Daniel Grumiller April 14^{th} 2021

Gravity and holography in lower dimensions II

(5.1) **Second order field equations for 2d dilaton gravity**
Derive the equations of motion by varying the 2d dilaton gravity action

$$S_{\text{2DG}} = \frac{1}{\kappa} \int \mathrm{d}^2 x \sqrt{-g} \left[XR + U(X)(\nabla X)^2 - 2V(X) \right]$$

with respect to the dilaton field X and the metric $g_{\mu\nu}$ (U, V are arbitrary functions of X). You may neglect surface terms in this exercise.

(5.2) **Higher curvature theories of gravity as dilaton gravity theories**
Show that a non-linear gravity theory in any dimension $D \geq 2$ with an action

$$\tilde{S} = \frac{1}{\tilde{\kappa}} \int \mathrm{d}^D x \sqrt{-g}\, R^{\tilde{\gamma}} \qquad \tilde{\gamma} \neq 0, 1$$

is equivalent to the following class of dilaton gravity models:

$$S = \frac{1}{\kappa} \int \mathrm{d}^D x \sqrt{-g} \left[XR - X^{\gamma} \right]$$

Derive a relation between the exponents γ and $\tilde{\gamma}$.
[Note: This result shows the equivalence of theories with $f(R)$ interactions to certain dilaton gravity models. The latter are also known as scalar-tensor theories, Jordan–Brans–Dicke theories or quintessence models. Both models have been used a lot in cosmology in the past 25 years.]

(5.3) **Spherical reduction**
Take Einstein gravity in $D > 2$ dimensions and assume spherical symmetry by considering adapted metrics

$$\mathrm{d}s^2 = g_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta + \frac{1}{\lambda}\, X^{2/(D-2)}\, \mathrm{d}^2\Omega_{S^{D-2}}$$

where $\alpha, \beta = 0, 1$, $\lambda = \text{const.}$ and $\mathrm{d}^2\Omega_{S^{D-2}}$ is the line-element of the round $(D-2)$-sphere. Insert the ansatz above into the D-dimensional Einstein–Hilbert action (disregarding boundary terms) and show that after integrating out the angular part you get a 2d dilaton gravity action like in exercise (5.1) with

$$U(X) = -\frac{D-3}{D-2}\,\frac{1}{X} \qquad\qquad V(X) \propto X^{(D-4)/(D-2)}\,.$$

These exercises are due on May 4^{th} 2021.

Hints/comments:

- The dilaton variation is straightforward. For the metric variation use the formula $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\,\delta g_{\mu\nu}$ (see Black Holes I). As we showed in Black Holes II the variation of the Ricci scalar yields

$$\delta R = -R^{\mu\nu}\,\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\,\delta g_{\mu\nu} - g^{\mu\nu}\nabla^2\,\delta g_{\mu\nu}$$

  Exploit also the fact that the 2d Einstein tensor vanishes identically for any 2d metric, $R_{\mu\nu} = \frac{1}{2}\, g_{\mu\nu} R$. Be careful with signs!

- Start with the dilaton gravity formulation and eliminate the dilaton X in terms of curvature R by means of its own equation of motion.

- This calculation might be lengthy. For me the most efficient way was to use the Cartan formulation and to make a $2+(D-2)$-split, but it is also ok if you just use computer algebra to calculate the D-dimensional Ricci scalar for metrics given in the ansatz of this exercise, expressing it in terms of the 2-dimensional Ricci scalar and additive extra terms involving the dilaton field and its derivatives. One thing you can calculate quickly and easily by hand is the volume form, just by taking the (square-root of minus the) determinant of the D-dimensional metric and expressing at as a product of the 2-dimensional volume form and a dilaton factor. Note that there is a simple geometric interpretation of the three terms in the Ricci scalar: the first one (containing R) comes from the intrinsic 2-dimensional curvature; the one proportional to $U(X)$ comes from spacetime-variations of the dilaton, which is essentially the curvature radius of the $(D-2)$-sphere; finally, the term with $V(X)$ comes from the intrinsic curvature of the $(D-2)$-sphere (this last term would be absent if you were to do a toroidal reduction instead of a spherical reduction). In terms of the Riemann-tensor (or curvature 2-form) the first term comes entirely from the first sector in the $2+(D-2)$-split, while the last term comes entirely from the second sector. The middle term comes from mixed terms involving both sectors.