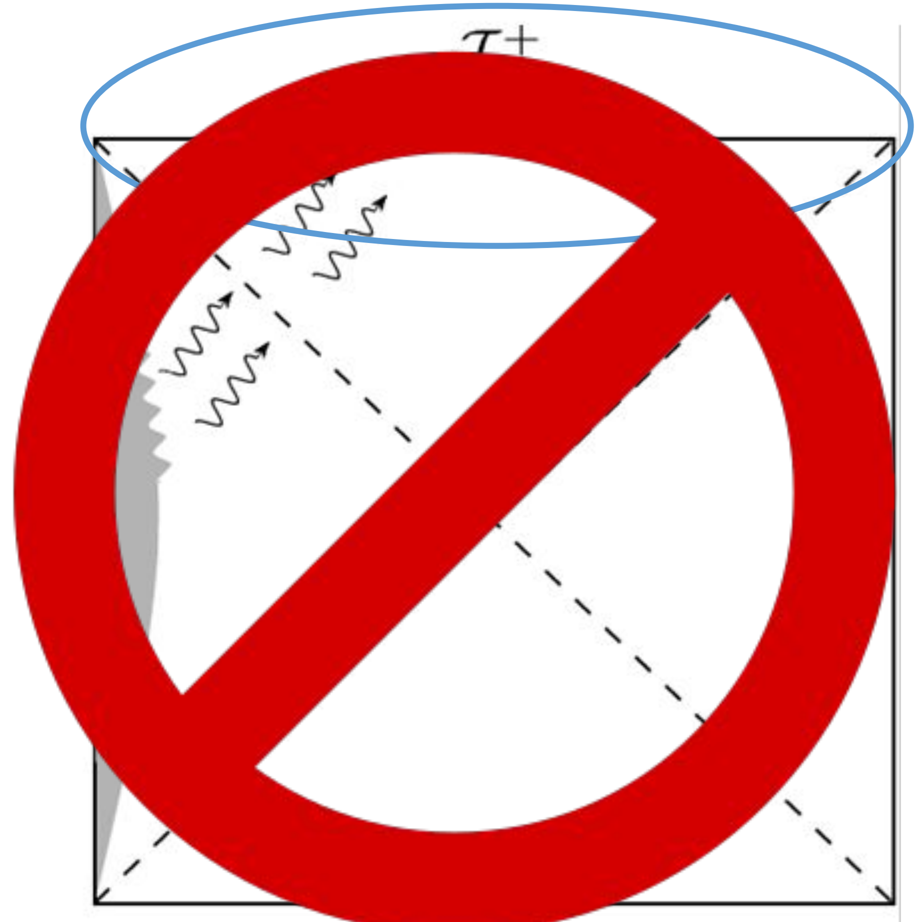
Expansion rates



Radiation zones



$\Lambda = 0$



$\Lambda > 0$

Goal

Study a general class of expanding spacetimes…

… but first look at the canonical example:

spatially flat decelerating Friedman-Lemaitre-Robertson-Walker spacetimes

Decelerating FLRW spacetimes

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 \delta_{AB} dx^A dx^B\right]$$

physical metric

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{\frac{s}{1-s}}$$

$$s = \frac{2}{3(1+w)}$$

$$0 \leq s < 1$$

$$-1/3 < w < \infty$$

$$P = w\rho$$

$w = 1$	stiff fluid
$w = 1/3$	radiation
$w = 0$	dust
$w = -1$	cosmological constant

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 \delta_{AB}\, dx^A dx^B\right]$$

$$\eta = \frac{\sin T}{\cos R + \cos T} = \frac{\sin\left(\frac{V+U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

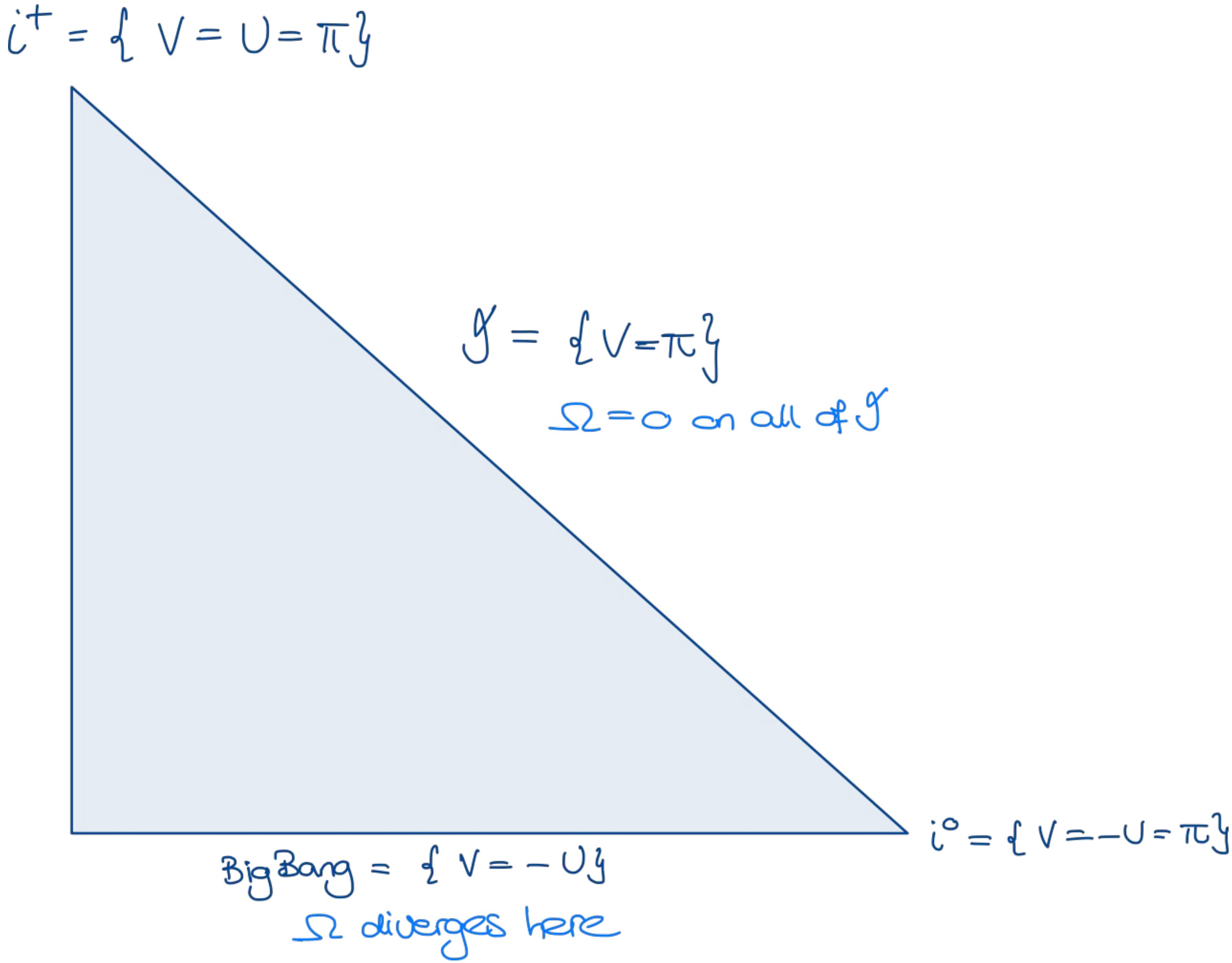
$$r = \frac{\sin R}{\cos R + \cos T} = \frac{\sin\left(\frac{V-U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

$$\begin{cases} U = T - R \\ V = T + R \end{cases} \Longleftrightarrow \begin{array}{l} -\pi < U < \pi \\ |U| < V < \pi \end{array}$$

Choose $\Omega = 2\left(\cos\frac{U}{2}\cos\frac{V}{2}\right)^{\frac{1}{1-s}}\left(\sin\frac{U+V}{2}\right)^{-\frac{s}{1-s}}$

$$ds^2 = \Omega^2 d\hat{s}^2 = -dU\,dV + \sin\left(\frac{V-U}{2}\right)^2 \delta_{AB}\, dx^A dx^B$$

$\longrightarrow$ can add $V = -U$ & $V = \pi$, because this metric is smooth everywhere including at the boundaries



$i^+ = \{V = U = \pi\}$
$\mathscr{I} = \{V = \pi\}$
$\Omega = 0$ on all of $\mathscr{I}$
$i^0 = \{V = -U = \pi\}$
Big Bang = $\{V = -U\}$
Ω diverges here

The conformal factor

But near $\mathscr{I}$...

$$\Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{1}{1-\delta}} \;\longrightarrow\; \text{NOT smooth!}$$

$$\nabla_a \Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{\delta}{1-\delta}}\,\nabla_a V \;\longrightarrow\; \hat{=}\, 0 \text{ unless } \delta = 0$$

Bad choice for Ω?

What to do?

$$\Omega' = \omega\Omega \quad \text{with } \omega \sim (\pi - V)^{-\frac{\delta}{1-\delta}} \;\Longrightarrow\;$$

- * Ω is smooth @ $\mathscr{I}$ ☑
- * $\nabla_a \Omega \,\hat{\neq}\, 0$ ☑

but then $g'_{ab} = \Omega'^2 \hat{g}_{ab}$

$$\sim (\pi - V)^{-\frac{2\delta}{1-\delta}}\,\hat{g}_{ab}$$

THIS DIVERGES @ $\mathscr{I}$!

Simple resolution

Ω^{1-s} is smooth @ γ

* $\Omega^{1-s} \hat{=} 0$
* $\nabla_a \Omega^{1-s} \hat{\neq} 0$

Define the normal to γ using Ω^{1-s}

$$\Rightarrow n_a = \frac{1}{1-s} \nabla_a \Omega^{1-s}$$

$$= \Omega^{-s} \nabla_a \Omega$$

$$\hat{=} -\frac{2^{-s}}{1-s} \left(\cos\frac{u}{2}\right)^{1-s} \nabla_a V$$

Presence of matter

For asymptotically flat spacetimes, $\Omega^{-2}\hat{T}_{ab}$ should have a limit to $\mathcal{I}$ but FLRW spacetimes are homogeneous, so there is matter *everywhere*!

$$\lim_{\to \mathcal{I}} 8\pi G\, g^{ab}\hat{T}_{ab} = \frac{6s(1-s)}{(1-s)^2}\left(\sec\frac{U}{2}\right)^2 \to \text{NON-VANISHING}$$

$$8\pi G\, \hat{T}_{ab} = \underbrace{2s\,\Omega^{2(s-1)} n_a n_b}_{\text{universal}} + 2s\,\Omega^{s-1}\, \tau_{(a} n_{b)} + \text{finite}$$

depends on choice Ω

$\tau_a \hat{=} \tan\frac{U}{2}\left(\nabla_a U + \nabla_a V\right)$

Spacetimes with a cosmological null asymptote

A physical spacetime $(\hat{M}, \hat{g}_{ab})$ admits a cosmological null asymptote if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{I} \cong \mathbb{R} \, x \, \mathbb{S}^2$ such that

(1) $\Omega \,\hat{=}\, 0$, Ω^{1-s} and $g_{ab} = \Omega^2 \, \hat{g}_{ab}$ is smooth on M,

$n_a = \Omega^{-s} \nabla_a \Omega$ is nowhere vanishing on $\mathcal{I}$ (for $0 \leq s < 1$)

(2) Einstein's equations are satisfied with $\hat{T}_{ab}$ such that

$\lim_{\to \mathscr{I}} g^{ab} \hat{T}_{ab}$ exists

$$\lim_{\to \mathscr{I}} \Omega^{1-s} \left[8\pi \hat{T}_{ab} - 2s\Omega^{2(s-1)} n_a n_b \right] \hat{=}\, 2s\tau_{(a} n_{b)}$$

Spacetimes with a cosmological null asymptote

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(2) Einstein's equations are satisfied with $\hat{T}_{ab}$ such that

$$\lim_{\to \mathscr{I}} g^{ab} \hat{T}_{ab} \quad \text{exists}$$

$$\lim_{\to \mathscr{I}} \Omega^{1-s} \left[8\pi \hat{T}_{ab} - 2s\Omega^{2(s-1)} n_a n_b \right] \mathrel{\hat{=}} 2s\tau_{(a} n_{b)}$$

Any other examples?

Class of spacetimes at least as big as asymptotically flat spacetimes

Linearization stability still open question!

Asymptotic symmetry algebra

All smooth vector fields that map

$$\{q_{ab}, n^a\} \longrightarrow \{q'_{ab} = \omega^2 q_{ab}, \; n'^a = \omega^{-1-s} n^a\}$$

$$\Longrightarrow \mathfrak{b}_s \cong \mathfrak{so}(1,3) \ltimes \mathfrak{S}_s$$

In terms of coordinates on $\mathcal{I}$

$$\xi^a \partial_a = \left(f(\theta, \varphi) + \frac{1+s}{2}\, u\, D_A Y^A \right) \partial_u + Y^A \partial_A$$

supertranslations with conformal weight $1 + s$

rotations

$$2D_{(A}Y_{B)} + q_{AB} D_C Y^C = 0$$

The critical s-dependence

A supertranslation can again be written as

$$\xi^a \partial_a = \underbrace{f(\theta,\varphi)}_{\text{has conformal weight } 1+s} \partial_u$$

For fixed $\xi^a = f n^a \longmapsto \xi^a = f' n'^a$

$$= f' \omega^{-1-s} n^a$$

$$\Longrightarrow \boxed{f' = \omega^{1+s} f}$$

Consequences s-dependence?

- Conformal Carroll algebra with N=2/(1+s)
  - Not clear how to interpret non-integer N

$$g \otimes n^{\otimes N}$$

- No translation subalgebra

Conclusion

Asymptotic symmetry algebra ~ Conformal Carroll algebra

- ➢ Asymptotic flat spacetimes
  - ■ BMS ⟷ Conformal Carroll algebra with N=2
- ➢ Asymptotic cosmological null asymptotes
  - ■ BMS-s ⟷ Conformal Carroll algebra with N=2/(1+s)
  - ■ There is no translation subalgebra


THANKS FOR LISTENING
TO MY PRESENTATION
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Geometric definition to coordinates

$$\tilde{r} = \Omega^{-1-s}$$

$$d\hat{s}^2 = \left(\frac{\tilde{r}^s}{1-s}\right)^{\frac{2}{1-s}} \left[-\frac{\tilde{V}}{\tilde{r}} e^{2\beta} du^2 - 2e^{2\beta} du d\tilde{r} + \tilde{r}^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du) \right]$$

$$\begin{aligned}
\frac{\tilde{V}}{\tilde{r}} &= W^{(2)} + \frac{W^{(3)}}{\tilde{r}} + \mathcal{O}(1/\tilde{r}^2) \\
e^{2\beta} &= 1 - \frac{s\tau}{\tilde{r}} + \frac{1}{\tilde{r}^2}\left(2\beta^{(2)} + \tfrac{1}{2}s^2\tau^2\right) + \mathcal{O}(1/\tilde{r}^3) \\
U^A &= \frac{1}{\tilde{r}} s\tau^A + \frac{1}{\tilde{r}^2} U^{(2)A} + \frac{U^{(3)A}}{\tilde{r}^3} + \mathcal{O}(1/\tilde{r}^4) \\
h_{AB} &= q_{AB} + \frac{1}{\tilde{r}} C_{AB} + \frac{1}{\tilde{r}^2} d_{AB} + \mathcal{O}(1/\tilde{r}^3) \ .
\end{aligned}$$

$\Longrightarrow$ physical area u & $\tilde{r}$ constant spheres

$$4\pi \left(\frac{\tilde{r}}{1-s}\right)^{\frac{2}{1-s}}$$

Another choice

$r = \Omega^{-1}$

$$d\hat{s}^2 = -\frac{V}{r}e^{2\beta}du^2 - 2r^s e^{2\beta}dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

$$\frac{V}{r} = \frac{r^{2s}}{(1-s)^2}\left[W^{(2)} - \frac{1}{(1-s)}\frac{W^{(3)}}{r^{1-s}} + \mathcal{O}(1/r^{2(1-s)})\right]$$

$$r^s e^{2\beta} = r^s\left[1 - \frac{1}{(1-s)}\frac{s\tau}{r^{1-s}} + \frac{1}{(1-s)^2}\frac{1}{r^{2(1-s)}}\left(2\beta^{(2)} + \tfrac{1}{2}s^2\tau^2\right) + \mathcal{O}(1/r^{3(1-s)})\right]$$

$$U^A = \frac{1}{(1-s)}\frac{1}{r^{1-s}}s\tau^A + \frac{1}{(1-s)^2}\frac{1}{r^{2(1-s)}}U^{(2)A} + \frac{1}{(1-s)^3}\frac{U^{(3)A}}{r^{3(1-s)}} + \mathcal{O}(1/r^{4(1-s)})$$

$$h_{AB} = q_{AB} + \frac{1}{(1-s)}\frac{1}{r^{1-s}}C_{AB} + \frac{1}{(1-s)^2}\frac{1}{r^{2(1-s)}}d_{AB} + \mathcal{O}(1/r^{3(1-s)})\ .$$

$\Longrightarrow$ physical area u & r constant spheres $4\pi r^2$

but fractional powers in r ...

Asymptotic symmetry algebra

All smooth vector fields that map

$$\{q_{ab}, n^a\} \longrightarrow \{q'_{ab} = \omega^2 q_{ab},\ n'^a = \omega^{-1-s} n^a\}$$

$$\begin{cases} \mathcal{L}_\xi q_{ab} \hat{=} 2\,\alpha_{(\xi)}\, q_{ab} \\ \mathcal{L}_\xi n^a \hat{=} -(1+s)\,\alpha_{(\xi)}\, n^a \end{cases}$$

for any $\alpha_{(\xi)}$ such that $\mathcal{L}_n \alpha_{(\xi)} \hat{=} 0$

What is the structure of this Lie algebra?

$$\star \begin{cases} \mathcal{L}_\xi q_{ab} \hat{=} 2\,\alpha_{(\xi)}\, q_{ab} \\ \mathcal{L}_\xi n^a \hat{=} -(1+s)\,\alpha_{(\xi)}\, n^a \end{cases} \quad \text{for any } \alpha_{(\xi)} \text{ such that } \mathcal{L}_n \alpha_{(\xi)} \hat{=} 0$$

* Take $\xi^a \hat{=} f n^a \overset{\star}{\Longrightarrow} \mathcal{L}_n f \hat{=} 0$, so "$f = f(\theta,\varphi)$"

The Lie bracket with an arbitrary element $\in \mathfrak{b}_s$

$$[\xi, fn]^a = \underbrace{\left(\mathcal{L}_\xi f - (1+s)\alpha f\right)}_{\text{again of the same form}} n^a$$

$\Longrightarrow$ $f n^a$ form a Lie ideal
$=$ super translations $\mathcal{S}_s$

* The quotient $\mathfrak{b}_s / \mathcal{S}_s \simeq$ vector fields X^a on S^2

$$\overset{\star}{\Longrightarrow} \underbrace{\mathcal{L}_X \underleftarrow{q_{ab}} \hat{=} 2\,\alpha_{(X)} \underleftarrow{q_{ab}}}$$

$=$ CKVF on S^2 $\simeq SO(1,3)$

$$\Longrightarrow \mathfrak{b}_s \cong SO(1,3) \ltimes \mathcal{S}_s$$