Gravity and holography in lower dimensions II

(8.1) **Summing melons in SYK**

Note: This exercise uses QFT slang that may be unfamiliar to you, but I define all quantities explicitly that you need for this exercise.

In order to obtain the SYK 2-point function at strong coupling,

$$G(\tau) \sim \frac{\operatorname{sign}(\tau)}{\sqrt{|\tau|}}$$

start with the free Euclidean propagator for Majorana fermions, $G_{\text{free}}(\tau) = \frac{1}{2}\operatorname{sign}(\tau)$. Next, define the self-energy $\Sigma(\tau,\tau')$ as all 1-particle irreducible contributions to the propagator, which in Fourier space yields $1/G(\omega) = 1/G_{\text{free}}(\omega) - \Sigma(\omega)$. The resummation of melon diagrams is captured by relating this self-energy to a melon, i.e., three insertions of the full 2-point function with a coupling strength J from each vertex, $\Sigma(\tau) = J^2\, G^3(\tau)$. At strong coupling you can neglect $1/G_{\text{free}}(\omega)$. Use this info to derive the result above for the 2-point function.

(8.2) **Schwarzian equations of motion**

After proving the identities (to reduce clutter we set $\beta = 2\pi$)

$$I[\phi] \sim \int \mathrm{d}\tau \left\{ \tan\frac{\phi}{2};\, \tau \right\} \sim \int \mathrm{d}\tau \left(\frac{1}{2}\,\phi'^2 + \{\phi;\,\tau\}\right) \sim \frac{1}{2}\int \mathrm{d}\tau \left(\phi'^2 - \frac{\phi''^2}{\phi'^2}\right)$$

derive the equations of motion descending from the Schwarzian action above. Show that for $\phi(\tau) = \tau + \epsilon(\tau)$ and small $\epsilon(\tau)$ you get a constant mode, a linear mode and two exponential modes with exponents $\pm i\tau$.

(8.3) **Schwarzian vs. black hole entropy at very low temperatures**

In the limit $\beta J \gg N$ the Schwarzian theory is 1-loop exact and has the partition function

$$Z_{\text{Schwarz}}(\beta) = \frac{\alpha_0}{(\beta J)^{3/2}}\, e^{\frac{2\pi^2 N \alpha_1}{\beta J}}$$

where α_i are $\mathcal{O}(1)$ coefficients. Derive the associated entropy and discuss under which conditions

$$S_{\text{Schwarz}} = S_{\text{Black}}$$

where S_{Black} is the entropy of black holes in JT gravity. In particular, establish a holographic dictionary relating the number N of Majorana fermions, the coupling strength J, the Goldstone bosons, the asymptotic symmetries and the Schwarzian action to corresponding quantities on the JT gravity side.

These exercises are due on May 25th 2021.

Hints/comments:

- Fourier-transform the relation between 2-point function and self-energy to obtain a corresponding integral relation.

$$\int \mathrm{d}\tau'\, G(\tau,\, \tau')\Sigma(\tau',\, \tau'') = -\delta(\tau - \tau'')$$

  Make an ansatz $G(\tau,\, \tau') = \text{sign}(\tau - \tau')/|\tau - \tau'|^h$, motivated by the facts that for $h = 0$ you recover the free 2-point function and for finite h you have an expression with nice scaling properties suitable for conformal symmetries. Finally, exploit the Fourier transformation

$$\int\limits_{-\infty}^{\infty} \mathrm{d}\tau\, e^{i\omega\tau}\, \frac{\text{sign}(\tau)}{|\tau|^h} \propto |\omega|^{h-1}\, \text{sign}(\omega)$$

  to cross the finishing line by determining the value of h needed to satisfy the melonic relation $\Sigma(\tau) = J^2\, G^3(\tau)$. You should find $h = \frac{1}{2}$.

- Recall that he Schwarzian derivative is defined as $\{x; \tau\} = x'''/x' - \frac{3}{2}\,(x''/x')^2$. For the identities of actions note that you can partially integrate, using that ϕ' is periodic. Then either vary the right action and insert the linearization of $\phi(\tau) = \tau + \epsilon(\tau)$ to derive an ODE for ϵ or insert first the linearization up to quadratic order and then vary w.r.t. ϵ. Note: by dimensional arguments re-instating β must yield the exponents $\pm 2\pi i\tau/\beta$.

- In the entropy $S = \partial_T[T \ln Z(T)]$ you can neglect all subleading terms in N, i.e., keep only the terms that grow in N. In this limit you should find a very simple entropy law. Compare this with the entropy of black holes in JT gravity, keeping all coupling constants. This comparison should relate N and J to the JT parameters k and $1/\bar{y}$. You can make the standard holographic observation that large N corresponds to small gravitational coupling to separate what corresponds to N and what to J. Concerning Goldstone bosons, recall what could be the boundary degrees of freedom from the gravity perspective. Similar remarks apply to asymptotic symmetries. Regarding the Schwarzian action you should know the answer already from the lecture videos.