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# Gravity and holography in lower dimensions II

(1.1) **Entanglement entropy for boost-invariant states**
Consider in a $\mathrm{CFT}_2$ a null deformed entangling region ($\Delta t = \lambda, \Delta x = \ell \pm \lambda$) where $\lambda$ is a small parameter and $\ell$ the original spatial entangling interval. Assuming you have a boost-invariant state, prove the relation

$$\frac{\mathrm{d}^2 S}{\mathrm{d}\lambda^2}\Big|_{\lambda=0} + \frac{6}{c}\left(\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right)^2\Big|_{\lambda=0} = \frac{\mathrm{d}^2 S}{\mathrm{d}\ell^2}\Big|_{\lambda=0} - \frac{1}{\ell}\,\frac{\mathrm{d}S}{\mathrm{d}\ell}\Big|_{\lambda=0} + \frac{6}{c}\left(\frac{\mathrm{d}S}{\mathrm{d}\ell}\right)^2\Big|_{\lambda=0}$$

where $S$ is entanglement entropy associated with the deformed interval, and compare with $\frac{1}{3\ell}\,\frac{\mathrm{d}c(\ell)}{\mathrm{d}\ell}$, where $c(\ell) = 3\ell\,\frac{\mathrm{d}S(\ell)}{\mathrm{d}\ell}$ is the Casini–Huerta $c$-function.

(1.2) **Near horizon expansion of BTZ**
Apply the near horizon expansion at small $\rho$

$$\mathrm{d}s^2 = \big(\,\mathrm{d}\rho^2 - \kappa^2\rho^2\,\mathrm{d}t^2 + \gamma^2\,\mathrm{d}\phi^2 + 2\kappa\,\omega\rho^2\,\mathrm{d}t\,\mathrm{d}\phi\big)\big(1 + \mathcal{O}(\rho^2)\big)$$

to the BTZ metric (expanding around the outer horizon $r = r_+$)

$$\mathrm{d}s^2_{\text{BTZ}} = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2}\,\mathrm{d}t^2 + \frac{\ell^2 r^2\,\mathrm{d}r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2\,\Big(\mathrm{d}\varphi - \frac{r_+ r_-}{\ell r^2}\,\mathrm{d}t\Big)^2$$

and determine $\kappa$, $\gamma$ and $\omega$ in terms of the horizon radii $r_\pm$.

(1.3) **Curvature, horizons and surface gravity in 2d**
Starting from two-dimensional metrics in Eddington–Finkelstein gauge,

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}r - K(u,\,r)\,\mathrm{d}u^2$$

calculate the Ricci scalar for any function $K(u,\,r)$. For which class of functions $K(u,\,r)$ is spacetime asymptotically flat? In the special case when you have a Killing vector $\partial_u$ determine the condition for a Killing horizon and calculate its surface gravity.

**These exercises are due on March 23rd 2021.**

Hints/comments:

- Remember how to boost to an equal time-slice. Prove the relation $\mathrm{d}S/\,\mathrm{d}\ell|_{\lambda=0} = S' = \mathrm{d}S/\,\mathrm{d}\lambda|_{\lambda=0}$ but note that $\mathrm{d}^2S/\,\mathrm{d}\ell^2|_{\lambda=0} = S'' \neq \mathrm{d}^2S/\,\mathrm{d}\lambda^2|_{\lambda=0}$. The comparison with the Casini–Huerta $c$-function should be straightforward.

- A convenient radial redefinition is $r^2 = r_+^2 + \rho^2(r_+^2 - r_-^2)/\ell^2 + \dots$, where you can neglect higher order terms. Inserting this into the BTZ line-element yields the desired $\mathrm{d}\rho^2$-term (up to negligible corrections). You also have to make a suitable shift of the angular coordinate, $\mathrm{d}\varphi = \mathrm{d}\phi + \text{const.}\,\mathrm{d}t$, in order to have the co-rotating form near the horizon. In the end you should find the BTZ surface gravity result for $\kappa$ (see previous semester) and the expressions $\gamma = r_+$ and $\omega = r_-/\ell$.

- You can either use the second order or the Cartan formulation to do the calculation — I'll leave it to you what you find simpler (you may use Computer algebra systems, but in that case make sure to have no typos when you write down the result for the Ricci scalar and/or write down some intermediate results, like the choice of zweibein or the non-zero Christoffel symbols). If you do this by hand in the second order formulation, note that the only Riemann-tensor component that you need simplifies to $R^u{}_{uur} = -\partial_r\Gamma^u{}_{uu}$ (all other non-zero components follow from this one using symmetries of the Riemann tensor); in particular, all bilinear terms of the form $\Gamma\Gamma$ necessarily cancel since we are in two dimensions where curvature becomes abelian.

  For the last part, consider first what the presence of a Killing vector $\partial_u$ implies for the function $K(u,\, r)$ and then use results from Black Holes I to determine the locus of possible Killing horizons and their surface gravities. [You compare particularly with exercise (8.2) from Black Holes I.]