Black Holes II — Exercise sheet 10

(20.1) **Brown–Henneaux boundary conditions**
Given some boundary conditions for the metric, the asymptotic Killing vectors generate those diffeomorphisms that preserve these boundary conditions. Consider (without further gauge-fixing) the Brown–Henneaux boundary conditions [J. D. Brown and M. Henneaux, *Commun. Math. Phys.* **104** (1986) 207] near $y = 0$,

$$g_{\alpha\beta} = \begin{pmatrix} g_{++} = \mathcal{O}(1) & \frac{\ell^2}{2y^2} + \mathcal{O}(1) & \mathcal{O}(y) \\ g_{-+} = g_{+-} & g_{--} = \mathcal{O}(1) & \mathcal{O}(y) \\ g_{y+} = g_{+y} & g_{y-} = g_{-y} & g_{yy} = \frac{\ell^2}{y^2} + \mathcal{O}(1) \end{pmatrix}$$

where the total metric $g = \bar{g} + \delta g$ consists of an asymptotic AdS_3 background

$$\bar{g}_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \ell^2\,\frac{\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}y^2}{y^2}$$

and of fluctuations δg that fall off near $y = 0$ according to the above boundary conditions. Show that the asymptotic Killing vectors are given by vector fields ξ of the form

$$\xi^+ = \varepsilon^+(x^+) - \frac{y^2}{2}\,\partial_-^2\varepsilon^- + \mathcal{O}(y^4)$$
$$\xi^- = \varepsilon^-(x^-) - \frac{y^2}{2}\,\partial_+^2\varepsilon^+ + \mathcal{O}(y^4)$$
$$\xi^y = \frac{y}{2}\,\big(\partial_+\varepsilon^+(x^+) + \partial_-\varepsilon^-(x^-)\big) + \mathcal{O}(y^3)$$

where $\varepsilon^\pm$ are arbitrary functions of their arguments.

(20.2) **BTZ boundary stress tensor**
Determine the holographically renormalized Brown–York stress tensor for all BTZ black hole solutions. Is it traceless?

(20.3) **Cardy formula as aspect of $\mathbf{AdS_3/CFT_2}$**
For unitary CFT_2's with a sufficiently sparse spectrum the Cardy formula relates the entropy S of thermal states to its eigenvalues of the Virasoro zero modes $L_0^\pm$,

$$S = 2\pi\sqrt{\frac{c^+L_0^+}{6}} + 2\pi\sqrt{\frac{c^-L_0^-}{6}}$$

where $c^\pm$ are the central charges appearing in the Virasoro algebras of the CFT_2. Show that the Cardy formula above recovers the Bekenstein–Hawking entropy formula by applying the $\mathrm{AdS}_3/\mathrm{CFT}_2$ dictionary.

These exercises are due on June 23rd 2020.

Hints:

- Recall that a diffeomorphism generated by a vector field ξ acts on the metric via the Lie derivative

$$\mathcal{L}_\xi \, g_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\mu\sigma} \partial_\nu \xi^\sigma + g_{\nu\sigma} \partial_\mu \xi^\sigma$$

  Check now that the diffeomorphisms generated by the vector field ξ given at the end of (20.1) preserve the Brown–Henneaux boundary conditions, $\mathcal{L}_\xi \, g_{\mu\nu} = \mathcal{O}(\delta g_{\mu\nu})$, where $\delta g_{\mu\nu}$ denotes all fluctuations allowed by the boundary conditions.

- Bring the BTZ metric (27) in section 11.5 into the Fefferman–Graham form (3) in section 11.1 and read off the coefficients $\gamma^{(0)}_{\mu\nu}$ and $\gamma^{(2)}_{\mu\nu}$. Then plug the result into the formula for the holographically renormalized Brown–York stress tensor, Eq. (14) in section 11.3. Equipped with this result it is easy to answer the final question.

- The results needed for this exercise are all contained in the Black Holes II lecture notes, section 11 and the remaining calculations you need to do are rather trivial. But conceptually this exercise may be non-trivial, so remember what is a thermal state in AdS_3, look up the Brown–Henneaux result for the central charges and recall that the sum of the Virasoro zero modes is the energy and the difference the angular momentum. To determine energy (or mass times AdS radius) and angular momentum of thermal states in AdS_3 you either use the result of exercise (20.2) or you look them up in the lecture notes.

Happy Summer holidays!

In Fall 2020 there will be a set of lectures "Gravity and holography in lower dimensions I" which may be of interest to many of you.
Topics covered in this course are

- *gravity with boundaries*
- *asymptotic Killing vectors*
- *Cartan formulation*
- *gravity in two and three dimensions*
- *BTZ black holes*
- *canonical boundary charges*
- *asymptotic symmetry algebras and central extensions*
- *AdS_3/CFT_2 and AdS_2/CFT_1*
- *holographic renormalization and correlation functions*
- *holographic entanglement entropy*