Rindler Holography

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FWF

based on work w. H. Afshar, S. Detournay, W. Merbis, (B. Oblak), A. Perez, D. Tempo, R. Troncoso

Simple punchline

Heisenberg algebra

$$[X_n,\, P_m] = i\,\delta_{n,\,m}$$

fundamental not only in quantum mechanics
but also in near horizon physics

Outline

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Bekenstein–Hawking

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- Generalizations in 2+1 gravity/gravity-like theories (Galilean CFT, warped CFT, ...)

warped CFT: Detournay, Hartman, Hofman '12
Galilean CFT: Bagchi, Detournay, Fareghbal, Simon '13; Barnich '13

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- Main idea: consider near horizon symmetries for non-extremal horizons

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- Generalizations in 2+1 gravity/gravity-like theories (Galilean CFT, warped CFT, ...)
- Main idea: consider near horizon symmetries for non-extremal horizons
- Near horizon line-element with Rindler acceleration a:

$$\mathrm{d}s^2 = -2a\rho\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}\rho + \gamma^2\,\mathrm{d}\varphi^2 + \dots$$

Meaning of coordinates:

- ρ: radial direction ($\rho = 0$ is horizon)
- $\varphi \sim \varphi + 2\pi$: angular direction
- v: (advanced) time

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suggestion in `1511.08687`

We make this choice in this talk!

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- Work in 3d Einstein gravity in Chern–Simons formulation

$$I_{\mathsf{CS}} = \pm \sum_{\pm} \frac{k}{4\pi} \int \langle A^\pm \wedge \mathrm{d}A^\pm + \tfrac{2}{3} A^\pm \wedge A^\pm \wedge A^\pm \rangle$$

with $sl(2)$ connections $A^\pm$ and $k = \ell/(4G_N)$ with AdS radius $\ell = 1$

Outline

Diagonal gauge

Standard trick: partially fix gauge

$$A^{\pm} = b_{\pm}^{-1}(\rho)\,\big(\mathrm{d}+\mathfrak{a}_{\pm}(x^0,\,x^1)\big)b_{\pm}(\rho)$$

with some group element $b \in SL(2)$ depending on radius ρ with $\delta b = 0$

Drop $\pm$ decorations in most of talk

Manifold topologically a cylinder or torus, with radial coordinate ρ and boundary coordinates $(x^0, x^1) \sim (v, \varphi)$

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- Standard AdS$_3$ approach: highest weight gauge

$$\mathfrak{a} \sim L_+ + \mathcal{L}(x^0,\, x^1) L_- \qquad b(\rho) = \exp\left(\rho L_0\right)$$

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- Precise boundary conditions (ζ: chemical potential):

  $$\mathfrak{a} = \left(\mathcal{J}\ \mathrm{d}\varphi + \zeta\ \mathrm{d}v\right)\, L_0 \qquad \delta\mathfrak{a} = \delta\mathcal{J}\ \mathrm{d}\varphi\, L_0$$

  and $b = \exp\left(\frac{1}{\zeta}\, L_+\right) \cdot \exp\left(\frac{\rho}{2}\, L_-\right)$. (assume constant ζ for simplicity)

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$$g_{\mu\nu} = \frac{1}{2} \left\langle \left(A_\mu^+ - A_\mu^-\right) \left(A_\nu^+ - A_\nu^-\right) \right\rangle$$

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$$\mathrm{d}s^2 = -2a\rho f\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}\rho - 2\omega a^{-1}\,\mathrm{d}\varphi\,\mathrm{d}\rho + 4\omega\rho f\,\mathrm{d}v\,\mathrm{d}\varphi + \left[\gamma^2 + \frac{2\rho}{a}f(\gamma^2 - \omega^2)\right]\mathrm{d}\varphi^2$$

state-dependent functions $\mathcal{J}^\pm = \gamma \pm \omega$, chemical potentials $\zeta^\pm = -a \pm \Omega$

For simplicity set $\Omega = 0$ and $a = \text{const.}$ in metric above

EOM imply $\partial_v \mathcal{J}^\pm = \pm\partial_\varphi \zeta^\pm$; in this case $\partial_v \mathcal{J}^\pm = 0$

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state-dependent functions $\mathcal{J}^\pm = \gamma \pm \omega$, chemical potentials $\zeta^\pm = -a \pm \Omega$
Neglecting rotation terms ($\omega = 0$) yields Rindler plus higher order terms:

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Comments:

- Recover desired near horizon metric

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- Rindler acceleration a indeed state-independent

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- Two state-dependent functions (γ, ω) as usual in 3d gravity

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- $\gamma = \gamma(\varphi)$: "black flower"

Canonical boundary charges

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- Zero mode charges: mass and angular momentum

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Background independent result for Chern–Simons yields

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- Finite
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Meaningful near horizon boundary conditions and non-trivial theory!

Outline

Near horizon symmetry algebra

- Near horizon symmetry algebra = all near horizon boundary conditions preserving trafos, modulo trivial gauge trafos

  Most general trafo

  $$\delta_\epsilon \mathfrak{a} = \mathrm{d}\epsilon + [\mathfrak{a},\, \epsilon] = \mathcal{O}(\delta\mathfrak{a})$$

  that preserves our boundary conditions for constant ζ given by

  $$\epsilon = \epsilon^+ L_+ + \eta L_0 + \epsilon^- L_-$$

  with

  $$\partial_v \eta = 0$$

  implying

  $$\delta_\epsilon \mathcal{J} = \partial_\varphi \eta$$

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- Expand charges in Fourier modes

$$J_n^{\pm} = \frac{k}{4\pi} \oint \mathrm{d}\varphi \, e^{in\varphi} \mathcal{J}^{\pm}(\varphi)$$

What should we expect?

  - Virasoro? (spacetime is locally AdS_3)
  - BMS_3? (Rindler boundary similar to scri)
  - warped conformal algebra? (this is what we found for Rindleresque holography and what Donnay, Giribet, Gonzalez, Pino found in their near horizon analysis)

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$$\left[J_n^{\pm},\, J_m^{\pm}\right] = \pm\tfrac{1}{2} k n \delta_{n+m,\,0} \qquad \left[J_n^{+},\, J_m^{-}\right] = 0$$

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- Map

$$P_0 = J_0^+ + J_0^- \qquad P_n = \tfrac{i}{kn}\,(J_{-n}^+ + J_{-n}^-) \text{ if } n \neq 0 \qquad X_n = J_n^+ - J_n^-$$

  yields Heisenberg algebra (with Casimirs X_0, P_0)

$$[X_n,\, X_m] = [P_n,\, P_m] = [X_0,\, P_n] = [P_0,\, X_n] = 0$$

$$[X_n,\, P_m] = i\delta_{n,m} \quad \text{if } n \neq 0$$

Soft hair

- Vacuum descendants $|\psi(q)\rangle$

$$|\psi(q)\rangle \sim \prod (J^+_{-n_i^+})^{m_i^+} \prod (J^-_{-n_i^-})^{m_i^-} |0\rangle$$

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$$H := Q[\epsilon^{\pm}|_{\partial_v}] = aP_0$$

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same as energy of vacuum

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$\boxed{\text{Soft hair} = \text{zero energy excitations on horizon}}$

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- Zero-mode solutions with constant chemical potentials: BTZ

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$$S = 2\pi(J_0^+ + J_0^-) = \frac{A}{4G_N}$$

calculated directly in Chern–Simons formulation

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Before addressing microstates consider map to aymptotic variables

Map to asymptotic variables

- Usual asymptotic AdS$_3$ connection with chemical potential μ:

$$\hat{A} = \hat{b}^{-1}\big(\mathrm{d}+\hat{\mathfrak{a}}\big)\hat{b} \qquad \hat{\mathfrak{a}}_\varphi = L_+ - \tfrac{1}{2}\,\mathcal{L}\,L_-$$
$$\hat{b} = e^{\rho L_0} \quad \hat{\mathfrak{a}}_t = \mu L_+ - \mu' L_0 + \big(\tfrac{1}{2}\,\mu'' - \tfrac{1}{2}\,\mathcal{L}\mu\big)\,L_-$$

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$$\mathcal{L} = \tfrac{1}{2}\mathcal{J}^2 + \mathcal{J}'$$

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$$\mathcal{L} = \tfrac{1}{2}\mathcal{J}^2 + \mathcal{J}'$$

- Get Virasoro with non-zero central charge $\delta\mathcal{L} = 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon - \varepsilon'''$

Remarks on asymptotic and near horizon variables

- Asymptotic spin-2 currents fulfill Virasoro algebra, but charges obey still Heisenberg algebra

$$\delta Q = -\frac{k}{4\pi} \oint \mathrm{d}\varphi\, \varepsilon\, \delta\mathcal{L} = -\frac{k}{4\pi} \oint \mathrm{d}\varphi\, \eta\, \delta\mathcal{J}$$

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  Solved automatically from map to asymptotic observables; reminder:

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Near horizon boundary conditions natural for near horizon observer

Cardy counting

- Idea: use map to asymptotic observables to do standard Cardy counting
- Twisted Sugawara construction expanded in Fourier modes

$$kL_n = \sum_{p\in\mathbb{Z}} J_{n-p}J_p + iknJ_n$$

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$$S_{\text{Cardy}} = 2\pi\sqrt{kL_0^+} + 2\pi\sqrt{kL_0^-} = 2\pi(J_0^+ + J_0^-) = \frac{A}{4G_N} = S_{\text{BH}}$$

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Precise numerical factor in twist term crucial for correct results

Warped CFT counting

- Map near horizon algebra $J_n^\pm = \frac{1}{2}(J_n \pm K_n)$

$$Y_n \sim \sum J_{n-p} K_p \qquad T_n \sim J_n$$

to centerless warped conformal algebra

$$[Y_n,\, Y_m] = (n-m) Y_{n+m}$$
$$[Y_n,\, T_m] = -m T_{n+m}$$
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- Assuming $J^{\text{vac}} = 0$ yields

$$S = \beta H = S_{\text{BH}}$$

Hamiltonian H is product of BH entropy and Unruh temperature

Outline

Comparison to related approaches

- Brown, Henneaux '86

  Our boundary conditions differ from Brown–Henneaux — their chemical potentials depend on our charges and chemical potentials!

> Virasoro composite in terms of Heisenberg algebra

Comparison to related approaches

- Brown, Henneaux '86
- Donnay, Giribet, González, Pino 1511.08687
  - Observed already $H = TS_{\mathrm{BH}}$
  - Changing our bc's to

$$\mathrm{d}s^2 = -2a\rho\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}\rho - 2\omega a^{-1}\,\mathrm{d}\varphi\,\mathrm{d}\rho + 4\omega\rho\,\mathrm{d}v\,\mathrm{d}\varphi + \big[\gamma^2 + \tfrac{2\rho}{a}(\gamma^2 - \omega^2)\big]\,\mathrm{d}\varphi^2 + \mathcal{O}(\rho^2)$$

yields AKVs

$$\xi = T(\varphi)\partial_v + Y(\varphi)\partial_\varphi + \mathcal{O}(\rho^3)$$

  - Up to subleading terms same AKVs as DGGP

But: T and Y state-dependent for our boundary conditions!

Comment: map to Brown–Henneaux variables requires second chemical potential, not just Rindler acceleration!

Warped CFT algebra composite in terms of Heisenberg algebra

Comparison to related approaches

- Brown, Henneaux '86
- Donnay, Giribet, González, Pino 1511.08687
- Afshar, Detournay, DG, Oblak 1512.08233

Rindler acceleration state-dependent in that approach

Twisted warped CFT algebra composite in terms of Heisenberg algebra

Comparison to related approaches

- Brown, Henneaux '86
- Donnay, Giribet, González, Pino 1511.08687
- Afshar, Detournay, DG, Oblak 1512.08233
- Hawking, Perry, Strominger 1601.00921
    - We constructed explicitly gravitational soft hair
    - We find no soft hair contribution to black hole entropy
    - BMS_3 follows from Sugawara-like construction from Heisenberg algebra

> BMS algebra (supertranslations + superrotation) composite in terms of near horizon Heisenberg algebra

Comparison to related approaches

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- Hawking, Perry, Strominger 1601.00921
- Comment on complementarity:

> - Asymptotic Virasoro algebra composite from near horizon perspective
> - Same physics described naturally in different variables for asymptotic and near horizon observers
> - In particular, asymptotic chemical potentials depend on near horizon charges and chemical potentials

Elaborations and generalizations

- More on dual field theory — to be done
- Flat space
  - Similar story works!
  - Get centerless BMS_3 as composite algebra from Heisenberg algebra!
  - Soft hairy flat space cosmologies
  - Asymptotic chemical potentials again depend on near horizon charges and chemical potentials
  - Obtain again Bekenstein–Hawking entropy with no soft hair contribution

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- Higher spins — with Stefan Prohazka: similar story works!
- Lower spins — lowest spin gravity! (see Hofman, Rollier `1411.0672`)
- 4d — Does it work? Is there soft Heisenberg hair? Is BMS_4 composite? What are near horizon symmetries?

> Near horizon symmetries shed new light on soft hair, microstate counting and complementarity

Thanks for your attention!

H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo and R. Troncoso
"Soft Heisenberg hair on black holes in three dimensions,"
Phys.Rev.**D** [R] (2016), in print; `1603.04824`.

H. Afshar, S. Detournay, D. Grumiller and B. Oblak
"Near-Horizon Geometry and Warped Conformal Symmetry,"
JHEP **1603** (2016) 187; `1512.08233`.

Thanks to Bob McNees for providing the LaTeX beamerclass!

Bonus level: exact metric with generic chemical potentials

Our bc's for the connection $A^{\pm} = b_{\pm}^{-1}(\rho)\,\big(\mathrm{d}+\mathfrak{a}_{\pm}(x^0,\,x^1)\big)b_{\pm}(\rho)$ with

$$\mathfrak{a}_{\pm} = \big(\mathcal{J}_{\pm}\,\mathrm{d}\varphi + \zeta^{\pm}\,\mathrm{d}v\big)\;L_0$$

and $b_{\pm} = \exp\big(\frac{1}{\zeta^{\pm}}\,L_+\big)\cdot\exp\big(\frac{\rho}{2}\,L_-\big)$ lead to the metric

$$\mathrm{d}s^2 = \tfrac{1}{2}\,\big\langle\big(A^+_\mu - A^-_\mu\big)\,\big(A^+_\nu - A^-_\nu\big)\big\rangle\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

$$\begin{aligned}
&= \Bigg(-\frac{(\zeta^{+2}+\partial_v\zeta^+)(\zeta^{-2}+\partial_v\zeta^-)}{\zeta^{+2}\zeta^{-2}}\rho^2 + \frac{\zeta^{+3}\zeta^{-2}+\zeta^{+2}\zeta^{-3}+\partial_v\zeta^+\zeta^{-3}+\zeta^{+3}\partial_v\zeta^-}{\zeta^{+2}\zeta^{-2}}\rho + \frac{1}{4}(\zeta^- - \zeta^+)^2\Bigg)\,\mathrm{d}v^2 \\
&+ \Bigg(\frac{(-\zeta^{+2}-\partial_v\zeta^+)\partial_\varphi\zeta^- + (-\zeta^{-2}-\partial_v\zeta^-)\partial_\varphi\zeta^+ - \mathcal{J}_+\zeta^+\partial_v\zeta^- + \zeta^-(\mathcal{J}_-\zeta^{+2} - \mathcal{J}_+\zeta^+\zeta^- + \mathcal{J}_-\partial_v\zeta^+)}{2\zeta^{+2}\zeta^{-2}}\rho^2 \\
&+ \frac{\partial_\varphi\zeta^-\zeta^{+3} + \partial_\varphi\zeta^+\zeta^{-3} + \mathcal{J}_+\zeta^{+2}\partial_v\zeta^- - \zeta^-\Big(\mathcal{J}_-\partial_v\zeta^+\zeta^- + \zeta^+(\zeta^- + \zeta^+)(\zeta^+\mathcal{J}_- - \zeta^-\mathcal{J}_+)\Big)}{2\zeta^{+2}\zeta^{-2}}\rho \\
&- \frac{1}{4}(\zeta^- - \zeta^+)(\mathcal{J}_- + \mathcal{J}_+)\Bigg)\,\mathrm{d}v\,\mathrm{d}\varphi + \Bigg(1 + \frac{\partial_v\zeta^-\zeta^{+2} + \partial_v\zeta^+\zeta^{-2}}{2\zeta^{+2}\zeta^{-2}}\Bigg)\,\mathrm{d}v\,\mathrm{d}\rho \\
&+ \Bigg(\frac{(\mathcal{J}_+\zeta^+ + \partial_\varphi\zeta_+)(\mathcal{J}_-\zeta^- - \partial_\varphi\zeta^-)}{\zeta^{+2}\zeta^{-2}}\rho^2 + \frac{\mathcal{J}_+\partial_\varphi\zeta^-\zeta^{+2} - \zeta^-\mathcal{J}_-(\zeta^-\partial_\varphi\zeta^+ + \mathcal{J}_+\zeta^+(\zeta^- + \zeta^+))}{\zeta^{+2}\zeta^{-2}}\rho \\
&+ \frac{1}{4}(\zeta^- + \zeta^+)^2\Bigg)\,\mathrm{d}\varphi^2 + \Bigg(\frac{\mathcal{J}_+\zeta^+\zeta^{-2} - \mathcal{J}_-\zeta^{+2}\zeta^- + \partial_\varphi\zeta^+\zeta^{-2} + \partial_\varphi\zeta^-\zeta^{+2}}{2\zeta^{+2}\zeta^{-2}}\Bigg)\,\mathrm{d}\varphi\,\mathrm{d}r
\end{aligned}$$