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Coadjoint representation of BMS4 on celestial Riemann surfaces

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Context

- asymptotically flat gravity at null infinity $\mathcal{J}^+$
- symmetry group : BMS_4
- why study coadjoint representation of BMS_4 ?
- momentum map $Q_\xi : M \times bms_4 \to \mathbb{R}$, $Q_a[z] \in bms_4^*$

$\delta_a = \{ Q_a , \cdot \}$, $\{ Q_a , Q_b \} = f_{ab}^{\ \ c} Q_c + C_{a,b}$

- relevant for celestial holography , amplitudes , (semi-classical) QG

Classify coadjoint orbits (of BMS_4)

symplectic manifolds can be quantized $\longrightarrow$ relation to UIRREPS (Kirillov)

Write geometric actions   Alekseev, Faddeev, Shatashvili J. Geom. Phys. 1988, Nucl. Phys. 1989

$$S = \int \mathrm{Tr}\,(g^{-1}\partial_\mu g\; g^{-1}\partial^\mu g)\, d^4x \longrightarrow S = \int \left(\langle \nu_0^*, g^{-1}\frac{dg}{dt}\rangle - H\right) dt$$

Effective actions for Goldstone bosons   no Killing metric needed.   correct global symmetries

PI quantization $\longrightarrow$ characters

for 3d gravity $\Leftrightarrow$ to actions constructed from

GB, Gonzalez, Salgado CQG 2018   CS $\to$ WZW   Elitzur et al. Nucl. Phys. 1989

Coussaert, Henneaux, Van Driel CQG 1995

related to Schwarzian actions

Contents

- Coadjoint representation for semi-direct product groups
- Construction for BMS_4 on sphere & punctured plane
- Identification in asymptotically flat spacetimes at $\mathscr{I}^+$
- Perspectives

in collaboration with R. Ruzziconi JHEP 2021

(C. Troessaert, B. Oblak, P. Mao)

Coadjoint representation & semi-direct product groups

adjoint $\mathfrak{g}: [e_a, e_b] = f^c_{ab}\, e_c \qquad (\mathrm{ad}\, e_a)^b{}_c = f^b_{ac} \iff \mathrm{ad}\, e_a(e_b) = f^c_{ab}\, e_c$

coadjoint $\mathfrak{g}^*: \quad \langle e^b_*, e_a \rangle = \delta^b_a \qquad (\mathrm{ad}^* e_a) = -(\mathrm{ad}\, e_a)^T \iff \mathrm{ad}^* e_a(e^b_*) = -f^b_{ac}\, e^c_*$

group $\quad \mathrm{Ad}_g\, e_a = g\, e_a\, g^{-1}, \qquad \mathrm{Ad}^*_g = g\, e^b_*\, g^{-1}$

semi-direct product $\quad G \ltimes_\sigma A \; : \; (f, \alpha) \cdot (g, \beta) = (f \cdot g,\ \alpha + \sigma_f(\beta))$ $\qquad \sigma$: representation

$ISO(3)$, $ISO(3,1)$, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ A: abelian ideal

BMS_3, BMS_4 ... $\qquad \mathfrak{g} \oplus_\Sigma A \; : \; [(X,\alpha), (Y,\beta)] = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$

$$\mathrm{Ad}_{(f,\alpha)}(X,\beta) = (\mathrm{Ad}_f X,\ \sigma_f \beta - \Sigma_{\mathrm{Ad}_f X}\, \alpha)$$

$$\mathrm{ad}_{(X,\alpha)}(Y,\beta) = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$$

dual space $\quad \mathfrak{g}^* \oplus A^* \qquad \langle (j,p), (X,\alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle$

terminology $\quad j$: angular momentum $\qquad p$: linear momentum $\qquad$ BMS: add "super"

X : inf. rotation $\qquad \alpha$: inf. translation

ingredients $\quad x : A \oplus A^* \longrightarrow \mathfrak{g}^* \quad : \quad \langle \alpha \times p, X \rangle = \langle p, \Sigma_X \alpha \rangle$

change in angular momentum due to a translation

$$\sigma^* : G \times A^* \longrightarrow A^* \quad : \quad \langle \sigma^*_f p, \alpha \rangle = \langle p, \sigma_{f^{-1}} \alpha \rangle$$

coadjoint representation $\quad \mathrm{Ad}^*_{(f,\alpha)} (j,p) = \left(\mathrm{Ad}^*_f j + \alpha \times \sigma^*_f p, \ \sigma^*_f p \right)$

$$\mathrm{ad}^*_{(X,\alpha)} (j,p) = \left(\mathrm{ad}^*_X j + \alpha \times p, \ \Sigma^*_X p \right)$$

Poincaré & BMS_4 algebra at $\mathcal{J}^+$

Poincaré generators $\mathbb{R}^{3,1}$ $\quad L^{ab} = -\left(x^a \frac{\partial}{\partial x_b} - x^b \frac{\partial}{\partial x_a}\right), \quad P^a = \frac{\partial}{\partial x_a} \qquad \eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$

structure constants
$$\begin{cases} [L^{ab}, L^{cd}] = -(\eta^{bc} L^{ad} - \eta^{ac} L^{bd} - \eta^{bd} L^{ac} + \eta^{ad} L^{bc}) \\ [P^a, L^{bc}] = -(\eta^{ab} P^c - \eta^{ac} P^b) \end{cases}$$

Boost & rotation generators
$$\begin{cases} L_z = L^{12}, \quad L^{\pm} = \pm i (L^{23} + L^{13}) \;; \quad K_z = L^{30}, \quad K^{\pm} = \mp i L^{20} - L^{10} \\ H = P^0, \quad P_z = -\frac{1}{2} P^3, \quad P^{\pm} = -\frac{1}{2}(i P^2 \pm P^1) \end{cases}$$

structure constants $\quad [L^+, L^-] = 2i L_z, \quad [L_z, L^{\pm}] = \pm i L^{\pm}, \;\dots$

spherical & retarded time $r=\sqrt{\sum_i (x^i)^2}$, $u=x^0-r$, $r\cos\theta = x^3$, $r\sin\theta e^{i\phi}=x^1+ix^2$

$$\boxed{L_z=\partial_\phi,\quad L^\pm = e^{\pm i\phi}\left[\partial_\theta \pm i\cot\theta\,\partial_\phi\right]},\quad K_z = -\cancel{\left(1+\frac{u}{r}\right)\cos\theta\, r\partial_r} + \cos\theta(u\partial_u) + \left(1+\cancel{\frac{u}{r}}\right)\sin\theta\,\partial_\theta$$

$$K^\pm = e^{\pm i\phi}\left[\cancel{\left(1+\frac{u}{r}\right)\sin\theta\, r\partial_r} - \sin\theta(u\partial_u) + \left(1+\cancel{\frac{u}{r}}\right)\cos\theta\,\partial_\theta \pm \left(1+\cancel{\frac{u}{r}}\right)\frac{i}{\sin\theta}\partial_\phi\right]$$

$$H=\partial_u,\quad -2P_z = \cos\theta(-\cancel{\partial_r}+\partial_u) + \cancel{\frac{1}{r}\sin\theta\,\partial_\theta},\quad \pm 2P^\pm = e^{\pm i\phi}\left[\sin\theta(\cancel{\partial_r}-\partial_u) + \cancel{\frac{1}{r}\cos\theta\,\partial_\theta} \pm \cancel{\frac{1}{r\sin\theta}\partial_\phi}\right]$$

Simplification 1: $\mathscr{I}^+$ $r=\text{cte}\to\infty$ Simplification 2: cut $u=0$ of $\mathscr{I}^+$

$$\boxed{\begin{array}{lll} K_z = \sin\theta\,\partial_\theta, & K^\pm = e^{\pm i\phi}\left[\cos\theta\,\partial_\theta \pm \frac{i}{\sin\theta}\partial_\phi\right] & \\ H=1 \sim {}_0Y_{00} & P_z = -\frac{1}{2}\cos\theta \sim {}_0Y_{10} & P^\pm = \mp\frac{1}{2}e^{\pm i\phi}\sin\theta \sim {}_0Y_{1\pm 1} \end{array}}$$

4 lowest harmonics

Poincaré algebra

$f = H, P_z, P^\pm$

$$[L_z, f] = L_z(f),\quad [L^\pm, f] = L^\pm(f)$$

$$[K_z, f] = K_z(f) - \cos\theta f,\quad [K^\pm, f] = K^\pm(f) + e^{\pm i\phi}\sin\theta f$$

BMS_4 algebra:

$f\in C^\infty(S^2)$

Sachs Phys Rev 1962

Simplification 3: stereographic coordinates on the sphere

$$\boxed{\zeta = \cot\frac{\theta}{2}\, e^{-i\phi} \qquad ds^2 = -2(P_S\bar{P}_S)^{-2}\, d\zeta\, d\bar{\zeta} \qquad P_S = \frac{1}{R\sqrt{2}}(1+\zeta\bar{\zeta})}$$

Lorentz algebra $\qquad l_m = \zeta^{1-m}\partial, \quad \bar{l}_m = \bar{\zeta}^{1-m}\bar{\partial} \qquad sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$

$$L_z = -i(l_0 - \bar{l}_0), \quad K_z = -(l_0 + \bar{l}_0), \quad L^+ = l_1 + \bar{l}_{-1}, \quad L^- = \bar{l}_1 + l_{-1}, \quad K^+ = -(\bar{l}_{-1} - l_1), \quad K^- = -(l_{-1} - \bar{l}_1)$$

action of Lorentz on (super)-translations

$$H = 1, \quad P_z = \frac{1-\zeta\bar{\zeta}}{2(1+\zeta\bar{\zeta})}, \quad P^+ = -\frac{\bar{\zeta}}{1+\zeta\bar{\zeta}}, \quad P^- = \frac{\zeta}{1+\zeta\bar{\zeta}}$$

$$[K_z, f] = K_z(f) + \frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}} f \qquad [K^+, f] = K^+(f) + \frac{2\bar{\zeta}}{1+\zeta\bar{\zeta}} f, \quad [K^-, f] = K^-(f) + \frac{2\zeta}{1+\zeta\bar{\zeta}} f$$

Coadjoint representation of BMS_4: general structure

2d conformally flat S aim: unified description for sphere & punctured plane

$$ds^2 = -2(P\bar{P})^{-1} d\xi\, d\bar{\xi}$$

$$\begin{cases} \xi' = \xi'(\xi) \quad \bar{\xi}' = \bar{\xi}'(\bar{\xi}) & \text{conformal coordinate trsf.} \\ P'(x) = P(x)\, e^{-E(x)}, \quad \bar{P}'(x) = \bar{P}(x)\, e^{-\bar{E}(x)} & \text{complex Weyl rescaling } \begin{cases} E_R & \text{standard Weyl} \\ iE_I & \text{local rotation} \end{cases} \end{cases}$$

$x = (\xi, \bar{\xi})$

zweibeins $\quad ds^2 = e^a{}_\mu dx^\mu \eta_{ab} e^b{}_\nu dx^\nu \qquad \eta_{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad e_1{}^\mu \frac{\partial}{\partial x^\mu} = P \frac{\partial}{\partial \xi} \qquad e_2{}^\mu \frac{\partial}{\partial x^\mu} = \bar{P} \frac{\partial}{\partial \bar{\xi}}$

conformal fields $\quad \phi'_{h,\bar{h}}(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)^h \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{\bar{h}} \phi_{h,\bar{h}}(x) \qquad P'(x') = P(x)\, e^{-E(x')} \frac{\partial \xi'}{\partial \xi} \quad$ combined trsf.

weighted scalars $\quad \eta'^{s,w}(x') = e^{w E_R(x')} e^{-is E_I(x')} \eta^{s,w}(x)$

Held, Posadas, Newman JMP 1970
Lorentz group & sphere

interpolation map $\quad \eta^{s,w} = P^h \bar{P}^{\bar{h}} \phi_{h,\bar{h}}$

D'Hoker, Phong Rev. Mod. Phys. 1988

$s = h - \bar{h}, \quad w = -(h + \bar{h}) \qquad h = \frac{s-w}{2}, \quad \bar{h} = -\frac{s+w}{2}$

covariant derivative $\quad \nabla: \quad \Gamma^{\xi}_{\xi\xi} = -\partial \ln(P\bar{P}) \qquad \Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}} = -\bar{\partial} \ln(P\bar{P})$

$$\Gamma^{\xi'}_{\xi'\xi'}(x') = \Gamma^{\xi}_{\xi\xi}(x) \frac{\partial \xi}{\partial \xi'} + \frac{\partial \xi'}{\partial \xi} \frac{\partial^2 \xi}{\partial \xi' \partial \xi'} + 2\partial' E_R(x')$$

introduce Weyl connection $\quad \mathcal{D}: \quad W'(x') = \frac{\partial \xi}{\partial \xi'} W + 2\partial' E_R(x'), \quad \bar{W}'(x') = \frac{\partial \bar{\xi}}{\partial \bar{\xi}'} \bar{W}(x) + 2\bar{\partial}' E_R(x')$

$$\underbrace{\mathcal{D}\phi_{h,\bar{h}}}_{(h+1,\bar{h})} = [\nabla + hW]\phi_{h,\bar{h}}, \qquad \underbrace{\bar{\mathcal{D}}\phi_{h,\bar{h}}}_{(h,\bar{h}+1)} = [\bar{\nabla} + \bar{h}\bar{W}]\phi_{h,\bar{h}}$$

$\partial, \nabla, \mathcal{D} = \partial_\xi, D_\xi, D_\xi$

$\bar{\partial}, \bar{\nabla}, \bar{\mathcal{D}} = \partial_{\bar{\xi}}, D_{\bar{\xi}}, D_{\bar{\xi}}$

weighted scalars
$$\eth \eta^{s,w} = P^{h+1}\bar{P}^{\bar{h}} \nabla \phi_{h,\bar{h}} = P\bar{P}^{-s}\partial(\bar{P}^{s}\eta^{s,w}), \qquad \bar{\eth}\eta^{s,w} = P^{h}\bar{P}^{\bar{h}+1}\bar{\nabla}\phi_{h,\bar{h}} = \bar{P}P^{s}\bar{\partial}(P^{-s}\eta^{s,w})$$

Weyl covariant
$$\underbrace{\mathcal{D}\eta^{s,w}}_{[s+1,\, w-1]} = P^{h+1}\bar{P}^{\bar{h}} \mathcal{D}\phi_{h,\bar{h}} = \left(\eth + \left(\frac{s-w}{2}\right)PW\right)\mathcal{D}\phi_{h,\bar{h}}, \qquad \underbrace{\bar{\mathcal{D}}\eta^{s,w}}_{[s-1,\, w-1]} = P^{h}\bar{P}^{\bar{h}+1}\bar{\mathcal{D}}\phi_{h,\bar{h}} = \left[\bar{\eth} - \left(\frac{w+s}{2}\right)\bar{P}\bar{W}\right]\phi_{h,\bar{h}}$$

$$[\mathcal{D}, \bar{\mathcal{D}}]\eta^{s,w} = -\frac{s}{2} R_s \eta^{s,w} - P\bar{P}\left(\frac{s-w}{2}\bar{\partial}W + \frac{s+w}{2}\partial\bar{W}\right)\eta^{s,w}$$

R_s : scalar curvature

Ingredients

(super-)translation $\quad \mathcal{T} : [0,1] \qquad \tilde{\mathcal{T}} : (-\tfrac{1}{2}, -\tfrac{1}{2}) \qquad$ real

(super-) rotation $\quad \mathcal{Y} : [-1,1] \qquad \tilde{\mathcal{Y}} : (-1,0) \qquad \bar{\mathcal{D}}\mathcal{Y} = 0 \Leftrightarrow \bar{D}\tilde{\mathcal{Y}} = 0$

$\bar{\mathcal{Y}} : [1,1] \qquad \tilde{\bar{\mathcal{Y}}} : (0,-1) \qquad \mathcal{D}\bar{\mathcal{Y}} = 0 \Leftrightarrow D\tilde{\bar{\mathcal{Y}}} = 0$

(super-) momentum $\quad \mathcal{P} : [0,-3] \qquad \tilde{\mathcal{P}} : (\tfrac{3}{2}, \tfrac{3}{2}) \qquad$ real

(super-) angular momentum $\quad \mathcal{J} : [-1,-3] \qquad \tilde{\mathcal{J}} : (1,2) \qquad \mathcal{J} \sim \mathcal{J} + \mathcal{D}\underset{[-2,-2]}{\mathcal{L}} \;, \quad \tilde{\mathcal{J}} \sim \tilde{\mathcal{J}} + D\underset{(0,2)}{\tilde{\mathcal{L}}}$

$\bar{\mathcal{J}} : [1,-3] \qquad \tilde{\bar{\mathcal{J}}} : (2,1) \qquad \bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\mathcal{D}}\underset{[2,-2]}{\bar{\mathcal{L}}} \;, \quad \tilde{\bar{\mathcal{J}}} \sim \tilde{\bar{\mathcal{J}}} + \bar{D}\underset{(2,0)}{\tilde{\bar{\mathcal{L}}}}$

In all relations, weights/dimensions are such that Weyl connection drops out!

$\mathcal{D} \to \delta \qquad \bar{\mathcal{D}} \to \bar{\delta}$ simplest description in terms of conformal fields

$\mathfrak{bms}_4$ algebra $\quad [(\mathcal{Y}_1, \bar{\mathcal{Y}}_1, \mathcal{J}_1), (\mathcal{Y}_2, \bar{\mathcal{Y}}_2, \mathcal{J}_2)] = (\hat{\mathcal{Y}}, \hat{\bar{\mathcal{Y}}}, \hat{\mathcal{J}})$

$$\hat{\mathcal{Y}} = \mathcal{Y}_1 \partial \mathcal{Y}_2 - \mathcal{Y}_2 \partial \mathcal{Y}_1 \qquad \hat{\mathcal{J}} = \mathcal{Y}_1 \partial \mathcal{J}_2 - \tfrac{1}{2} \partial \mathcal{Y}_1 \mathcal{J}_2 - (1 \leftrightarrow 2) + c.c.$$

subalgebra $\mathfrak{g}$ (Lorentz, witt⊕witt) $\quad (\mathcal{Y}, \bar{\mathcal{Y}}, 0) \qquad (\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}}, 0)$

representation of $\mathfrak{g}$ on $\eta^{s,w}$ $\qquad$ on $\phi_{h,\bar{h}}$

$$\mathcal{Y} \cdot \eta^{s,w} = \mathcal{Y} \partial \eta^{s,w} + \frac{s-w}{2} \partial \mathcal{Y} \, \eta^{s,w} \qquad \tilde{\mathcal{Y}} \cdot \phi_{h,\bar{h}} = \tilde{\mathcal{Y}} \partial \phi_{h,\bar{h}} + h \partial \tilde{\mathcal{Y}} \phi_{h,\bar{h}}$$

$$\bar{\mathcal{Y}} \cdot \eta^{s,w} = \bar{\mathcal{Y}} \bar{\partial} \eta^{s,w} - \frac{s+w}{2} \bar{\partial} \bar{\mathcal{Y}} \, \eta^{s,w} \qquad \tilde{\bar{\mathcal{Y}}} \cdot \phi_{h,\bar{h}} = \tilde{\bar{\mathcal{Y}}} \bar{\partial} \phi_{h,\bar{h}} + \bar{h} \bar{\partial} \tilde{\bar{\mathcal{Y}}} \phi_{h,\bar{h}}$$

$$\mathcal{E}_X \alpha = (\mathcal{Y}, \bar{\mathcal{Y}}) \cdot \mathcal{T}^{[0,1]} \qquad \mathcal{E}_X \alpha = (\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}}) \cdot \tilde{\mathcal{T}}^{(-1/2,-1/2)}$$

action of inf rotation on translations

$\mathfrak{bms}_4^*$ dual space $([\mathcal{J}], [\bar{\mathcal{J}}], \mathcal{P})$ $([\tilde{\mathcal{J}}], [\tilde{\bar{\mathcal{J}}}], \tilde{\mathcal{P}})$

$(0,0)$; $[0,-2]$

pairing $\langle([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P});(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})\rangle = \int_S d\mu\,[\bar{\mathcal{J}}\mathcal{Y} + \mathcal{J}\bar{\mathcal{Y}} + \mathcal{P}\mathcal{T}]$, $d\mu(\xi,\bar{\xi}) = \frac{iC}{P\bar{P}}\, d\xi \wedge d\bar{\xi}$

$\langle([\tilde{\mathcal{J}}],[\tilde{\bar{\mathcal{J}}}],\tilde{\mathcal{P}}),(\tilde{\mathcal{Y}},\tilde{\bar{\mathcal{Y}}},\tilde{\mathcal{T}})\rangle = \int_S d\tilde{\mu}\,[\tilde{\bar{\mathcal{J}}}\tilde{\mathcal{Y}} + \tilde{\mathcal{J}}\tilde{\bar{\mathcal{Y}}} + \tilde{\mathcal{P}}\tilde{\mathcal{T}}]$ $d\tilde{\mu} = iC\, d\xi \wedge d\bar{\xi}$

assumption : pairing annihilates total $\eth, \bar{\eth}$ $(\partial, \bar{\partial})$ derivatives
non-degenerate $\rightarrow$ integrations by parts

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{J} = \bar{\mathcal{Y}}\bar{\eth}\mathcal{J} + 2\bar{\eth}\bar{\mathcal{Y}}\mathcal{J} + \underbrace{\eth(\mathcal{Y}\mathcal{J})}_{=\mathrm{ad}^*_{\mathcal{Y}}\mathcal{J}\sim 0} + \underbrace{\tfrac{1}{2}\mathcal{T}\bar{\eth}\mathcal{P} + \tfrac{3}{2}\bar{\eth}\mathcal{T}\mathcal{P}}_{\alpha\times P},$$

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{P} = \underbrace{\mathcal{Y}\eth\mathcal{P} + \tfrac{3}{2}\eth\mathcal{Y}\,\mathcal{P} + \text{c.c.}}_{\mathcal{E}^*_X P}$$

work out formulas for the group ✓

Realization on the sphere

stereographic coord. $\xi = \cot\frac{\theta}{2}\, e^{-i\phi}$ $\quad ds^2 = -2(P_S\bar{P}_S)\, d\xi\, d\bar{\xi}$ $\quad P_S = \frac{1}{R\sqrt{2}}(1+\xi\bar{\xi})$

globally well-defined conf. coord. trsf $\quad \xi' = \frac{a\xi+b}{c\xi+d}$, $\quad ad-bc=1$, $\quad a,b,c,d \in \mathbb{C}$ $\quad \frac{\partial\xi}{\partial\xi'} = (c\xi+d)^2$

compensating Weyl trsf. $\quad e^{E_R(x')} = \frac{1+\xi\bar{\xi}}{|a\xi+b|^2+|c\xi+d|^2}$ $\quad e^{iE(x')} = \frac{\bar{c}\bar{\xi}+\bar{d}}{c\xi+d}$ $\quad w$: boost weight

Pairing $\quad \langle K^{s,-w-2}, \eta^{s,w}\rangle = \frac{1}{4\pi R^2}\int_{S^2}\frac{i\, d\xi \wedge d\bar{\xi}}{P_S\bar{P}_S}\, \overline{K^{s,-w-2}}\, \eta^{s,w}$ $\quad C=(4\pi R^2)^{-1}$

assumptions ✓

$$\frac{1}{4\pi R^2}\int_{S^2}\frac{i\, d\xi \wedge d\bar{\xi}}{P_S\bar{P}_S} = \frac{1}{4\pi}\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi = 1$$

adjoint repres. group

$$\mathcal{Y}'(x') = e^{E_R(x')}\, e^{iE_I(x')}\, \mathcal{Y}(x)$$

$$\beta'(x') = e^{E_R(x')}\left(\beta - (\mathcal{Y}\partial\alpha - \tfrac{1}{2}\partial\alpha\,\partial\mathcal{Y} + c.c.)\right)(x)$$

coadjoint repres. group

$$\mathcal{J}'(x') = e^{-3E_R(x')}\, e^{iE_I(x')}\left(\mathcal{J} + \tfrac{1}{2}\mathcal{J}\bar{\partial}\mathcal{P} + \tfrac{3}{2}\bar{\partial}\mathcal{J}\mathcal{P}\right)(x)$$

$$\mathcal{P}'(x') = e^{-3E_R(x')}\, \mathcal{P}(x)$$

in terms of conf. fields

$$\tilde{\mathcal{Y}}'(\xi') = (c\xi+d)^{-2}\, \tilde{\mathcal{Y}}(\xi)$$

$$\tilde{\beta}'(x') = (c\xi+d)^{-1}(\bar{c}\bar{\xi}+\bar{d})^{-1}\left(\tilde{\beta} - \tilde{\mathcal{Y}}\partial\tilde{\alpha} - \tfrac{1}{2}\tilde{\alpha}\,\partial\tilde{\mathcal{Y}} + c.c.\right)(x)$$

$$\tilde{\mathcal{J}}'(x') = (c\xi+d)^{2}(\bar{c}\bar{\xi}+\bar{d})^{4}\left(\tilde{\mathcal{J}}(x) + (\tfrac{1}{2}\tilde{\mathcal{J}}\,\mathcal{J}\,\tilde{\mathcal{P}} + \tfrac{3}{2}\mathcal{J}\tilde{\mathcal{J}}\,\tilde{\mathcal{P}})\right)(x)$$

$$\tilde{\mathcal{P}}'(x') = (c\xi+d)^{3}(\bar{c}\bar{\xi}+\bar{d})^{3}\, \tilde{\mathcal{P}}(x)$$

Expansions : spin weighted spherical harmonics: ${}_sZ_{jm}$ un normalized ${}_sY_{jm}$ normalized

Gelfand, Minlos, Shapiro (1958) ; Wu & Yang, Nucl. Phys. B (1976)
Newman, Penrose, JMP (1966) ; Thorne, Rev. Mod. Phys (1980)

conformal Killing eq. on S^2 $\quad \bar{\eth} \mathcal{Y}^{[-1,1]} = 0 = \eth \bar{\mathcal{Y}}^{[1,-1]}$

$$\mathcal{Y}_m = -R\sqrt{2}\ {}_{-1}Z_{1,m} \qquad m = -1, 0, 1 \qquad \mathcal{Y} = \sum_{m=-1}^{1} y_m \mathcal{Y}_m$$

$$\mathcal{T}_{jm} = {}_0Z_{j,m} \qquad \mathcal{T} = \sum_{j,|m|\leq j} t_{j,m} \mathcal{T}_{j,m}, \qquad \bar{t}_{jm} = (-1)^m t_{j,-m}$$

dual basis $\quad \mathcal{Y}_*^m = \frac{-6}{R\sqrt{2}(1+m)!(1-m)!}\ {}_{-1}Z_{1,m} \qquad \mathcal{J} = \sum_{m=-1}^{1} j_m \mathcal{Y}_*^m$

$$\mathcal{T}_*^{j,m} = \frac{(2j+1)!(2j)!}{j!\,j!\,(j+m)!(j-m)!}\ {}_0Z_{j,m} \qquad \mathcal{P} = \sum_{j,|m|\leq j} p_{j,m} \mathcal{T}_*^{j,m}, \qquad \bar{p}_{jm} = (-1)^m p_{j,-m}$$

NB: conformal fields : $\tilde{\mathcal{Y}}_m = \mathcal{Y}_m P_S = \zeta^{1-m} \Rightarrow [\tilde{\mathcal{Y}}_m, \tilde{\mathcal{Y}}_n] = (m-n)\tilde{\mathcal{Y}}_{m+n}$

$\longrightarrow$ all other structure constants can be worked out explicitly (ugly)

Remark (i) Penrose & Rindler Vol I, section 4.15

$w \geq |s|$ $\quad \eth^{w+s+1}\,\eta^{s,w}$ $\quad [w+1, s-1]$ $\quad \bar\eth^{w+s+1}\,\eta^{s,w}$ $\quad [-w-1, -s-1]$

definite boost weight

In the study of spin-weighted spherical harmonics it is useful to contemplate the following array:

j													
0							1						
$\frac{1}{2}$						2		2					
1			$\overleftarrow{\eth'}$		3		3		3		$\overrightarrow{\eth}$		
$\frac{3}{2}$				4		4		4		4			
2			5		5		5		5		5		
$\frac{5}{2}$		6		6		6		6		6		6	
$\vdots$	$\ldots$												$\ldots$
$s =$	$\ldots$	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\ldots$

(4.15.60)

$\bar\eth\eta = 0 \iff \eth^3\eta = 0$

$\eth\bar\eta = 0 \iff \bar\eth^3\bar\eta = 0$ same solutions

The numbers in this triangular array (which extends indefinitely downwards) represent the complex *dimensions* of the various spaces of spin-weighted spherical harmonics, as discussed in (4.15.43) *et seq.* Each of these spaces is characterized by its values of s and j, as shown. The dimension *zero* is assigned wherever a blank space appears in the array. The operator $\eth$ carries us a step of one s-unit to the right and $\eth'$ one s-unit to the left. (From our earlier discussion, the j-value is not affected by $\eth$ or $\eth'$.) Whenever such a step carries us off the array, the result of the operator $\eth$ or $\eth'$ is zero. Note that the dimension remains constant whenever it does not drop to, or increase from, zero.

dual situation $w \leq -|s|-2$

$\eth^{s-w-1}\,\kappa^{w+1,s-1}$ $\quad \bar\eth^{-s-w-1}\,\kappa^{-w-1,-s-1}$

$[s,w]$ $\quad [s,w]$ definite boost weight

$[2,-2]$ $\quad [-2,0]$

$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar\eth\,\bar{\mathcal{I}} \iff \tilde{\bar{\mathcal{J}}} \sim \tilde{\bar{\mathcal{J}}} + \eth^3 \mathcal{M}$

same equivalence classes

Remark (ii) reduction to Poincaré

$\eth^2\mathcal{J} = 0 = \bar\eth^2\bar{\mathcal{J}}$ $\quad \mathcal{P} \sim \mathcal{P} + \eth^2\mathcal{N} + \bar\eth^2\bar{\mathcal{N}}$

$[-2,-1]$ $\quad [2,-1]$

$j \leq w$: finite dim. repres of Lorentz, "heads"

$j > w$: ∞ dim, "tails"

Remark (iii)

$$\tilde{\mathcal{Y}}_m \cdot \phi_{h,\bar{h}} = \xi^{-m}\left(\xi \partial \phi_{h,\bar{h}} + h(l-m)\, \phi_{h,\bar{h}}\right)$$
$$\bar{\tilde{\mathcal{Y}}}_m \cdot \phi_{h,\bar{h}} = \bar{\xi}^{-m}\left(\bar{\xi}\bar{\partial}\, \phi_{h,\bar{h}} + \bar{h}(l-m)\, \phi_{h,\bar{h}}\right)$$

Goldberg et al. JMP 1967 $\quad {}_sY_{j,m} \quad j \leq L \quad \longleftrightarrow \quad {}_sZ^L_{m_1,m_2} = (1+\xi\bar{\xi})^{-L}\, \xi^{L-s-m_1}\, \bar{\xi}^{L+s-m_2}$

invertible $\qquad 0 \leq m_1 \leq L-s, \quad 0 \leq m_2 \leq L+s$

overcomplete set of functions, look like expansions on the punctured plane

when transforming to associated conformal fields structure constants look like those on the punctured plane, up to corrections.

Realization on punctured plane

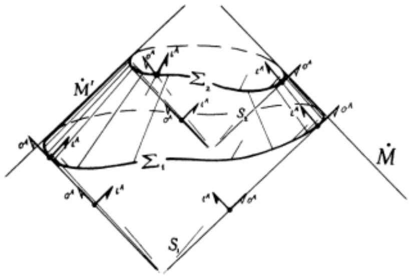
- Weyl trsf $e^{-\Phi(\xi,\bar\xi)} = \frac{\sqrt{2}}{1+\xi\bar\xi}$ $\quad \xi = R^{-1} z \quad ds^2 = -2\, dz\, d\bar z$

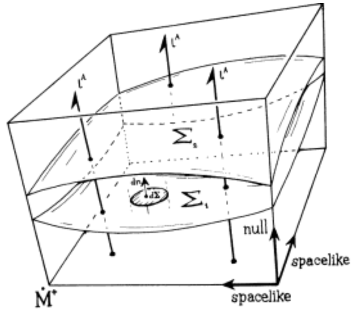
- 2-punctures: remove points at origin & infinity $\mathbb{C}_0$

- on the level of the algebra, look at the algebra of all infinitesimal local conformal trsf.

Not the Lie algebra of globally well-defined trsf.

For asymptotically flat spaces, $\dot M$ is in fact a null hypersurface [7]. The structure of $\dot M$ is essentially the same as for Minkowski space (Figure 4). We shall omit the three points I^-, I^0, I^+ here. Then $\dot M$ consists of two portions, each of which is topologically a "cylinder" $S^2 \times E^1$. We are concerned, here, only with the future portion $\dot M^+$, and by judicious choice of conformal factor Ω, we can ensure that the geometry of $\dot M^+$ is as simple as possible. In fact, by taking one generator of $\dot M^+$ "back to infinity" we can open out the cylinder into a space with Euclidean three-space topology. Furthermore, it turns out that we can also make this three-space metrically *flat* (Figure 6). This will simplify matters considerably.





Penrose 1967 AMS

$P=1 \Rightarrow$ weighted scalars = conformal fields

gravity : sphere $\to$ plane

CFT : plane $\to$ sphere

Coulomb gas ?

$$e^{\Phi(x')} = \frac{\partial z'}{\partial z} \qquad e^{\Phi_R(x')} = \left(\frac{\partial z'}{\partial z}\frac{\partial \bar z'}{\partial \bar z}\right)^{1/2}, \quad e^{i\Phi_I(x')} = \left(\frac{\partial z'/\partial z}{\partial \bar z'/\partial \bar z}\right)^{1/2}$$

Expansions

$$\phi_{h,\bar h}(z,\bar z) = \sum_{k,l} a_{k,l}\; \tilde{\mathcal{Z}}_{h,\bar h\, k,l}\,, \qquad \tilde{\mathcal{Z}}_{h,\bar h\, k,l} = z^{-h-k}\, \bar z^{-\bar h - l}$$

$h,\bar h \in \mathbb{N} \Rightarrow k,l \in \mathbb{Z}$

$h,\bar h \in \frac{\mathbb{N}}{2} \Rightarrow k,l \in \frac{1}{2}+\mathbb{Z}$

(NS)

Pairing $\langle \psi_{-\bar h+1,-h+1}, \phi_{h,\bar h} \rangle = \mathrm{Res}_z\, \mathrm{Res}_{\bar z} \left[\overline{\psi_{-\bar h+1,-h+1}}\; \phi_{h,\bar h} \right]$

assumptions ✓

$\mathrm{Res}_z(\partial\phi) = 0 = \mathrm{Res}_{\bar z}(\bar\partial \phi)$

adjoint repr. group $\tilde{\mathcal{Y}}'(z') = \left(\frac{\partial z}{\partial z'}\right)^{-1} \tilde{\mathcal{Y}}(z)$

$$\tilde\beta'(x') = \left(\frac{\partial z}{\partial z'}\right)^{-1/2} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{-1/2} \left(\tilde\beta - \left(\tilde{\mathcal{Y}}\, \partial\tilde\alpha - \tfrac{1}{2}\tilde\alpha\, \partial \tilde{\mathcal{Y}} + c.c.\right)\right)(x)$$

coadjoint repr group $\mathcal{J}'(x') = \left(\frac{\partial z}{\partial z'}\right)^{1} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{2} \left(\mathcal{J} + \tfrac{1}{2}\tilde{\mathcal{P}}\,\partial\tilde{\mathcal{P}} + \tfrac{3}{2}\,\partial\tilde{\partial}\tilde{\mathcal{P}}\right)(x)$

$$\tilde{\mathcal{P}}'(x) = \left(\frac{\partial z}{\partial z'}\right)^{3/2} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{3/2} \tilde{\mathcal{P}}(x)$$

to be used for conformal mapping.

Expansions $\qquad \langle {}_{-\bar{h}+1,-h+1}\tilde{Z}_{k',l'},\ {}_{h,\bar{h}}\tilde{Z}_{k,l} \rangle = \delta^0_{l'+k}\,\delta^0_{k'+l}$

$$\tilde{y}_m = z^{1-m}, \qquad \tilde{J}_{k,l} = z^{1/2-k}\,\bar{z}^{1/2-l} \qquad m,\ \tfrac{1}{2}+k,\ \tfrac{1}{2}+l \in \mathbb{Z}$$

$$\tilde{y}^m_* = z^{-1}\,\bar{z}^{-2+m} \qquad \tilde{J}^{k,l}_* = z^{-3/2+l}\,\bar{z}^{-3/2+k}$$

$$\tilde{y}_m \cdot {}_{h,\bar{h}}\tilde{Z}_{k,l} = -(hm+k)\,{}_{h,\bar{h}}\tilde{Z}_{k+m,l}, \qquad \tilde{\bar{y}}_m \cdot {}_{h,\bar{h}}\tilde{Z}_{k,l} = -(\bar{h}m+l)\,{}_{h,\bar{h}}\tilde{Z}_{k,l+m}$$

Structure constants $\qquad [\tilde{y}_m, \tilde{y}_n] = (m-n)\,\tilde{y}_{m+n} \qquad [\tilde{y}_m, \tilde{J}_{k,l}] = (\tfrac{1}{2}m-k)\,\tilde{J}_{k+m,l}$

$$[\tilde{\bar{y}}_m, \tilde{J}_{k,l}] = (\tfrac{1}{2}m-l)\,\tilde{J}_{k,l+m}$$

$$[\tilde{y}_m, \tilde{\bar{y}}_n] = 0 = [\tilde{J}_{k,l}, \tilde{J}_{r,s}]$$

coadjoint repr. algebra

$$\mathrm{ad}^*_{\tilde{y}_m} \tilde{y}^n_* = (-2m+n)\, \tilde{y}^{n-m}_* \,, \qquad \mathrm{ad}^*_{\tilde{y}_m} \tilde{J}^{k,l}_* = \left(-\tfrac{3}{2}m+k\right) \tilde{J}^{k-m,l}_* \,,$$

$$\mathrm{ad}^*_{\tilde{J}_{k,l}} \tilde{J}^{r,s}_* = \left(\frac{r-3k}{2}\right) \delta^s_l\, y^{r-k}_* + \left(\frac{s-3l}{2}\right) \delta^r_k\, \tilde{y}^{s-l}_*$$

Realization on cylinder

$$z = e^{-i\frac{2\pi}{L_1} w} \,, \quad w = w_1 + i w_2 \,, \quad w_1 \sim w_1 + L_1 \,, \quad \phi^C_{h,\bar{h}}(w,\bar{w}) = \left(-i\frac{2\pi}{L_1} z\right)^h \left(i\frac{2\pi}{L_1}\bar{z}\right)^{\bar{h}} \phi_{h,\bar{h}}(z,\bar{z})$$

use formulas for the group to map generators $\quad y^C_m = i\left(\frac{2\pi}{L_1}\right)^{-1} e^{i\frac{2\pi}{L_1} m w} \ldots$

same structure constants, obtained from ad^* still provide a representation

but pairing issues ....

(Torus: $\quad e^{i\frac{2\pi}{L_1}(w_1 + i w_2)} \rightarrow e^{i\frac{2\pi}{L_1} w_1^T} e^{i\frac{2\pi}{L_2} w_2^T}$

$w_2^T \sim w_2^T + L_2$

$$w_2 = +i\frac{L_2}{L_1} w_2^T \;)$$

Identification in non-radiative asymptotically flat spacetimes at $\mathscr{I}^+$

Back to S^2 & GR: BMS metric $\Longleftrightarrow$ $\boxed{\text{NP first order}}$ (similar analysis at $\mathscr{I}^-$)

Solution space, free data at $\mathscr{I}^+$: $\boxed{\psi_2^0 + \bar{\psi}_2^0 , \quad \psi_1^0 , \quad \sigma^0}$ undetermined u-dependence

$\dot{\sigma}^0$ news

evolution equations. $\partial_u \psi_2^0 = \eth \psi_3^0 + \sigma^0 \psi_4^0$, $\quad \partial_u \psi_1^0 = \eth \psi_2^0 + 2\sigma^0 \psi_3^0$

constraints $\quad \psi_2^0 - \bar{\psi}_2^0 = \bar{\eth}^2 \sigma^0 - \eth^2 \bar{\sigma}^0 + \dot{\sigma}^0 \bar{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0$

$\psi_3^0 = -\eth \dot{\bar{\sigma}}^0$, $\quad \psi_4^0 = -\ddot{\bar{\sigma}}^0$

additional data to construct solutions

$$\psi_0 = \sum_{n \geq 0} \psi_0^n (\zeta, \bar{\zeta}, u_0)\, r^{-5-n}$$

Transformation of (relevant) free data at $\mathcal{J}^+$ $\qquad s = (\mathcal{Y}, \bar{\mathcal{Y}}, \mathcal{T})$, $\quad f = \mathcal{T} + \frac{1}{2}u(\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}})$

$$\delta_s \sigma^0 = [\, f \partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \tfrac{3}{2}\eth\mathcal{Y} - \tfrac{1}{2}\bar{\eth}\bar{\mathcal{Y}}\,]\sigma^0 - \eth^2 f$$

$$\delta_s \psi_2^0 = [\quad \text{''} \qquad \text{''} \qquad \text{''} \quad + \tfrac{3}{2}\eth\mathcal{Y} + \tfrac{3}{2}\bar{\eth}\bar{\mathcal{Y}}\,]\psi_2^0 + 2\eth f\, \psi_3^0$$

(constraints to be imposed)

$$\delta_s \psi_1^0 = [\quad \text{''} \qquad \text{''} \qquad \text{''} \quad + 2\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}}\,]\psi_1^0 + 3\eth f\, \psi_2^0$$

broken current algebra

$$J_s = \frac{i}{R^2}\left[(P_s\bar{P}_s)^{-1} J^u_s\, d\xi \wedge d\bar{\xi} + P_s^{-1} J^{\bar{\xi}}_s\, du \wedge d\xi - \bar{P}_s J^{\xi}_s\, du \wedge d\bar{\xi}\right]$$

$$\delta_{s_1} J_{s_2} + \boxed{\Theta_{s_2}(\delta_{s_1} X)} \approx -J_{[s_1, s_2]} + d\, L_{s_1, s_2}$$

non-conservation $\qquad d J_s + \Theta_s(\delta_{(0,0,1)} X) \approx 0$

$s_1 : (\mathcal{Y}_1, \bar{\mathcal{Y}}_1, \mathcal{T}_1) = (0,0,1)$ $\qquad \Theta_s(\delta X) \sim \dot{\sigma}^0, \dot{\bar{\sigma}}^0$ $\quad$ vanishes in the absence of news time components

$$J^u_s = -\frac{1}{8\pi G}\Big\{\big[\underbrace{\Psi^0_2+\bar\Psi^0_2}_{BH}+\underbrace{\sigma^0\dot{\bar\sigma}^0+\bar\sigma^0\dot\sigma^0}_{\text{gravitons}}\big]f+\big[\Psi^0_1+\sigma^0\eth\bar\sigma^0+\tfrac{1}{2}\eth(\sigma^0\bar\sigma^0)\big]\mathcal{Y}+\big[\bar\Psi^0_1+\bar\sigma^0\bar\eth\sigma^0+\tfrac{1}{2}\bar\eth(\sigma^0\bar\sigma^0)\big]\bar{\mathcal{Y}}\Big\}$$

$$\Theta^u_s(\delta\chi) = \frac{1}{8\pi G}\big[\dot{\bar\sigma}^0\delta\sigma^0+\dot\sigma^0\delta\bar\sigma^0\big]f$$

charges $\displaystyle Q_s = \int_{S^2,\,u=cte}\frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}J^u_s$. $\qquad \displaystyle \Theta_s[\delta\chi] = \int_{S^2,\,u=cte}\frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}\Theta^u_s[\delta\chi]$

algebra $\qquad \delta_{s_1}Q_{s_2}+\Theta_{s_2}[\delta_{s_1}\chi] = -Q_{[s_1,s_2]}$

(non-)conservation of BMS_4 charges

G.B. & C. Troessaert JHEP (2011)

JHEP (2013)

$$\frac{d}{du}Q_s = \underbrace{-\int_{S^2,\,u=cte}\frac{i}{R^2\,8\pi G}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}\big[\dot{\bar\sigma}^0\delta_s\sigma^0+\dot\sigma^0\delta_s\bar\sigma^0\big]}_{\text{fluxes}\qquad\text{generalizes Bondi mass loss}}$$

$\boxed{\text{non-radiative spacetimes (no news)}}$ $\sigma^0 = \sigma^0(\xi, \bar\xi, \cancel{u})$ $\left(\Rightarrow \dot\sigma^0 = 0 = \psi_3^0 = \psi_4^0, \quad \mathcal{O}_s[\delta \chi] = 0 \right)$

compare "abstract" construction of $\mathfrak{bms}_4^*$

identification at $u = 0$ $\quad P = -\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)$ $\quad \bar J = -\frac{1}{2G}\underbrace{\left(\psi_1^0 + \sigma^0 \eth \bar\sigma^0 + \frac{1}{2}\eth(\sigma^0\bar\sigma^0)\right)}_{\psi^0_{1\bar J}}$

super-momentum = Bondi mass aspect

~~super~~- angular momentum = Bondi angular momentum aspect

(pre)- momentum map: $\mathcal{F}$. algebra of non-radiative free data

$\mathfrak{bms}_4$ representation δ_s, $[\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1, s_2]}$

$$\mu : \mathcal{F} \longrightarrow \mathfrak{bms}_4^*$$

$$\mu\left(-\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)\right) = P, \qquad \mu\left(-\frac{1}{2G}\psi^0_{1\bar J}\right) = [\bar J], \qquad \mu \circ \delta_s = \mathrm{ad}_s^* \circ \mu$$

transformation laws at $u=0$ $\quad \delta_s(\psi_2^0+\bar\psi_2^0)=(\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar\eth+\frac{3}{2}\eth\mathcal{Y}+\frac{3}{2}\bar\eth\bar{\mathcal{Y}})(\psi_2^0+\bar\psi_2^0)$ ✓

$$\delta_s\psi_1^0=[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar\eth+2\eth\mathcal{Y}+\bar\eth\bar{\mathcal{Y}}]\psi_1^0+\frac{1}{2}\mathcal{J}\eth(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma^0}-\eth^2\cancel{\bar\sigma^0})$$
$$+\frac{3}{2}\eth\mathcal{J}(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma^0}-\eth^2\cancel{\bar\sigma^0})$$

$$\delta_s\psi_{1\mathcal{J}}^0=[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar\eth+2\eth\mathcal{Y}+\bar\eth\bar{\mathcal{Y}}]\psi_{1\mathcal{J}}^0+\frac{1}{2}\mathcal{J}\eth(\psi_2^0+\bar\psi_2^0)+\frac{3}{2}\eth\mathcal{J}(\psi_2^0+\bar\psi_2^0)$$
$$+\frac{1}{2}\bar\eth(\mathcal{J}\bar\eth\eth\sigma^0-\bar\eth\mathcal{J}\eth\sigma^0+3\eth\mathcal{J}\bar\eth\eth\sigma^0-3\bar\eth\eth\mathcal{J}\sigma^0-\frac{3}{2}\mathcal{J}\sigma^0)\ \boxed{-\frac{1}{2}\eth^3(\mathcal{J}\bar\sigma^0)}$$

$\boxed{\text{trivial !}}$

Remark: electric case $\bar\eth^2\sigma_e^0=\eth^2\bar\sigma_e^0 \iff \sigma_e^0=\eth^2\chi_e$

$$\delta_s\chi_e=[\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar\eth-\frac{1}{2}\eth\mathcal{Y}-\frac{1}{2}\bar\eth\bar{\mathcal{Y}}]-\mathcal{J}$$

Newman Penrose JMP 1966

Strominger et al. 2015-

Compère et al. 2016

simplified pre-momentum map $\mu':\mathcal{F}_e\longrightarrow\mathfrak{bms}_4^*$ (not physically relevant!)

$$\mu'[-\frac{1}{2G}(\psi_2^0+\bar\psi_2^0)]=\mathcal{P},\quad \mu'[-\frac{1}{2G}\psi_1^0]=[\mathcal{J}],\quad \mu'\circ\delta_s=\mathrm{ad}_s^*\circ\mu'$$

Perspectives

1) on punctured plane $\eth^3 \tilde{\mathcal{Y}} \neq 0$

$\exists$ super-rotations & super-angular momentum $K_{s_1,s_2} = \mathrm{Res}_z \mathrm{Res}_{\bar z} [\, r^0 f_1 \eth^3 Y_2 - (1,2) + \text{c.c.} \,]$

$\exists$ field-dependent central extension & associated Souriau cocycle

GB & Troessaert JHEP 2016 GB JHEP 2017 mapping from plane to cylinder to make r^0 non-zero for Kerr

2) coadjoint repres. of generalized BMS_4 Campiglia & Laddha Phys. Rev. 2014

$\mathrm{Diff}(S^2) \ltimes C^\infty(S^2)$ on S^2 drop $\eth \bar{\mathcal{Y}} = 0 = \bar\eth \mathcal{Y}$ $\eth^3 \mathcal{Y} = 0 = \bar\eth^3 \bar{\mathcal{Y}}$

dual also equivalence relations

$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \eth \bar{\mathcal{Y}}$, $\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \eth^3 \mathcal{M}$

related groups
Donnelly et al. 2020

simply expand everything in spin-weighted spherical harmonics

3) Geometric actions for BMS_4 (K. Nguyen, R. Ruzziconi, in progress)

4) Complete pre-momentum map to bona fide one connection to spatial infinity Henneaux & Troessaert JHEP 2018

Torre CQG 1986

Oliveri & Speziale 2019

Wieland 2020

5) Study interactions of this group theory sector with radiative DoF at $\mathcal{J}^+$

Ashtekar & Streubel Proc. Roy. Soc. 1981

Ashtekar (1984)

6) Krichever-Novikov algebras for more than 2 punctures?

Parity conditions & connection to i^0 (preliminary)

spin-weighted spherical harmonics ${}_s Z_{j,m}$: polynomials $\alpha, \beta, \gamma, \delta$ $\quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2} \sin\frac{\theta}{2} & e^{-i\phi/2}\cos\frac{\theta}{2} \\ -e^{i\phi/2}\cos\frac{\theta}{2} & e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$

- anti-podal map : $\begin{cases} \phi \mapsto \phi + \pi \\ \theta \longmapsto \pi - \theta \end{cases}$ $\quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto i \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}$ $\quad {}_s Z_{j,m} \longmapsto i^{2j}\, {}_s Z_{j,-m}$

- $\overline{{}_s Z_{jm}} = (-1)^{m+s}\, {}_{-s} Z_{j-m}$, $\quad {}_s Z_{jm} \longmapsto i^{2j}\, {}_{-s} Z_{jm}$ $\quad$ real spaces of definite parity

$\Rightarrow$

even integer $j = 2n$		odd integer $j = 2n+1$	
${}_0 Z_{00}$	time translation	${}_0 Z_{1m}$	space translations
${}_0 Z_{2n\,m}$	"T" even supertranslations	${}_0 Z_{2n+1\;m}$	"W" odd supertranslations