Daniel Grumiller March 17^{th} 2021

Gravity and holography in lower dimensions II

(3.1) **Virasoro algebra from near horizon symmetries**
Starting with the $u(1)_k$ current algebra

$$[J_n,\, J_m] = \frac{k}{2}\, n\, \delta_{n+m,\,0}$$

prove that the twisted Sugawara construction

$$L_n = \frac{1}{k}\sum_{p\in\mathbb{Z}} J_{n-p}J_p + in\, J_n$$

yields a Virasoro algebra $[L_n,\, L_m]$ with central charge $c = 6k$. Moreover, calculate the commutator $[L_n,\, J_m]$. From a CFT_2 perspective, is J a primary field?

(3.2) **Global AdS_3 in near horizon state space?**
Consider exact solutions of AdS_3 Einstein gravity (with unit AdS radius) adapted to a near horizon expansion,

$$\mathrm{d}s^2 = \mathrm{d}r^2 - \kappa^2\, \sinh^2 r\ \mathrm{d}t^2 + 2\kappa\omega\, \sinh^2 r\ \mathrm{d}t\,\mathrm{d}\varphi + \big(\gamma^2 \cosh^2 r - \omega^2 \sinh^2 r\big)\ \mathrm{d}\varphi^2$$

with arbitrary (but fixed) κ and arbitrary (state-dependent) ω and γ. Show that these spacetimes, despite of being locally AdS_3, are not locally asymptotically AdS_3 in the Brown–Henneaux sense. Determine the values for κ, ω and γ such that the metric above is global AdS_3.

(3.3) **Near horizon microstates for BTZ**
Consider as BTZ microstates $u(1)_k$ descendants of the vaccum,

$$|\mathcal{B}(\{n_i^\pm\})\rangle = \prod_{\{n_i^\pm>0\}} \big(J^+_{-n_i^+} J^-_{-n_i^-}\big)|0\rangle$$

labelled by positive integers $n_i^\pm$, where the vacuum obeys the usual highest weight conditions $J_n^\pm|0\rangle = 0$ for $n \geq 0$. Assuming the integers obey the constraints

$$\sum_i n_i^\pm = c\,\Delta^\pm \gg 1$$

count the number of different descendants in the limit where $c\,\Delta^\pm$ are large positive integers. Finally, take the logarithm of that number and compare with Cardy's entropy formula, where c is the central charge and $\Delta^\pm$ the eigenvalues of the Virasoro zero mode.

These exercises are due on April 20^{th} 2021.

Hints/comments:

- To calculate $[L_n,\, L_m]$ it is useful to first calculate $[L_n,\, J_m]$ (which you need anyhow for this exercise), exploiting the commutator identity $[AB, C] = A\,[B,\, C]+[A,\, C]\,B$. For the last question recall that a weight h primary field Φ must obey the commutation relations $[L_n,\, \Phi_m] = \big((h-1)n - m\big)\,\Phi_{n+m}$.

- For the first part expand at large r and check whether or not you obtain the Fefferman–Graham form $\mathrm{d}r^2 + e^{2r}\,\gamma^{(0)}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + \dots$ with fixed $\gamma^{(0)}_{ij}$. For the last part recall that global AdS_3 has the line-element

$$\mathrm{d}s^2 = \mathrm{d}r^2 - \cosh^2 r\;\mathrm{d}\bar{t}^2 + \sinh^2 r\;\mathrm{d}\bar{\varphi}^2$$

  and note that 'values' may be interpreted generously, in particular allowing for complex numbers.

- The number $p(N)$ of partitions of the integer N into positive integers in the limit of large N is given by the Hardy–Ramanujan formula

$$p(N) \simeq \frac{1}{4N\sqrt{3}}\,\exp\left(2\pi\sqrt{\frac{N}{6}}\right).$$

  Knowing this, the exercise should be straightforward. If you need a reminder of the Cardy formula, here it is:

$$S = 2\pi\left(\sqrt{\frac{c\Delta^+}{6}} + \sqrt{\frac{c\Delta^-}{6}}\right)$$

  For more physics background on these proposed BTZ black hole microstates see `1607.00009` and `1705.06257`.