Asymptotic Chern-Simons formulation of spacelike stretched AdS gravity

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Outline

Introduction

$SL(2,R) \times U(1)$ CS gauge theory

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The canonical generator, gauge conditions and asymptotic symmetries

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Specific asymptotic conditions, new form of surface terms, boundary degrees of freedom

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The talk is based on paper:

- M. Blagojević and B. Cvetković, Class. Quantum Grav. **27** (2010) 185022

- The AdS asymptotic structure of three-dimensional (3D) Einstein's gravity with a cosmological constant (GR$_\Lambda$) is described by two independent Virasoro algebras with classical central charges.
- Of particular importance for our understanding of the dynamics of 3D GR$_\Lambda$ is the fact that it can be represented as an ordinary gauge theory—the $SL(2,R) \times SL(2,R)$ Chern-Simons (CS) gauge theory.
- Topologically massive gravity with a cosmological constant, denoted shortly as TMG$_\Lambda$, is an extension GR$_\Lambda$ by a gravitational CS term.
- While GR$_\Lambda$ is a topological theory, TMG$_\Lambda$ is a dynamical theory with *one propagating mode, the massive graviton*.
- In the AdS sector, TMG$_\Lambda$ contains a maximally symmetric vacuum solution, known as AdS_3, and BTZ black hole.

- The AdS sector of TMG_Λ around AdS_3 is not consistent, so Anninos et al. proposed to choose a new vacuum, the so-called spacelike streched AdS_3.
- This choice reduces the $SL(2,R) \times SL(2,R)$ isometry group of AdS_3 to its subgroup $U(1) \times SL(2,R)$.
- An important step towards an understanding of the spacelike stretched AdS asymptotic structure was showing that the resulting asymptotic symmetry is centrally extended semidirect sum of a Virasoro and a $u(1)$ Kac-Moody algebra, Virasoro $\oplus_{\mathrm{sd}} u(1)_{\mathrm{KM}}$.
- We want to further improve our understanding of the spacelike stretched AdS gravity by showing that its *asymptotic* structure can be faithfully represented by an $SL(2,R) \times U(1)$ CS gauge theory.

- We consider the CS gauge theory defined by the action

$$I_{\mathrm{CS}} = -\kappa \int_{\mathcal{M}} \left(A^i dA_i + \frac{1}{3} \varepsilon_{ijk} A^i A^j A^k \right) + \bar{\kappa} \int_{\mathcal{M}} \bar{A} d\bar{A} \,, \qquad (2.1)$$

  where manifold $\mathcal{M}$ has the topology $R \times \Sigma$, where R is time and Σ is a spatial manifold ($\partial\Sigma$ is topologically a circle), A^i and $\bar{A}$ are the $SL(2,R)$ and $U(1)$ gauge potentials.
- The action is invariant under the gauge transformations:

$$\delta_0 A^i = \nabla u^i := du^i + \varepsilon^i{}_{jk} A^j u^k \,, \qquad \delta_0 \bar{A} = d\bar{u} \,.$$

- The asymptotic parameters in the spacelike stretched AdS sector of TMG$_\Lambda$ are *time independent*, so we choose the CS boundary conditions as

$$A^i{}_0 = 0 \,, \qquad \bar{A}_0 = \bar{a}_0 \qquad \text{at } \partial\Sigma \,, \qquad (2.2)$$

  since they imply $\partial_0 u^i = 0$, $\partial_0 \bar{u} = 0$ at the boundary.

- The canonical gauge generator is given by:

$$G = \int d^2x \left[(\nabla_0 u^i)\pi_i{}^0 + u^i\mathcal{H}_i\right] + \int d^2x \left[(\partial_0 \bar{u})\bar{\pi}^0 + \bar{u}\bar{\mathcal{H}}\right] ,$$

$$\mathcal{H}_i := \kappa\varepsilon^{0\alpha\beta}F_{i\alpha\beta} \approx 0 , \qquad \bar{\mathcal{H}} := -\bar{\kappa}\varepsilon^{0\alpha\beta}\bar{F}_{\alpha\beta} \approx 0 ,$$

where $(\pi_i{}^0, \bar{\pi}^0)$ are canonical momenta corresponding to Lagrangian variables $(A^i{}_0, \bar{A}_0)$.

- The first set of the corresponding gauge conditions is chosen in accordance with boundary conditions:

$$A^i{}_0 = 0 , \qquad \bar{A}_0 = \bar{a}_0 . \tag{2.3a}$$

- The second set of gauge conditions is defined by:

$$A_1 = a_1 , \quad \bar{A}_1 = \bar{a}_1 , \tag{2.3b}$$

where $a_1 = a_1^i T_i$ and $\bar{a}_1$ are constant elements of the corresponding Lie algebras.

- The field equations imply:

$$A_2 \approx b^{-1}\hat{A}_2(\varphi)b\,, \qquad \bar{A}_2 \approx \bar{A}_2(\varphi)\,. \tag{2.4}$$

- With the adopted gauge conditions is not differentiable:

$$\delta G = -\delta\Gamma_L[u] - \delta\Gamma_R[\bar{u}]\,.$$

- We adopt the following boundary conditions for u^i and $\bar{u}$:
  - $u^i = -\theta^i - \xi^\rho A^i{}_\rho$ and $\bar{u} = -\xi^\rho \bar{A}_\rho$ (at the boundary),

  which in conjunction with additional requirements:

$$\hat{A}^1{}_2 = 0\,, \qquad \hat{A}^-{}_2 = -2C\,, \qquad \bar{a}_1 = 0\,, \tag{2.5}$$

  lead to the following form of the surface term:

$$\Gamma[\xi] := \Gamma_L[\xi] + \Gamma_R[\xi] = -\int d\varphi \xi^0 \mathcal{E} - \int d\varphi \xi^2 \mathcal{M}\,,$$

$$\mathcal{E} = 2\bar{\kappa}\bar{a}_0\bar{A}_2\,, \qquad \mathcal{M} = \mathcal{M}_L + \bar{\kappa}(\bar{A}_2)^2\,. \tag{2.6}$$

- The PB algebra of the improved canonical generators expressed in terms of the the Fourier modes takes the form of the semidirect sum Virasoro $\oplus_{\mathrm{sd}}\, u(1)_{\mathrm{KM}}$:

$$\begin{aligned} i\{L_m, L_n\} &= (m-n)L_{m+n} + 4\pi\kappa m^3 \delta_{m,-n}\,, \\ i\{L_m, K_n\} &= -nK_{m+n}\,, \\ i\{K_m, K_m\} &= -4\pi\bar{\kappa}\bar{a}_0^2 m\delta_{m,-n}\,. \end{aligned} \tag{2.7}$$

- The gauge conditions in conjunction with the additional requirements imply that the original set of $9+3$ gauge potentials is reduced to just *two independent boundary degrees of freedom*, $\hat{A}^+{}_2(\varphi)$ and $\bar{A}_2(\varphi)$.
- The basic content of our analysis is encoded in the form of the *surface term* and the *PB algebra of the asymptotic generators* and are to be compared to those found in the asymptotic region of the spacelike stretched AdS gravity.

- The Lagrangian of TMG$_\Lambda$ is given by:

$$L_{\mathrm{TMG}} = 2ab^i R_i - \frac{\Lambda}{3}\varepsilon_{ijk} b^i b^j b^k + a\mu^{-1} L_{\mathrm{CS}}(\omega) + \lambda^i T_i\,, \quad (3.1)$$

  where ω^i is the Lorentz connection and b^i the orthonormal coframe, R^i and T^i are curvature and torsion, $L_{\mathrm{CS}}(\omega)$ is the CS Lagrangian for the connection, λ^i is the Lagrange multiplier and $a = 1/16\pi G$.
- The improved canonical for the spacelike stretched AdS gravity is given by:

$$\tilde{G} = G + \Gamma\,, \quad \Gamma := -\int_0^{2\pi} d\varphi \left(\ell T \mathcal{E}^1 + S\mathcal{M}^1\right)\,,$$

- The corresponding PB algebra takes the centrally extended Virasoro $\oplus_{\mathrm{sd}}\, u(1)_{\mathrm{KM}}$ form with central charges $c_V = (5\nu^2+3)\ell/[G\nu(\nu^2+3)]$ and $c_K = (\nu^2+3)\ell/(G\nu)$.

- We introduce a set of the *specific* (refined) asymptotic conditions, *compatible* with the general asymptotic structure.
- Neither the black hole solution nor the leading order asymptotic parameters depend on time. Hence, we introduce the following refined asymptotic conditions:

$$(b^i{}_\mu, \omega^i{}_\mu, \lambda^i{}_\mu) \quad \text{and} \quad (\xi^\mu, \theta^i) \quad \text{are time independent}\,. \tag{3.2}$$

- Motivated again by the properties of the black hole solution we adopt the following conditions:

$$\frac{\nu}{\ell} b^i{}_0 + \omega^i{}_0 = 0\,, \qquad \frac{a}{\mu\ell^2}(4\nu^2 - 3) b^i{}_0 - \lambda^i{}_0 = 0\,. \tag{3.3}$$

- These conditions can be considered as the gauge conditions that are compatible with the spacelike stretched AdS asymptotics.

- The surface terms $\mathcal{E}^1$ and $\mathcal{M}^1$ can be written in a form:

$$\begin{aligned}\mathcal{E}^1 &= -\frac{a[3(\nu^2-1)]^{3/2}}{3\nu\ell}\hat{B}^2{}_0\,, \quad \mathcal{M}^1 = \mathcal{M}^+ + \frac{12\pi\ell^2}{c_K}(\mathcal{E}^1)^2\,,\\ \mathcal{M}^+ : &= -\frac{a\sqrt{3(\nu^2-1)}}{2}\left(\frac{\hat{\hat{B}}^2{}_2}{\nu\ell}+\frac{2\sqrt{\nu^2+3}}{3\ell}\hat{B}^0{}_2\right.\\ &+ \left.\frac{4}{3}\hat{\hat{\Omega}}^2{}_2+\frac{1}{a}\hat{\hat{\Lambda}}^2{}_2+\frac{2\sqrt{\nu^2+3}}{3\nu}\hat{\hat{\Omega}}^0{}_2\right)\,. \qquad (3.4)\end{aligned}$$

- The first/second order sub-leading terms in the asymptotic expansion of the fields are denoted by single/double hats.
- One can prove that in the spacelike stretched AdS sector of TMG$_\Lambda$, there are two independent boundary degrees of freedom.

- Asymptotic structures of the spacelike stretched AdS gravity and the $SL(2,R) \times U(1)$ CS gauge theory can be identified by adopting a natural asymptotic correspondence between their field variables and coupling constants.
  - To prove the statement, we compare the *asymptotic canonical algebras* and the corresponding *surface terms*:

$$4\pi\bar{\kappa}\bar{a}_0^2 \sim \frac{c_K}{12}\,,\; 4\pi\kappa \sim \frac{c_V}{12}\,,\; 2\bar{\kappa}\bar{a}_0\bar{A}_2 \sim \mathcal{E}^1\,,\; 2\kappa C\hat{A}^+{}_2 \sim \mathcal{M}^+\,.$$

  - Under the adopted gauge and boundary conditions:

$$A^i{}_\mu \sim \omega^i{}_\mu + \frac{3\nu}{2(2\nu+\sqrt{\nu^2+3})}\left(\frac{3+2\nu\sqrt{\nu^2+3}}{3\ell\nu}b^i{}_\mu + \frac{1}{a}\lambda^i{}_\mu\right),$$

$$\bar{A}_\mu \sim \frac{\ell}{2\bar{\kappa}\bar{a}_0}b^i{}_0\left(\frac{4a}{3}\omega_{i\mu} + \lambda_{i\mu} - \frac{a}{3\ell}\frac{2\nu^2+3}{\nu}b_{i\mu}\right)\,. \qquad (4.1)$$

- To illustrate practical aspects of the correspondence established above, we note that in thermodynamic applications, one needs a finite and differentiable action.
- The improved actions $\tilde{I}_{\rm CS}$ and $\tilde{I}_{\rm TMG}$ are:

$$\tilde{I}_{\rm CS} = I_{\rm CS} + B_{\rm CS} \,, \qquad B_{\rm CS} := -\bar{\kappa}\bar{a}_0 \int_{\partial\mathcal{M}} dt d\varphi \bar{A}_2 \,,$$

$$\tilde{I}_{\rm TMG} = (I_{\rm TMG})_{\rm r} + B_{\rm TMG} \,, \qquad B_{\rm TMG} = -\frac{1}{2}\int_{\partial\mathcal{M}} dt d\varphi \mathcal{E}^1 \,,$$

- In view of the asymptotic correspondence, the CS boundary term is seen to coincide with TMG$_\Lambda$ one:

$$B_{\rm CS} = B_{\rm TMG} \,, \qquad (4.2)$$

and the on-shell values of the improved actions are identical giving a deeper insight into the correspondence of the two theories.