10 Action principle and boundary issues

The first variation of the action should vanish on all solutions to the equations of motion allowed by the boundary conditions. Interestingly, this does not happen automatically. In particular, it does not happen for the Einstein–Hilbert action with the most common boundary conditions (asymptotically flat, asymptotically (A)dS). To resolve this issue we need to first understand what the issue is and how it arises. This, in turn necessitates to take a closer look at the variational principle of Einstein gravity in the presence of (actual or asymptotic) boundaries. In order to be able to do so we need to introduce such boundaries, which in turn requires techniques to decompose "bulk quantities" (such as the metric or the Riemann tensor) into "boundary quantities" plus extra stuff. In this section we give these words a precise mathematical meaning, starting with a canonical decomposition of the metric and related quantities.

10.1 Canonical decomposition of the metric

The canonical decomposition of a D-dimensional metric into a $(D-1)$-dimensional metric and a normal vector was already used in our derivation of the Raychaudhuri equations. Such a decomposition is useful in initial value formulations/Hamiltonian formulations of gravity. For our purposes we need a slightly different decomposition, where the normal vector is not time-like (as it would be for Raychaudhuri's equation or the initial value formulation) but rather spacelike. Thus, our primary data are some D-dimensional metric $g_{\mu\nu}$ (often referred to as "bulk metric") and some spacelike normal vector n^μ, normalized to unity, $n^\mu n_\mu = +1$.

With these data we can define a $(D-1)$-dimensional metric (often referred to as "boundary metric", "induced metric" or "first fundamental form"),

$$h_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu \tag{1}$$

which is still a D-dimensional symmetric tensor, but projects out the normal component,

$$h_{\mu\nu} n^\nu = 0 \qquad\qquad h^\mu_\mu = D - 1\,. \tag{2}$$

It is also useful to define the projected velocity with which the normal vector changes (often referred to as "extrinsic curvature" or "second fundamental form"),

$$K_{\mu\nu} := h^\alpha_\mu h^\beta_\nu \, \nabla_\alpha n_\beta = \frac{1}{2}\left(\mathcal{L}_n h\right)_{\mu\nu} \tag{3}$$

which can be recast as (one half of) the Lie-variation of the boundary metric along the normal vector. Note that also extrinsic curvature is a symmetric tensor and has vanishing contraction with the normal vector,

$$K_{\mu\nu} = K_{\nu\mu} \qquad\qquad K_{\mu\nu} n^\mu = 0 \tag{4}$$

We shall also need the contraction (or trace) of extrinsic curvature,

$$K := K^\mu_\mu = \nabla_\mu n^\mu\,. \tag{5}$$

Projection with the boundary metric yields a boundary-covariant derivative

$$\mathcal{D}_\mu := h^\nu_\mu \nabla_\nu \tag{6}$$

that leads to standard (pseudo-)Riemann tensor calculus at the boundary when acting on tensors projected to the boundary.

Note that in a canonical context extrinsic curvature also can be interpreted as velocity of the boundary metric, since in that case $\mathcal{L}_n h \sim \dot{h}$, where dot denotes derivative with respect to time, so that derivative of the Lagrange density with respect to extrinsic curvature yields the canonical momentum density. Beware: in such a context there are also various sign changes as compared to these lecture notes since the normal vector in that case would be normalized to -1 instead of $+1$.

10.2 Boundary action for Dirichlet boundary value problem

Often a Dirichlet boundary value problem is desired where the metric is fixed at the boundary $\partial\mathcal{M}$, while its normal derivative is free to fluctuate,

$$\delta g_{\mu\nu}\big|_{\partial\mathcal{M}} = 0 \qquad\qquad n^\alpha\nabla_\alpha\delta g_{\mu\nu}\big|_{\partial\mathcal{M}} \neq 0\,. \tag{7}$$

We show now that the Einstein–Hilbert action is incompatible with such a boundary value problem.

As we have shown previously [see section 4.3, Eq. (14)], first variation of the Einstein–Hilbert action leads to the Einstein equations in the bulk plus total derivative terms,

$$\delta I_{\text{EH}}\big|_{\text{EOM}} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}\,\nabla^\mu\big(\nabla^\nu\delta g_{\mu\nu} - g^{\alpha\beta}\nabla_\mu\delta g_{\alpha\beta}\big) \tag{8}$$

where the subscript 'EOM' indicates that we drop terms that vanish when the bulk equations of motion hold. Using Stokes theorem the total derivative terms in (8) are converted into boundary terms,

$$\delta I_{\text{EH}}\big|_{\text{EOM}} = \frac{1}{16\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^{D-1} x\sqrt{-h}\,n^\mu\big(\nabla^\nu\delta g_{\mu\nu} - g^{\alpha\beta}\nabla_\mu\delta g_{\alpha\beta}\big)\,. \tag{9}$$

Using $n^\mu\nabla^\nu\,\delta g_{\mu\nu} = n^\mu(h^{\nu\alpha}+n^\nu n^\alpha)\nabla_\alpha\,\delta g_{\mu\nu}$ and $n^\mu h^{\nu\alpha}\nabla_\alpha\,\delta g_{\mu\nu} = n^\mu h^{\nu\alpha}\nabla_\alpha[(h^\gamma_\mu + n_\mu n^\gamma)(h^\beta_\nu + n_\nu n^\beta)\,\delta g_{\beta\gamma}] = -K^{\mu\nu}\,\delta g_{\mu\beta} + K n^\mu n^\nu\delta g_{\mu\nu}$ + total boundary derivative, the result (10) can be reformulated as[1]

$$\delta I_{\text{EH}}\big|_{\text{EOM}} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^{D-1} x\sqrt{-h}\,\big(h^{\mu\nu}n^\alpha\nabla_\alpha\delta g_{\mu\nu} + (K^{\mu\nu} - K n^\mu n^\nu)\,\delta g_{\mu\nu}\big)\,. \tag{10}$$

The first term in (9) generically is non-zero for the Dirichlet boundary value problem (7). Thus, the Einstein–Hilbert action is inconsistent with (7).

To resolve this issue we add suitable boundary terms to the bulk action, since they do not affect the bulk equations of motion, but may convert the result for the variation (9) into something compatible with the boundary value problem (7). Specifically, we need a boundary term that preserves diffeomorphisms along the boundary and that is capable of canceling the normal derivative of the fluctuations of the metric in (9). Like in the bulk, we can do a derivative expansion of the boundary action,

$$I_{\partial\mathcal{M}} = \frac{1}{16\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^{D-1} x\sqrt{-h}\,\big(b_0 + b_1\mathcal{R} + b_2 K + \dots\big) \tag{11}$$

where the ellipsis refers to terms with higher derivatives (e.g. $K^{\mu\nu}K_{\mu\nu}$ or $K\mathcal{R}$) and $\mathcal{R}$ is the boundary Ricci scalar (constructed from the boundary metric $h_{\mu\nu}$ and the boundary covariant derivative (6)). It is now easy to see that terms intrinsic to the boundary (like the boundary cosmological constant b_0 or the boundary Einstein–Hilbert term $b_1\mathcal{R}$) will not help us, since they cannot produce normal derivatives $n^\mu\nabla_\mu$. Thus, we set $b_0 = b_1 = 0$, focus on the term $b_2 K$ and vary it. Using the definition (5) as well as $\delta n_\mu = \frac{1}{2}\,n_\mu n^\alpha n^\beta\,\delta g_{\alpha\beta}$ yields

$$\delta K = \frac{1}{2}\,h^{\mu\nu}n^\alpha\nabla_\alpha\delta g_{\mu\nu} - \frac{1}{2}\,K n^\mu n^\nu\,\delta g_{\mu\nu} + \text{total boundary derivative} \tag{12}$$

Comparing with the variation (10) we deduce that we should choose $b_2 = 2$ to get consistency with the Dirichlet conditions (7).

[1] We assume here that the boundary $\partial\mathcal{M}$ has no boundary; if this assumption is relaxed the total derivative term is converted into a 'corner' contribution $1/(16\pi G)\int_{\partial^2\mathcal{M}}\mathrm{d}^{D-2}x\sqrt{|\sigma|}\,n^\mu_\sigma n^\nu\,\delta g_{\mu\nu}$, where n^μ_σ is the outward pointing unit normal of the corner.

The full action for Einstein gravity (at this stage of our discussion) compatible with a Dirichlet boundary value problem (7) thus consists of the bulk action I_{EH} plus a boundary action I_{GHY}, known as Gibbons–Hawking–York boundary term.

$$\boxed{I = I_{\text{EH}} + I_{\text{GHY}} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}\big(R-2\Lambda\big) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^{D-1}x\sqrt{-h}\,K} \tag{13}$$

Its first variation (assuming a smooth boundary) is given by

$$\delta I = -\frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}\,\big(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R + \Lambda g^{\mu\nu}\big)\,\delta g_{\mu\nu} \\ - \frac{1}{16\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^{D-1}x\sqrt{-h}\,\big(K^{\mu\nu} - h^{\mu\nu}K\big)\,\delta g_{\mu\nu} \tag{14}$$

The tensor multiplying the variation $\delta g_{\mu\nu}$ at the boundary is known as **Brown–York stress tensor**,

$$T^{\mu\nu}_{\text{BY}} := \frac{1}{8\pi G}\,\big(K^{\mu\nu} - h^{\mu\nu}K\big)\,. \tag{15}$$

It is important to realize that further boundary terms can be added to the action (13) without spoiling the Dirichlet boundary value problem (7), for instance by choosing $b_0 \neq 0$ or $b_1 \neq 0$ in (11). As we shall see later in these lecture notes these terms are actually necessary in many applications. The reason for this is that even though we have a well-defined Dirichlet boundary value problem we still may not have a well-defined action principle, in the sense that there could be allowed variations of the metric that do not lead to a vanishing first variation (14) on some solutions of the equations of motion. We show now an example for this.

10.3 Action principle in mechanics

Before dealing in the next section with Einstein gravity we consider a much simpler example where the same boundary issues can arise, namely a classical field theory in 0+1 dimensions, also known as mechanics.

Consider specifically the conformal mechanics Hamiltonian

$$H(q,\,p) = \frac{p^2}{2} + \frac{1}{q^2} \tag{16}$$

in the bulk action (chosen on purpose with a $-q\dot{p}$-term to make it more similar to Einstein–Hilbert)

$$I_{\text{bulk}} = \int\limits_0^{t_c} \mathrm{d}t\,\big(-q\dot{p} - H(q,\,p)\big) \tag{17}$$

and a Dirichlet boundary problem, $q(0) = q_0$, $q(t_c) = q_c$. The first variation of the action (17) leads to a boundary term $-q\delta p$, so we introduce a mechanics version of the Gibbons–Hawking–York boundary term

$$I_{\text{GHY}} = qp\big|_0^{t_c}\,. \tag{18}$$

The variation of the full action $I = I_{\text{bulk}} + I_{\text{GHY}}$ yields

$$\delta I = \int\limits_0^{t_c} \mathrm{d}t\,\Big[\Big(-\dot{p} - \frac{H(q,\,p)}{\partial q}\Big)\,\delta q + \Big(\dot{q} - \frac{H(q,\,p)}{\partial p}\Big)\,\delta p\Big] + p\,\delta q\big|_{t=t_c} - p\,\delta q\big|_{t=0}\,. \tag{19}$$

Assuming the initial value q_0 is finite we have $\delta q\big|_{t=0} = 0$ and the last term drops. The bulk terms yield the (Hamilton) equations of motion. Thus, the first variation of the action (19) vanishes on-shell if it were true that $p\,\delta q\big|_{t=t_c} = 0$. For finite t_c and vanishing δq this is obviously the case, but we are interested in the limit $t_c \to \infty$ to mimic typical gravity systems where the range of the coordinates is non-compact.

Now comes the key observation: if we consider $t_c \to \infty$ the correct boundary value is $q_c \to \infty$ (if you look at the form of the potential in (16) you can see this — a ball in that potential just rolls all the way to infinity given infinite amount of time). Thus, finite variations,

$$\lim_{t_c\to\infty} \delta q|_{t=t_x} = \mathcal{O}(1) \tag{20}$$

preserve the asymptotic boundary condition that q_c tends to infinity. But if we allow such variations then **the action I does not have a well-defined variational principle since (19) does not vanish for all variations that preserve our boundary conditions.**

The resolution of this profound problem is to add another boundary term to the action (or to "holographically renormalize it") that does not spoil our Dirichlet boundary value problem. The most general such action is given by

$$\Gamma = \lim_{t_c\to\infty} \big(I_{\text{bulk}} + I_{\text{GHY}} - S(q,\,t)\big|^{t_c}\big) \tag{21}$$

where the counterterm $S(q,\,t)$ needs to be chosen such that the problem above goes away, i.e., the first variation of the full action Γ,

$$\delta\Gamma\big|_{\text{EOM}} = \lim_{t_c\to\infty} \Big(p - \frac{\partial S}{\partial q}\Big)\,\delta q\Big|^{t_c} \tag{22}$$

has to vanish on-shell for all variations preserving our boundary conditions, including finite variations δq.

Thus, we are looking for some function depending on the boundary values that is on-shell equivalent to the momentum, so that the term in parenthesis vanishes in (22). Actually, classical mechanics provides us with a natural candidate, namely Hamilton's principal function which is a solution to the Hamilton–Jacobi equation,

$$H\Big(q,\,\frac{\partial S}{\partial q}\Big) + \frac{\partial S}{\partial t} = 0\,. \tag{23}$$

For the potential (16) the solution is given by the expansion (if you want to see the exact solution look at (11) in `0711.4115`)

$$S(q,\,t) = \frac{q^2}{2t} + \mathcal{O}(1/t)\,. \tag{24}$$

Solving the equations of motion for large time yields

$$p = \frac{q}{t} + \mathcal{O}(1/t^2)\,. \tag{25}$$

Plugging these asymptotic expansions into the variation (22) establishes

$$\delta\Gamma\big|_{\text{EOM}} = \lim_{t_c\to\infty} \Big(\frac{q_c}{t_c} + \mathcal{O}(1/t_c^2) - \frac{q_c}{t_c} + \mathcal{O}(1/t_c)\Big)\,\delta q\Big|^{t_c} = \lim_{t_c\to\infty} \mathcal{O}(1/t_c)\,\delta q\Big|^{t_c} = 0\,. \tag{26}$$

Thus, the action (21) with (17) and (18) has a well-defined variational principle.

Let us finally address another issue with the unrenormalized action. Evaluating I on-shell yields a result that diverges in the limit $t_c \to \infty$. This is problematic insofar as the on-shell action provides the leading order contribution to the semi-classical partition function, which should not be singular. Fortunately, this problem is solved here automatically once we use the action Γ that has a well-defined variational principle. Indeed, evaluating Γ on-shell shows that the result is always finite, even when the upper boundary tends to infinity, $t_c \to \infty$.