Daniel Grumiller April 28th 2021

Gravity and holography in lower dimensions II

(7.1) **Design your own 2d dilaton gravity model!**
Write down one (or more) dilaton gravity model functions $U(X)$ and $V(X)$ (or equivalently, $Q(X)$ and $w(X)$ as defined in the lectures) such that the associated linear dilaton solutions have all of the following geometric properties:

I. For vanishing mass the metric asymptotes to (A)dS$_2$ for large values of the dilaton.
II. For non-vanishing mass the metric asymptotes to (A)dS$_2$ for large values of the dilaton.
III. For vanishing mass and any values of the dilaton (including $X \to 0$) the Ricci scalar remains finite.
IV. For non-vanishing mass the Ricci scalar tends to $\pm\infty$ for $X \to 0$.
V. There are up to 42 Killing horizons, but not more than that.

(7.2) **Gauge trafos vs. diffeomorphisms**
Consider generic dilaton gravity in the PSM formulation (see lecture notes for the choice of the Poisson tensor and field content, as well as for the equations of motion). Show that the non-linear gauge transformations

$$\delta_\lambda X^I = \lambda_J P^{JI} \qquad \delta_\lambda A_I = \mathrm{d}\lambda_I + \partial_I P^{JK} A_J \lambda_K$$

on-shell correspond to local Lorentz transformations for the choice $\lambda_I = (\lambda_X, \lambda_a) = (\gamma, 0)$ and to diffeomorphisms for the choice $\lambda_I = A_{I\mu}\xi^\mu$, where ξ^μ is the vector field generating the diffeomorphism.

(7.3) **Natural boundary conditions for dilaton?**
Consider locally asymptotically AdS$_2$ boundary conditions

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - \big(e^{2\rho} + \mathcal{L}(t)\big)\,\mathrm{d}t^2 + \mathcal{O}(e^{-2\rho})$$

and natural boundary conditions for the dilaton, $X = e^\rho + e^{-\rho}\mathcal{X}(t) + \mathcal{O}(e^{-3\rho})$, where $\mathcal{L}$ and $\mathcal{X}$ are state-dependent functions. Which asymptotic Killing vectors preserve the asymptotic form of the metric? How does the state-dependent function $\mathcal{L}$ transform under the asymptotic symmetries? Which of the asymptotic Killing vectors also preserve the natural boundary conditions for the dilaton?

These exercises are due on May 18th 2021.

Hints/comments:

- You need only three ingredients: the exact form of the metric, (49) in the lecture notes, the condition for Killing horizons, (50) in the lecture notes, and the exact form of the Ricci scalar, (51) in the lecture notes. Since these quantities depend on $Q(X)$ and $w(X)$ it is easier to phrase your answer in terms of these quantities.

- Recall that under local Lorentz transformations the spin connection transforms inhomogeneously, $\delta_\gamma \omega = \mathrm{d}\gamma$, while Lorentz scalars are invariant, $\delta_\gamma X = 0$. Lorentz vector components must transform covariantly (the sign depends on conventions, but otherwise is fixed),

$$\delta_\gamma X^a = (\pm)\gamma \epsilon^a{}_b X^b$$

  and analogously for e^a. Under diffeos you must have the standard transformation behavior for scalar fields, e.g.

$$\delta_\xi X = \xi^\mu \partial_\mu X$$

  (and analogously for X^a) and the standard transformation behavior for 1-forms, e.g.

$$\delta \omega_\mu = \xi^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu \xi^\nu \,.$$

  (and analogously for e^a). Derive the transformation equations above starting with the PSM gauge symmetries, using on-shell conditions when necessary (for the local Lorentz trafos they will *not* be needed). You will need the Poisson tensor (35) and the equations of motion. [If it makes your life easier you are allowed to restrict the discussion to dilaton gravity models with vanishing kinetic potential, $U(X) = 0$, for the diffeo part of the exercise, in which case you can directly use the equations of motion (41)-(44).]

- The first part should be straightforward for you by now and will be much shorter than a corresponding 3-dimensional calculation that most likely you did already. You should find one Witt algebra worth of asymptotic Killing vectors and that $\mathcal{L}$ transforms with an infinitesimal Schwarzian derivative. Once you have answered the last question you will understand why natural boundary conditions on the dilaton are not so useful in holographic contexts.