Black Holes I VU

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Contents

Preface

These lecture notes were written as part of three student projects at TU Wien (by Stefan Prohazka, Max Riegler and Sebastian Singer, supervised by Daniel Grumiller) in 2009/2010. The current version 0.0 was edited by Stefan Prohazka and corrected by Daniel Grumiller. However, the authors would be surprised if this first public version was free of mistakes.

If you find typos or errors please contact grumil@hep.itp.tuwien.ac.at.

Note on units

If not noted otherwise this script will use natural units (also known as Planck units) where we set human conversion factors equal to one, i.e. $c = \hbar = G_N = k_B = 1$, where c is the speed of light (Einstein's constant), $\hbar$ is Planck's constant, G_N is Newton's constant and k_B is Boltzmann's constant.

Note that this is not neglecting anything, it just amounts to a more convenient choice of units than the historically grown ones. $c = 1$ means we measure time in the same units as distances; $\hbar = 1$ means we measure additionally energy in inverse units of time; $G_N = 1$ then means that we set the Planck mass (and thus Planck length and Planck time) to unity and measure everything else in Planck units; $k_B = 1$ means we measure information in e-bits and that energy and temperature have the same units. For an enjoyable paper on dimensionfull and dimensionless constants see http://arxiv.org/abs/1412.2040 (it is interesting to note how many well-known physicists appear to be confused about units).

Further reading

Information about running lectures by Daniel Grumiller and additional resources can be found at the teaching webpage http://quark.itp.tuwien.ac.at/ grumil/teaching.shtml.

Here is further selected literature:

- Einstein gravity in a nutshell, (A. Zee, 2013, Princeton U. Press)
- Spacetime and Geometry: An Introduction to General Relativity, (S. Carroll, 2003, Addison Wesley)
- Gravitation und Kosmologie, (R.U. Sexl and H.K. Urbantke, 1987, Wissenschaftsverlag, Mannheim/Wien/Zürich)
- General Relativity, (R. Wald, 1984, U. Chicago Press, Chicago)
- Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (S. Weinberg, 1972, John Wiley)
- The large scale structure of space-time, (S.W. Hawking and G.F.R. Ellis, 1973, Cambridge University Press, Cambridge)
- Accretion Power in Astrophysics (J. Frank, A. King and D. Raine, 2002, Cambridge University Press, Cambridge)
- Active galactic nuclei: from the central black hole to the galactic environment (J. Krolik, 1998, Princeton University Press, Princeton)
- Black Hole Physics: Basic Concepts and New Developments (V.P. Frolov and I.D. Novikov, 1998, Springer, New York)
- Gravitation, (C. Misner, K.S. Thorne and J.A. Wheeler, 1973)

1. Historical Overview

As an introduction to black hole physics we want to start our lecture notes by a brief historical overview on black hole science.

- **O.C. Rømer (1676):** the speed of light is finite
- **I. Newton (1686):** the law of gravity

$$F_r = -G_N \frac{mM}{r^2} \tag{1.1}$$

- **J. Michell (1783):** referring to Newtonian black holes, “All light emitted from such a body would be made to return towards it by its own proper gravity”
- **P.S. Laplace (1796):** “Exposition du systéme du Monde” (“dark stars”)
- **T. Young (1801):** interference experiments confirm Huygens’ theory of the wave nature of light; Newton’s theory of light is dead, and so are dark stars
- **A. Einstein (1905):** Special Relativity
- **A. Einstein (1915):** General Relativity (GR)
- **K. Schwarzschild (1916):** first exact solution of GR is a black hole!
- **S. Chandrasekhar (1931):** gravitational collapse of Fermi gas
- **A. Eddington (1935):** regarded the idea of black holes with skepticism, “I think there should be a law of Nature to prevent a star from behaving in this absurd way!”
- **M. Kruskal; G. Szekeres (1960):** global structure of Schwarzschild spacetime
- **R. Kerr (1963):** exact (and essentially unique) rotating (and charged) black hole solution sparks interest of astrophysics community
- **Cygnus X-1 (1964):** first detection of X-ray emission from a black hole in a binary system (though realized only in 1970ties that it might be black hole; conclusive evidence only in 1990ies)
- **J. Wheeler (December 1967):** invention of the term “black hole”
- **S. Hawking and R. Penrose (1970):** black holes contain singularities
- **J. Bekenstein (1972):** speculation that black holes might have entropy
- **N.I. Shakura and R.A. Sunyaev (1972):** first accretion disk model
- **J. Bardeen, B. Carter and S. Hawking (1973):** four laws of black hole mechanics
- **S. Hawking (1974):** black holes evaporate due to quantum effects
- **W. Unruh (1981):** black hole analogs in condensed matter physics
- **S. Deser, R. Jackiw, C. Teitelboim et al. (1982):** gravity in lower dimensions
- **E. Witten et al. (1984):** first superstring revolution

- **H.-P. Nollert; N. Andersson (1992):** quasinormal modes of a "ringing" Schwarzschild black hole
- **M. Bañados, C. Teitelboim and J. Zanelli (1992):** black holes in $2+1$ dimensions
- **M. Choptuik (1993):** Critical collapse in numerical relativity discovered
- **G. 't Hooft and L. Susskind (1993):** holographic principle
- **M. Veltman (1994):** black holes still sometimes regarded with skepticism, "Black holes are probably nothing else but commercially viable figments of the imagination."
- **J. Polchinski (1995):** p-branes and second superstring revolution
- **A. Strominger and C. Vafa (1996):** microscopic origin of black hole entropy
- **J. Maldacena (1997):** AdS/CFT correspondence
- **S. Dimopoulos and G.L. Landsberg; S.B. Giddings and S. Thomas (2001):** black holes at the LHC?
- **Saggitarius A* (2002):** supermassive black hole in center of Milky Way
- **R. Emparan and H. Reall (2002):** black rings in five dimensions
- **G. 't Hooft (2004):** "It is however easy to see that such a position is untenable." (comment on Veltman a decade earlier)
- **S. Hawking (2004):** concedes bet on information paradox; end of "black hole wars"
- **P. Kovtun, D. Son and A. Starinets (2004):** viscosity in strongly interacting quantum field theories from black hole physics
- **F. Pretorius (2005):** breakthrough in numerical treatment of binary problem
- **C. Barcelo, S. Liberati, and M. Visser (2005):** "Analogue gravity" - black hole analogon in condensed matter physics
- **J.E. McClintock et al. (2006):** measuring of spin of GRS1915+105 – nearly extremal Kerr black hole!
- **E. Witten (2007) and W. Li, W. Song and A. Strominger (2008):** quantum gravity in three dimensions?
- **S. Hughes (2008):** "Unambiguous observational evidence for the existence of black holes has not yet been established."
- **S. Hughes (2008):** "Most physicists and astrophysicists accept the hypothesis that the most massive, compact objects seen in many astrophysical systems are described by the black hole solutions of general relativity."
- **S. Gubser; S. Hartnoll, C. Herzog and G. Horowitz (2008):** "holographic superconductors"
- **D. Son; K. Balasubramanian and J. McGreevy (2008):** black hole duals for cold atoms proposed

More recent developments are not included in this list, but will be updated in the list presented in the first lectures of "Black holes I".

2. Gravitational Collapse – Chandrasekhar Limit

Four fundamental interactions are known to physicists today - *gravitational*, *electromagnetic*, *strong* and *weak interaction*.
On large (cosmological) scales, only gravity – the weakest of these forces – plays a major role. This is easily understood by the facts that the nuclear forces are extremely short ranged (about the radius of nuclei) and that our Universe is electrically neutral on large scales.

During the history of the Universe it was gravity that intensified local density fluctuations. This process finally led to the formation of planets, stars, galaxies and even black holes.

So, in order to understand the formation of a black hole, we must investigate the influence of gravity in stellar dynamics. Accordingly, the aim of this chapter will be to derive approximate stability limits for stars.

2.1. Chandrasekhar Limit

As long as a star can fusion lighter elements into more heavy ones, the thermal and radiation outward pressure counteracts gravitational collapse. Only when the end of stellar fusion is reached, gravitational collapse can begin. This process continues until all energy levels up to the Fermi level are occupied by the star's electrons. At that point the resulting Fermi pressure (caused by Pauli's exclusion principle) of the degenerate Fermi gas prevents further collapse. Now, we derive a limit where the Fermi pressure balances the gravitational force. In our derivation F_G is the force of gravity, ρ the density of the star, M its mass, R its radius, P is (gravitational) pressure, A an area element of the star's surface; finally E_F denotes the Fermi energy and m_N the nucleon mass. f is the equation of state of the given system.) Also, we drop most factors of the order of unity, because we are solely interested in orders of magnitude.

$$F_G \sim \frac{M\rho R^3}{R^2} \qquad P \sim \frac{F_G}{A} \sim \frac{F_G}{R^2} \tag{2.1}$$

$$\frac{P}{\rho} \sim \frac{M}{R} \tag{2.2}$$

$$\frac{P}{\rho} \sim f(\rho) \sim \frac{E_F}{m_N} \tag{2.3}$$

Equation (2.3) is valid because in the star we can consider a degenerate Fermi gas, where the equation of state is independent of the temperature T. We distinguish between relativistic and non-relativistic case for the Fermi energy:

$$E_F = \begin{cases} \text{non-rel.} & \frac{p_F^2}{2m_e} \\ \text{relat.} & p_F \end{cases} \tag{2.4}$$

Here p_F is the Fermi momentum, which is in the same order of magnitude as the de-Broglie wavelength. Therefore it is proportional to $\frac{1}{d}$, where d is the typical distance between two electrons in the collapsing star. Additionally, we get for ρ

$$\rho \propto \frac{m_N}{d^3} \qquad \Rightarrow \qquad p_F \sim \left(\frac{\rho}{m_N}\right)^{1/3}. \tag{2.5}$$

While the conclusions would not change drastically, it is a good assumption to consider the electrons as relativistic. Then, with (2.2)-(2.5) we get

$$\frac{M}{R} \sim \frac{P}{\rho} \sim \frac{p_F}{m_N} \sim \left(\frac{\rho}{m_N}\right)^{\frac{1}{3}} \frac{1}{m_N} \tag{2.6}$$

Using

$$R \sim \left(\frac{M}{\rho}\right)^{\frac{1}{3}} \tag{2.7}$$

yields

$$\frac{M}{M^{\frac{1}{3}}} \rho^{\frac{1}{3}} \sim \rho^{\frac{1}{3}} \frac{1}{(m_N)^{4/3}} \, . \tag{2.8}$$

Thus we establish our estimate

$$\boxed{M \sim \frac{1}{m_N^2}} \tag{2.9}$$

Our estimate is independent of the electron mass—so the mass of the particles that cause the Fermi pressure are not relevant for a stellar mass limit estimate. To get a grasp of the magnitude of the Chandrasekhar limit, let us insert the neutron mass $m_N \approx 10^{-19}$

$$\mathrm{M_{Ch}} \sim 10^{38} \sim 10^{30}\mathrm{kg} \sim \mathrm{M}_{\odot} \tag{2.10}$$

where $\mathrm{M}_{\odot}$ is the Sun's mass. More detailed calculations show for the Chandrasekhar limit:

$$\mathrm{M_{Ch}} \approx 1.4 \ \mathrm{M}_{\odot} \tag{2.11}$$

We now discuss what happens when a star collapses to a neutron star. There are two major nuclear reactions ongoing while a neutron star is formed:

$$p^+ + e^- \to n + \nu_e \qquad n \to p^+ + e^- + \overline{\nu_e} \tag{2.12}$$

These are inverse and "normal" β-decay, respectively. Whilst the second reaction is favored in vacuum the first one is predominant in neutron stars since almost every energy level up to the Fermi niveau is filled with electrons. Hence, the second reaction is forbidden by Pauli's principle and all electron-proton pairs are subsequently converted into neutrons.

Up until now we only talked about "normal" stars collapsing into neutron stars; but we have made no statements about the possible collapse of a neutron star into a black hole. Precisely the same estimation we made above can be conducted for neutron stars – only the m_e-terms have to be exchanged with m_N-terms in all formulas that led to the estimate (2.9). But since the latter does not depend on m_e, we obtain the same estimate (2.9) for the mass limit of the neutron stars.

A far more exact (but as well more complicated) way of determining the neutron-star mass limit is solving the Tolman–Oppenheimer–Volkov equation. This results in:

$$\mathrm{M_{TOV}} \sim (1.5 - 3)\mathrm{M}_{\odot} \tag{2.13}$$

The large error bars of the result (2.13) originate in the fact that the equations of state governing neutron-stars are not fully understood in detail yet.

In conclusion, cold matter stars slightly heavier than the sun collapse to neutron stars.[1] Neutron stars that exceed the TOV limit (2.13), $M > M_{\mathrm{TOV}}$, then collapse to a black hole. Therefore, black holes emerge from common objects in our Universe, namely old stars that are not so different from our Sun, but slightly heavier.

[1]Stars, like the sun, that are still in the process of burning hydrogen to helium need at this stage at least a mass of approximately $15M_{\odot}$ to form (after a supernova explosion) a neutron star and at least a mass of $20M_{\odot}$ to form (again after a supernova explosion) a black hole.

3. Phenomenology of and Experiments with Black Holes

3.1. “Fishy” Gedankenexperiment

Imagine a pond populated by two kinds of fish — the fast gammas and the slower alphas. The tranquility of the pond is only disturbed by a small creek that flows out of it. Since our fish are curious, they send an alpha explorer down the creek. Unfortunately they do not know that the speed of the water-flow continuously increases down the creek. So our poor alpha ultimately gets to a point where the water-speed exceeds his maximum swimming capability, a point of no-alpha-return. We call this the α-horizon. From the viewpoint of the fish in the pond something really bad must happen to the alpha at that point; maybe a bigger fish is swallowing it there or a fisherman is catching it. They do not know for sure what is going on but it definitely must be something “fishy”. But, and that is the curious thing, for the alpha fish only the water-speed increases a little there. As the pond fish get no sign of life from the brave alpha explorer, they send a fast gamma to look for it. The gamma finally reaches the alpha thereby crossing the α-horizon unharmed.

The gamma returns to the pond, with considerable effort due to the meanwhile quite high current and tells the pond-dwellers alpha’s fate. Matter-of-factly these fish are quite high on the evolutionary ladder, so their curiousness beats the concerns about risking the life of another fish. So the heroic gamma again throws itself down the creek. Again, it passes the α-horizon and again he meets alpha. But now, something has happened: the current has increased so much that even the fastest fish in pond, the gamma-explorer, can not swim back anymore. We call the point at which the current speed is equal to gamma’s speed the “black-hole-horizon”. Here again, something “fishy” is noticed by the pond fish, whereas gamma and alpha only feel the slight increase in water-speed. For a picture of the pond with all its interesting points, see figure 3.1.

So both alpha and gamma are doomed to travel on downwards. Unlucky as they are, the creek ends in a ripping waterfall! With the help of a great portion luck, both our fish survive their ride on the waterfall. They discover that the creek continues to a new pond — obviously getting slower and slower as they get nearer to the new pond.

After living there a while, our fish get homesick and try to swim back up the creek. But, and that is the sad ending of our short story, the countercurrent is too strong for our gamma at a certain point. So alpha and gamma have to stay in the new pond ...

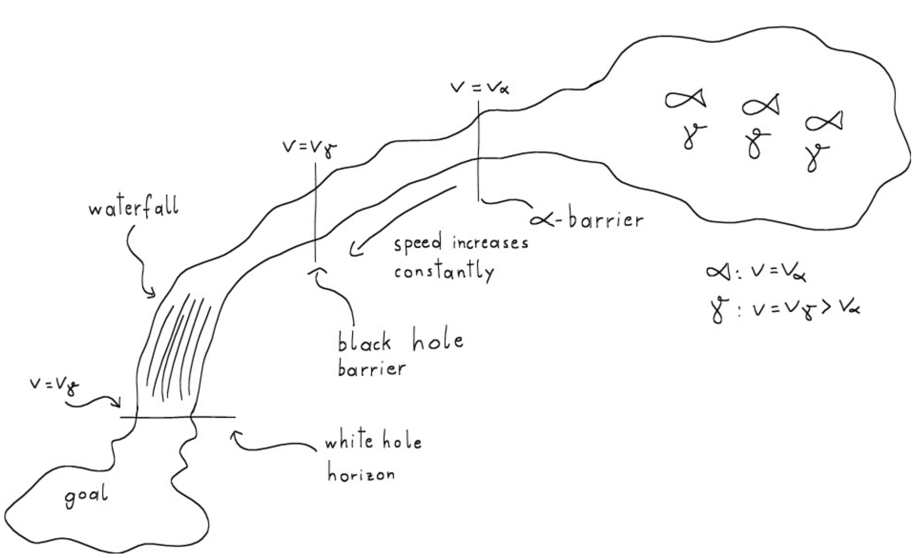

Figure 3.1: The pond and the waterfall

Now, back to physics: As the attentive reader might have noticed, alpha and gamma fish are just the analogs of alpha and gamma particles, respectively. Furthermore, the speed of the current flowing from our pond (the "flat" Universe) is nothing more than gravitational strength. And the waterfall just resembles the singularity of the black hole. Therefore, an observer located anywhere between the α-horizon and the black-hole (event) horizon is able to communicate with and return to the "outside" (i.e. the flat Universe). But if the event horizon is crossed, no interaction whatsoever with anything outside this boundary is possible. Ultimately, the observer is drawn into the singularity.

The reversed process, the current flowing from the waterfall, resembles a "white hole" - a (hypothetical) stellar object which is just the time-reversed of a black hole. No matter, however fast, can move *into* it. Matter is only allowed to travel outwards.

3.2. Brief Review of Special Relativity

As we see special relativity as a prerequisite to this course, this chapter is going to be quite short emphasizing only the most important aspects.

In the Newtonian Universe, all changes in force at a specific point effects the rest of the Universe instantaneously. This means that information travels at infinite speed in this Universe. Speaking in other terms, there is only one global, unique time throughout the whole Universe.

The finite speed of light and its invariance under chance of inertial frame contradicts the Newtonian world-view. Albert Einstein, in his 1905 paper "*Zur Elektrodynamik bewegter Körper*" ("*On the Electrodynamics of Moving Bodies*"), was able to solve the various contradictions by abandoning a global time and an invariant length. Both quantities are now dependent on the observer's state of motion. Both time dilation and Lorentz contraction are consequences of Einstein's possibly most severe assumptions: no particle is allowed to cross the speed of light in either direction "normal" particles can not go faster than c and hypothetical "tachyons" can not be decelerated to speeds lower than c and the speed of light is constant and thus equal in every inertial system.

This new "relative" Universe is best described by the so-called Minkowski-spacetime. This 4-dimensional spacetime consists of three spatial dimensions and time as fourth dimension.

The relativity of time leads us to a new definition of causality and simultaneity. Mathematically the coordinate relationships between two moving observers are given by the Lorentz-transformations, which will be discussed in a moment. Due to the fact that the speed of light is constant and equal in every inertial system, we can draw a light cone at every point of a given objects world-line.

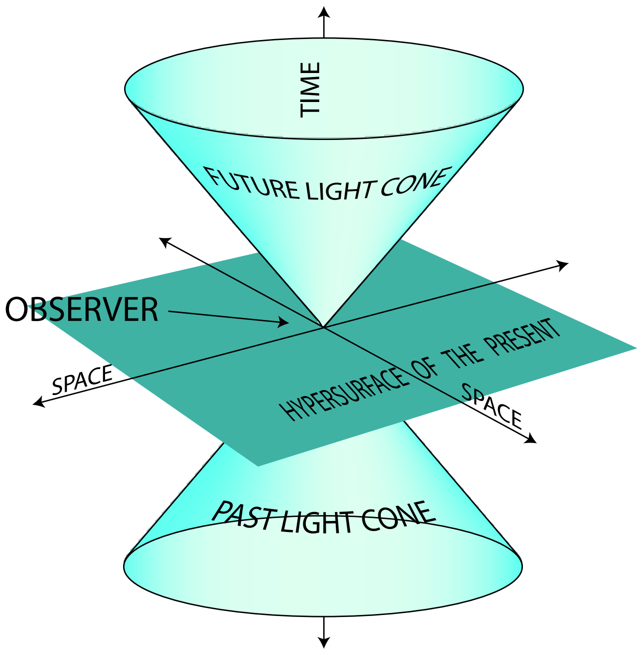

Figure 3.2: The Light Cone (Source: Aainsqatsi/Stib, CC-BY-SA-3.0, via Wikimedia Commons)

As we see, the light cone divides the spacetime into three different regions with respect to any given observer.

1. causal past and future.

   All points enveloped by the light cone can be reached with speeds $\leq c$ by the observer at a given time - likewise, all points in the observers causal past could have affected him. We call straight lines connecting the observer and any point in the interior of this region timelike.

2. the light cone itself.

   A light ray sent outwards by the observer travels on the light cone - therefore the light cone itself is the boundary of the region an observer can send messages to (again, at a given time). Accordingly, he could only have received messages that originated within or on the boundary of the past light cone. We call straight lines connecting the observer and any point on the light cone lightlike.

3. the “elsewhere”.

   The region which lies outside the light cone can not influence and be influenced by the observer, because to do so, traveling at speeds greater then c would be necessary. We call straight lines connecting the observer and any point in the interior of this region spacelike.

3.3. Mathematical Aspects of Special Relativity

To describe the special causality structure of the Minkowski-spacetime we need a pseudo-Euclidean metric. With $\mu, \nu = t, x, y, z$, it reads:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

Accordingly,with a^μ,$b^\nu \in \mathbb{R}^4$ we define the inner product:

$$a \cdot b = a^\mu b^\nu \eta_{\mu\nu} \tag{3.2}$$

The Minkowski metric is called a pseudo-Euclidean metric because its norm is not positive definite.

$$\|a\| = a^\mu a^\nu \eta_{\mu\nu} \begin{cases} > 0 & : \text{spacelike} \\ = 0 & : \text{lightlike; with } a^\mu \neq 0 \\ < 0 & : \text{timelike} \end{cases} \tag{3.3}$$

In Euclidean space we can change from on set of relative coordinates to another using arbitrary rotations (here, for simplicity, in a 2-dimensional form):

$$\Lambda = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \in SO(2) \tag{3.4}$$

$$\Lambda^T \delta \Lambda = \delta \tag{3.5}$$

In 2-dimensional Minkowski spacetime we have to use a hyperbolic rotation matrix to take respect of the different metric:

$$\Lambda = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \in SO(1,1) \tag{3.6}$$

With the quantity ξ, sometimes called “rapidity” being defined as

$$\cosh\xi = \frac{1}{\sqrt{1-v^2}} = \gamma \tag{3.7}$$

To emphasize the fact that time plays no “special” role (i.e. it is just another coordinate) in special relativity, an exemplary coordinate transformation is shown here:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \gamma \begin{pmatrix} t - vx \\ x - vt \end{pmatrix} \tag{3.8}$$

4. Metric and Geodesic Equation

In this chapter we will recall metrics in different coordinate systems and we are going to derive the geodesic equation which represents the equation of motion for a point particle in curved spacetime.

If we perform an arbitrary change of coordinates in special relativity then the Minkowski metric $\eta_{\mu\nu}$ is transformed into a new metric $g_{\mu\nu}$. In order to find this new metric $g_{\mu\nu}$ we have to perform an appropriate sufficiently smooth coordinate transformation by mapping the old coordinates to the new ones

$$x^i \to \tilde{x}^i(x^k) \tag{4.1}$$

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^k} dx^k \qquad \tilde{\partial}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \partial_j \tag{4.2}$$

Therefore an infinitesimal line element in the new coordinates can be written as

$$ds^2 = \eta_{ij} dx^i dx^j = g_{ij} d\tilde{x}^i d\tilde{x}^j = g_{ij} \frac{\partial \tilde{x}^i}{\partial x^k} dx^k \frac{\partial \tilde{x}^j}{\partial x^l} dx^l \tag{4.3}$$

Since ds^2 has to be invariant due to coordinate transformations we obtain the following relation between the components of the new metric components g_{ij} and the ones of the old metric components η_{ij} by comparing the coefficients of (4.3)

$$\eta_{kl} = g_{ij} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^l} \tag{4.4}$$

4.1. Euclidean Coordinate Transformation

Let us consider a 2-dimensional euclidean metric $\delta_{ij} \to ds^2 = dx^2 + dy^2$ and a coordinate transformation to polar coordinates $r = \sqrt{x^2 + y^2}$ and $\varphi = \arctan(\frac{y}{x})$. Using (4.3) we obtain for the line element

$$ds^2 = g_{ij} d\tilde{x}^i d\tilde{x}^j = g_{rr} dr^2 + 2 g_{r\varphi} dr d\varphi + g_{\varphi\varphi} d\varphi^2 \tag{4.5}$$

After evaluating the total derivatives of the new coordinates the line element can be written as

$$ds^2 = g_{rr} \left(\frac{xdx + ydy}{\sqrt{x^2+y^2}} \right)^2 + 2 g_{r\varphi} \left(\frac{xdx + ydy}{\sqrt{x^2+y^2}} \right) \left(\frac{xdy - ydx}{x^2+y^2} \right) + g_{\varphi\varphi} \left(\frac{xdy - ydx}{x^2+y^2} \right)^2 \tag{4.6}$$

By rearranging the right hand side of (4.6) we get

$$ds^2 = dx^2 \underbrace{\left(g_{rr} \frac{x^2}{x^2+y^2} - 2 g_{r\varphi} \frac{xy}{(x^2+y^2)^{\frac{3}{2}}} + g_{\varphi\varphi} \frac{y^2}{(x^2+y^2)^2} \right)}_{A} + \tag{4.7a}$$

$$dy^2 \underbrace{\left(g_{rr} \frac{y^2}{x^2+y^2} + 2 g_{r\varphi} \frac{xy}{(x^2+y^2)^{\frac{3}{2}}} + g_{\varphi\varphi} \frac{x^2}{(x^2+y^2)^2} \right)}_{B} + \tag{4.7b}$$

$$dxdy \underbrace{\left(g_{rr} \frac{2xy}{x^2+y^2} + 2 g_{r\varphi} \frac{x^2 - y^2}{(x^2+y^2)^{\frac{3}{2}}} - g_{\varphi\varphi} \frac{2xy}{(x^2+y^2)^2} \right)}_{C} \tag{4.7c}$$

Since ds^2 is invariant under coordinate transformations it follows that $A = B = 1$ and $C = 0$. This yields three linear equations in three variables

$$A + B = 2 = g_{rr} + g_{\varphi\varphi}\frac{1}{r^2} \tag{4.8a}$$

$$A = 1 = (g_{rr}x^2 + g_{\varphi\varphi}\frac{y^2}{r^2})\frac{1}{r^2} - 2g_{r\varphi}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} \tag{4.8b}$$

$$C = 0 = (g_{rr} - g_{\varphi\varphi}\frac{1}{r^2})\frac{2xy}{x^2+y^2} + 2g_{r\varphi}\frac{x^2-y^2}{(x^2+y^2)^{\frac{3}{2}}} \tag{4.8c}$$

$$\Rightarrow g_{rr} = 1 \qquad g_{\varphi\varphi} = r^2 \qquad g_{r\varphi} = 0 \tag{4.8d}$$

Hence the line element in polar coordinates is given by

$$ds^2 = g_{ij}dx^i dx^j = dr^2 + r^2 d\varphi^2 \tag{4.9}$$

4.2. The Geodesic Equation

In order to derive the geodesic equation we will consider two arbitrary points in spacetime. There are two possible ways to find the minimal distance between the two points. The first is the so-called parallel transport where all possible vectors are drawn outwards from one point, then they are parallel transported until one of these vectors finally “hits” the target[2]. The other method is to find a curve of minimal length connecting the two points by variational calculus. We will use this way to derive the geodesic equation, with the locally shortest connection of two points being called a geodesic.
For an arbitrary curve the arc length can be obtained by evaluation of the following integral in the special case of an euclidean metric

$$\int_{\tau_0}^{\tau_1} d\tau \sqrt{\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} \tag{4.10}$$

Similarly for a Minkowski metric and spacelike line elements with $ds^2 > 0$ the arc length is given by

$$\int_{\tau_0}^{\tau_1} d\tau \sqrt{\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} \tag{4.11}$$

and for timelike line elements with $ds^2 < 0$

$$\int_{\tau_0}^{\tau_1} ds \sqrt{-\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} \tag{4.12}$$

4.2.1. Geodesics in Euclidean Space

Consider a line element in 2 dimensional euclidean space and $y = y(x) : ds^2 = dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$. Hence the arc length of this line element can be written as

$$s = \int_{x_0}^{x_1} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{4.13}$$

[2] In order to be able to parallel transport a vector one has to find of course a satisfying definition for parallel transported vectors in curved spacetime

By variation of the arc length (4.13) we can find the path $y(x)$ with minimal arc length such that $\delta s = 0$.

$$\delta s = \int_{x_0}^{x_1} dx \left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \frac{dy}{dx} \frac{d}{dx} \delta y \right) \tag{4.14}$$

In order to get rid of the derivative acting on the variation partial integration can be used

$$\delta s = \underbrace{\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \frac{dy}{dx} \delta y \Big|_{x_0}^{x_1}}_{0} - \int_{x_0}^{x_1} dx \delta y \underbrace{\frac{1}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}}_{>0} \left(\frac{d^2 y}{dx^2} \right) \tag{4.15}$$

The boundary term can be dropped by choosing appropriate boundary conditions. Since the square root is greater than zero for arbitrary $y(x)$ the variation can only be zero for

$$\left(\frac{d^2 y}{dx^2} \right) = 0 \qquad \text{or} \qquad \frac{dy}{dx} = \pm\infty \tag{4.16}$$

which is the equation of a straight line in euclidean space in either of the cases. This is indeed the shortest path between two points in euclidean space.

4.2.2. Timelike Geodesics

For an arbitrary metric, a geodesic minimizes the arc length S which for timelike curves is given by

$$S = \int_{s_0}^{s_1} ds = \int_{\tau_0}^{\tau_1} d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \tag{4.17}$$

The minus sign in front of the metric ensures reality of S for timelike curves. In order to get rid of the square root in S we use a little trick by introducing the einbein. The einbein is a variable which can be viewed as a parameter "measuring" how fast the curve is being traversed as a function of the parameter. Hence the arc length (4.17) can be rewritten as

$$S = \frac{1}{2} \int_{\tau_0}^{\tau_1} d\tau \, e \left(1 - e^{-2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) \tag{4.18}$$

In order to show that this rewritten arc length (4.18) is indeed equal to the original form of the arc length (4.17) the variation with respect to the einbein of the rewritten arc length (4.18) has to vanish

$$\frac{\delta S}{\delta e} = \frac{1}{2} \int_{\tau_0}^{\tau_1} d\tau \left[1 + e^{-2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] = 0 \tag{4.19a}$$

$$\Rightarrow e = \pm \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \tag{4.19b}$$

Since the rewritten arc length containing the einbein (4.18) equals the original expression of the arc length (4.17), the rewritten arc length (4.18) can be varied instead of the original one (4.17) (∂_α denotes $\frac{\partial}{\partial x^\alpha}$).

$$\begin{aligned}\delta S =& \frac{1}{2}\int_{\tau_0}^{\tau_1} d\tau \Bigg[-e^{-2}\left(\partial_\alpha g_{\mu\nu}\right)\delta x^\alpha \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \\ & \frac{d}{d\tau}\left(e^{-2} g_{\mu\nu} \underbrace{\delta x^\mu}_{\delta_\alpha{}^\mu \delta x^\alpha} \frac{dx^\nu}{d\tau} + e^{-2} g_{\mu\nu} \underbrace{\delta x^\nu}_{\delta_\alpha{}^\nu \delta x^\alpha} \frac{dx^\mu}{d\tau}\right)\Bigg] = 0 \end{aligned} \tag{4.20}$$

The variation of the arc length (4.20) has to be zero for arbitrary δ_α (($\ddot{x}^\mu$) denotes $\frac{d^2x^\mu}{d\tau^2}$).

$$\begin{aligned} & e^{-2} g_{\alpha\nu}\ddot{x}^\nu + e^{-2} g_{\mu\alpha}\ddot{x}^\mu + e^{-2}\left(\partial_\beta g_{\alpha\nu}\right)\dot{x}^\beta \dot{x}^\nu + e^{-2}\left(\partial_\beta g_{\mu\alpha}\right)\dot{x}^\beta\dot{x}^\mu - \\ & e^{-2}\left(\partial_\alpha g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu + \frac{de^{-2}}{d\tau}\left(g_{\mu\nu}\delta_\alpha{}^\mu \frac{dx^\nu}{d\tau} + g_{\mu\nu}\delta_\alpha{}^\nu\frac{dx^\mu}{d\tau}\right) = 0 \end{aligned} \tag{4.21}$$

The term $\frac{de^{-2}}{d\tau}(\dots)$ can be eliminated by a reparametrization $d\tau' = e\,d\tau$ which is called affine parametrization. Since we want to have an equation of the form $\ddot{x}^\mu + (\dots) = 0$ we multiply the reparametrized expression of the variated arc length (4.21) with $e^2 g^{\alpha\gamma}$, rename some indices ($\ddot{x}^\mu \to \ddot{x}^\nu$ and $\beta \to \mu$) and use $g_{\alpha\gamma}g^{\gamma\delta} = \delta_\alpha{}^\delta$.

$$\begin{aligned} & 2\delta_\nu{}^\gamma \ddot{x}^\nu + \dot{x}^\mu\dot{x}^\nu\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)g^{\alpha\gamma} = 0 \\ \Rightarrow\ & \ddot{x}^\gamma + \frac{1}{2}\dot{x}^\mu\dot{x}^\nu\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)g^{\alpha\gamma} = 0 \end{aligned} \tag{4.22}$$

With

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) \tag{4.23}$$

being the Christoffel symbols of the first kind. By contracting one index with the metric and multiplying the factor $\frac{1}{2}$ we obtain the Christoffel symbols of the second kind $\Gamma^\gamma{}_{\mu\nu}$

$$\boxed{\Gamma^\gamma{}_{\mu\nu} = g^{\alpha\gamma}\Gamma_{\alpha\mu\nu}} \tag{4.24}$$

Hence minimum condition for a geodesic (4.22) can be written as

$$\boxed{\ddot{x}^\gamma + \Gamma^\gamma{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0} \tag{4.25}$$

Equation (4.25) is called geodesic equation and determines the equations of motion in curved spacetime, or more general it defines the geodesics in a curved space. To quote John Archibald Wheeler: “Space tells matter how to move and matter tells space how to curve”. The geodesic equation does indeed relate to the first part of this quote, i.e that the movement of a point particle can be determined by the curvature of the spacetime. We will derive the equations that will motivate the second part of this quote in chapter 7.

4.2.3. Geodesics in a Special Metric: The Newton Limit

Consider a metric of the form

$$ds^2 = -\left(1 + 2\phi(x^i)\right)dt^2 + dx^2 + dy^2 + dz^2 \quad x^i = (x, y, z) \tag{4.26}$$

First we have to calculate the Christoffel symbols of the first kind for the metric given for the line element (4.26)

$$\begin{aligned} &\Gamma_{ijk} = 0 \qquad \Gamma_{tij} = 0 \qquad \Gamma_{tti} = \frac{1}{2}\left(\partial_i g_{tt}\right) = -\partial_i\phi \\ &\Gamma_{itt} = -\frac{1}{2}\left(\partial_i g_{tt}\right) = \partial_i\phi \qquad \Gamma_{ttt} = 0 \qquad \Gamma_{ijt} = 0 \end{aligned}$$

Since $\Gamma^{\gamma}{}_{\mu\nu} = g^{\gamma\alpha}\Gamma_{\alpha\mu\nu}$ the geodesic equation is given by

$$\ddot{x}^{\gamma} + g^{\gamma\alpha}\Gamma_{\alpha\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \begin{cases} \ddot{x}^{t} + g^{tt}\Gamma_{t\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \\ \ddot{x}^{i} + \underbrace{g^{ij}}_{\delta^{ij}}\Gamma_{jtt}\dot{x}^{t}\dot{x}^{t} = 0 \end{cases} \tag{4.27}$$

With $\dot{x}^i = v^i$, $\ddot{x}^i = a^i$ and $v \ll 1$, v^{μ} can be approximated by $v^{\mu} = \begin{pmatrix} 1 + O(v^2) \\ v^i \end{pmatrix}$ and the second equation in (4.27) is simplified to

$$\begin{aligned} a^i + \delta^{ij}\partial_j\phi &= 0 \\ \vec{a} = -\vec{\nabla}\phi \quad &\Rightarrow \quad m\vec{a} = -m\vec{\nabla}\phi \end{aligned} \tag{4.28}$$

Since we can neglect higher order terms of v we can also neglect the first equation given by the geodesic equation in (4.27) because it contains such higher order terms of v.
For ϕ one could use $-\frac{M}{r}$ for example. This choice for ϕ would lead to Newton's gravity law. Hence the interaction between particles with masses can be ascribed to the curvature of spacetime. Since mass deforms spacetime — a result we are going to derive in chapter 7 — the geodesics aren't straight lines anymore as they would be in flat spacetime and the equations of motion are given by the geodesic equation. That's a quite extraordinary result, since we only used geometrical principles and were hence able to ascribe gravitation as a geometric phenomenon without the need of a special force. Gravitation can therefore be "reduced" to a fictitious force. An observer on earth for example seems to be attracted by some kind of gravitational force just because the ground on earth prevents the observer from following a geodesic path along the curved spacetime. Without the ground the observer would follow his geodesic path and would therefore "feel" no force! Or take for example an elevator. An observer resting in an elevator which is relatively accelerating with respect to a chosen rest frame at 9.81 $\frac{m}{s^2}$ would not be able to tell the difference of being in an relatively accelerating elevator, or being in an relatively resting elevator in a gravitational field. This equivalence of a gravitational field and a corresponding acceleration of the reference system is a manifestation of the equivalence of gravitational and inertial mass and therefore the mass independence of relative acceleration in a gravitational field.

4.2.4. General Geodesics

If the curve whose length we extremize is not timelike, but instead spacelike or lightlike, we have to make minor adjustments to the geodesic action (4.18). The most general case is covered by extremizing the action

$$S = k\int_{\tau_0}^{\tau_1} d\tau\, g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \tag{4.29}$$

with some irrelevant normalization constant k and the additional normalization condition

$$g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = \begin{cases} -1 : \text{timelike} \\ 0 : \text{lightlike} \\ +1 : \text{spacelike} \end{cases} \tag{4.30}$$

Note the action (4.29) is essentially equivalent to the action (4.18) provided we choose $e = -1$. In that case τ is the proper time. In the lightlike or spacelike cases it makes no sense to call τ "proper time", so in those cases (and in full generality) τ is referred to as "affine parameter". The action (4.29) is a 1-dimensional analog of the 2-dimensional Polyakov action of string theory.

5. Geodesics for Schwarzschild Black Holes

After Einstein published the Einstein field equations, Schwarzschild was the first who found a nontrivial exact solution. We are going to introduce the Schwarzschild solution and show some important physical results.

After the definition of the Schwarzschild metric we look at the asymptotic behavior of light rays and try to interpret them. The redshift of photons, the perihelion shift of mercury and the bending of light are important tests of general relativity (especially of the Schwarzschild solution) and will be discussed in this chapter. The geodesic equations are going to tell us something about the trajectories of test particles and the differences to the Newtonian world.

The Schwarzschild solution is not only of great importance in black hole physics, it also describes the gravitational field in the region outside of ordinary spherically symmetric stars.

5.1. Schwarzschild Solution: Asymptotic Behavior, Light in Radial Motion

The Schwarzschild metric in natural units has the form

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2} \tag{5.1}$$

The relativistic Schwarzschild solution describes the gravitational field around a spherical symmetric mass M which is placed at $r = 0$.

Limits of the Schwarzschild solution:

- $r \to \infty$: Asymptotically flat space in spherical coordinates.
- $r \to 0$: True singularity of the spacetime structure.
- $r \to 2M$: The singularity is caused by a breakdown of the coordinates (5.1). The spacetime is not singular at $r = 2M$.
- $M \to 0$: Flat space in spherical coordinates.

The only difference of the Schwarzschild solution (5.1) to the Newtonian approximation (4.26) is the dr^2 coefficient which asymptotes to the Newtonian result for $r \to \infty$. That means as long as we are staying far away of the central mass there are only marginal differences to Newton's law of gravity. The closer we get and the heavier the central mass becomes the more our classical approach fails.

Let us now derive how light behaves under radial motion. For photons we have to set $ds = 0$ and since we are looking at radial motion we also have to set $d\varphi = d\theta = 0$. Substituting this into equation (5.1) we obtain the coordinate velocity

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right) \tag{5.2}$$

If the light is far away ($r \to \infty$) the coordinate velocity takes the expected value 1. Recalling section 3.1 this is the case where the gamma fish are in the pond and do not feel the flow of the water. At $r = 2M$ the coordinate velocity is 0. Here the gamma fish want to swim back but they do not get closer to the pond. So $r = 2M$ is the already mentioned event horizon of the black hole.

Now we want to see what happens to the light ray in its local coordinate system. Here we have to differentiate the proper time $d\tau$ with respect to the proper length dx

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 \qquad\qquad dx^2 = \frac{dr^2}{1 - \frac{2M}{r}} \tag{5.3}$$

and using (5.2) we get

$$\frac{dx}{d\tau} = \frac{dr}{dt}\frac{1}{1-\frac{2M}{r}} = \pm 1 \tag{5.4}$$

So the light ray has in its local coordinate system the expected velocity 1 (c in SI units).

5.2. Gravitational Redshift (equivalence principle)

Consider two static observers O_A and O_B with the radial coordinates r_A and r_B in a Schwarzschild geometry. O_A sends light signals with the wavelength τ_A to observer O_B.

$$ds^2 = -\left(1-\frac{2M}{r_A}\right)dt^2 = -\,d\tau_A{}^2 \qquad ds^2 = -\left(1-\frac{2M}{r_B}\right)dt^2 = -\,d\tau_B{}^2 \tag{5.5}$$

The ratio of the frequency ω_A (measured by the emitter) and the frequency ω_B (measured by the observer who receives the signal) results in

$$\frac{\omega_B}{\omega_A} = \frac{d\tau_A}{d\tau_B} = \frac{\sqrt{1-\frac{2M}{r_A}}}{\sqrt{1-\frac{2M}{r_B}}} \tag{5.6}$$

The closer the emitter comes to $r_A = 2M$ the more the frequency ω_B gets redshifted (we assume that $r_A < r_B$). So for an observer who is looking at an object which is falling into a black hole it looks like the object moves slower and slower and the frequency gets redder and redder. The observer would never see the emitter reach $r_A = 2M$.

We assume that Observer O_B is in the asymptotic flat region ($r_B \to \infty$) and $\frac{M}{r_A} \ll 1$ (which is the case for an “ordinary body”) we obtain

$$\frac{\omega_B}{\omega_A} \approx 1 - \frac{M}{r_A} \tag{5.7}$$

$$\frac{\Delta\omega}{\omega_A} = \frac{\omega_B - \omega_A}{\omega_A} \approx -\frac{M}{r_A} = \phi_A \tag{5.8}$$

$$\boxed{\frac{\Delta\omega}{\omega_A} \approx \phi_A} \tag{5.9}$$

Hence the frequency change equals to the change in potential energy. This effect is known as gravitational redshift and was observed by Pound and Rebka in 1960 (see figure 5.1).

For a stable static spherical body (with $d\rho/dr \leq 0$ everywhere inside the body) the theoretical minimal radius r_{star} for a given mass M_{star} is given by

$$r_{\text{star}} \geq \frac{9}{4} M_{\text{star}}\,. \tag{5.10}$$

This minimum radius is valid independently of the specific equation of state of the star. We can now use equation (5.6) to estimate what the maximum redshift of light emitted from the surface of such a star is

$$\frac{\omega_\infty}{\omega_{\text{star}}} = \frac{\sqrt{1-\frac{2M_{\text{star}}}{r_{\text{star}}}}}{\sqrt{1-\frac{2M}{\infty}}} \to \omega_{\text{star}} = 3 \tag{5.11}$$

The redshift factor is in general given by

$$z = \frac{\lambda_B - \lambda_A}{\lambda_A} = \frac{\omega_A}{\omega_B} - 1 \tag{5.12}$$

and leads in our current estimation to an maximal redshift of

$$z_{\max} = \frac{\omega_{\text{star}}}{\omega_\infty} - 1 = 2\,. \tag{5.13}$$

This means that observed redshifts of greater than 2 (as measured for example for Quasars) can not arise solely from gravitational redshift of a static spherical body.

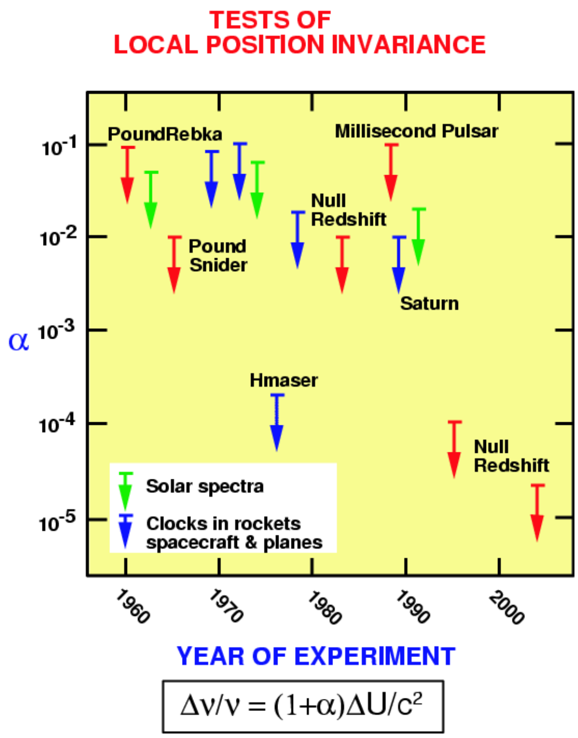

Figure 5.1: Tests of gravitational redshift (Source: Will - The Confrontation between General Relativity and Experiment)

5.3. Geodesic Equation of the Schwarzschild Solution

Now we want to derive the timelike ($M \neq 0$) and the null ($M = 0$) geodesics of the Schwarzschild solution. One possibility is to substitute (5.1) into the geodesic equation (4.25) and solve the differential equations. A faster way is to use the geodesic action (4.18) and parameterize it by the proper time τ ($d\tau = eds$). Since m is constant it does not contribute to the variation. So we are allowed to drop the first term of the geodesic action (4.18) which leads to

$$S = \frac{1}{2}\int d\tau\,[-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu] \tag{5.14}$$

$$= \frac{1}{2}\int d\tau \left[\left(1-\frac{2M}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2M}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2\right] \tag{5.15}$$

The functional can be parameterized[3] in such a way that

$$k = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \begin{cases} -1 & \text{timelike geodesics} \\ 0 & \text{lightlike geodesics} \end{cases} \tag{5.16}$$

Without loss of generality we look at the case $\dot{\theta} = 0$. Now we vary the geodesic action (5.14) with respect to θ and obtain the corresponding Euler–Lagrange equation

$$\frac{\partial L}{\partial\theta} - \frac{d}{d\tau}\frac{\partial L}{\partial\dot{\theta}} = -2r^2\sin\theta\cos\theta\dot{\varphi}^2 = 0 \tag{5.17}$$

[3]It also follows by the definition of the general relativistic proper time $\tau = \int(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}dt$. See also (4.11)

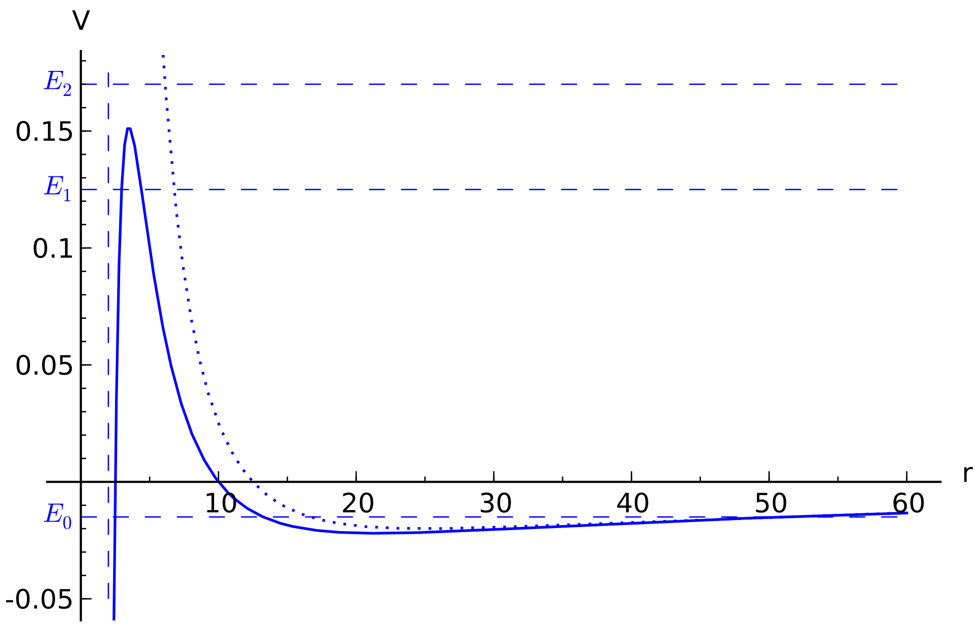

Figure 5.2: The effective potential for the timelike (solid, in the case of $L^2 > 12M^2$) and the Newtonian (dotted) trajectory. L=5,M=1

In general $r \neq 0 \neq \dot{\varphi}$ thus $2\sin\theta\cos\theta = \sin(2\theta) = 0$. Without loss of generality we consider $\theta = \frac{\pi}{2}$. The Euler–Lagrange equations for t and φ define two constants of motion

$$\frac{d}{d\tau}\left(\left(1-\frac{2M}{r}\right)2\dot{t}\right) = 0 \Longrightarrow \left(1-\frac{2M}{r}\right)\dot{t} = F = \text{const} \tag{5.18}$$

$$\frac{d}{d\tau}\left(r^2\dot{\varphi}\right) = 0 \Longrightarrow r^2\dot{\varphi} = l = \text{const} \tag{5.19}$$

Substituting (5.18) and (5.19) into (5.16) gives

$$K = \frac{F^2}{1-\frac{2M}{r}} - \frac{\dot{r}^2}{1-\frac{2M}{r}} - \frac{l^2}{r^2} \tag{5.20}$$

Since the problem is equal to the Kepler problem in the Newtonian case we want to get an equation that looks like

$$\frac{\dot{r}^2}{2} + V^{\text{eff}} = E \tag{5.21}$$

5.3.1. Timelike Geodesic

For timelike geodesics ($k = -1$) we get from (5.20) and (5.21) the effective potential of the timelike geodesic

$$\frac{\dot{r}^2}{2} - \frac{M}{r} + \frac{l^2}{2r^2} - \frac{l^2 M}{r^3} = E = \frac{F^2-1}{2} \tag{5.22}$$

$$V^{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{l^2 M}{r^3} \tag{5.23}$$

The only difference between the relativistic and the Newtonian trajectory of a massive particle is the $-\frac{l^2 M}{r^3}$ term. The trajectories for different energy levels are (see figure 5.2):

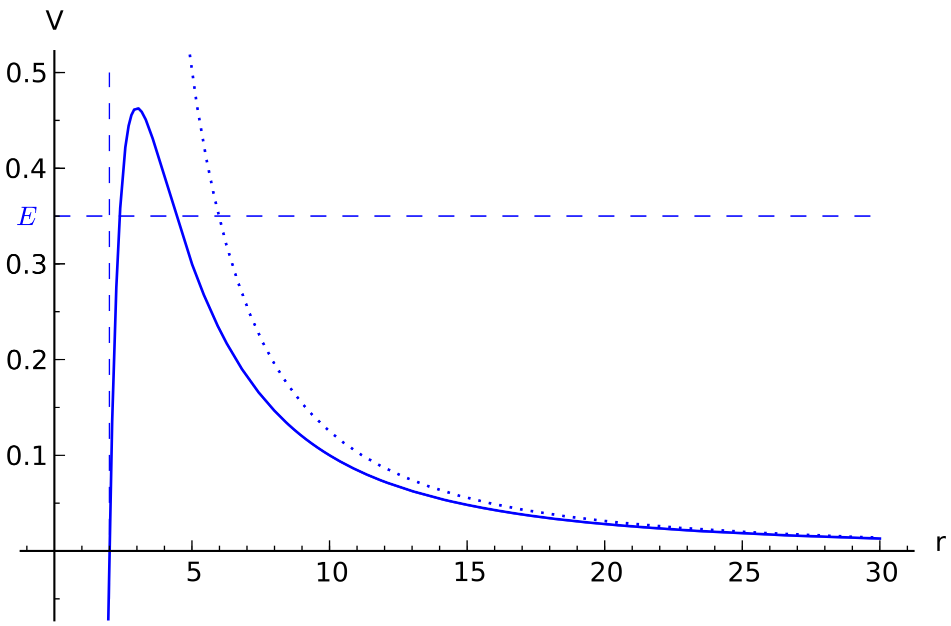

Figure 5.3: The effective potential for the lightlike (solid) and the Newtonian (dotted) trajectory. L=5,M=1

E_0 At the right side of the maximum of V^{eff} there are stable circular states. If the energy is equal to the minimum of the potential the motion is circular. When that circular state is slightly perturbed the motion leads to a perihelion shifted elliptic trajectory (see figure 5.4).

E_1 Particles left of the maximum will bounce against the potential barrier and fall into the black hole. Particles on the right side behave similar to the Newtonian case and are able to escape to infinity.

E_2 Contrary to the classical physical expectations the particle falls directly towards $r = 0$. In Kepler's problem that is only possible for $L = 0$. If the energy equals to V^{eff} at the maximum $\dot{r}$ is zero and the mass point moves on an unstable circular orbit.

5.3.2. Lightlike Geodesic

For lightlike geodesics ($k = 0$) we get

$$\frac{\dot{r}^2}{2} + \frac{l^2}{2r^2} - \frac{l^2 M}{r^3} = E = \frac{F^2}{2} \tag{5.24}$$

$$V^{\text{eff}} = \frac{l^2}{2r^2} - \frac{l^2 M}{r^3} \tag{5.25}$$

The trajectories is similar to the timelike case except that there are no stable circular orbits. Also mind the scale factor of the two figures.

5.4. Orbits of the Schwarzschild Black Hole

The stable circular orbits of the timelike trajectories are the minima of the timelike effective potential (5.23)

$$\frac{dV^{\text{eff}}}{dr} = 0 \qquad \frac{d^2 V^{\text{eff}}}{dr^2} > 0 \tag{5.26}$$

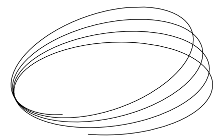

Figure 5.4: Perihelion shift

The extrema are

$$r_\pm = \frac{l^2}{2M}\left(1 \pm \sqrt{1-\frac{12M^2}{l^2}}\right) \qquad \left(r_+r_- = 3l^2\right) \tag{5.27}$$

where r_+ is the stable orbit and r_- is the unstable orbit (the formula in the parentheses will be usefull in the next section). Since the square root of the potential should not be negative the bound states are restricted to the condition $l^2 \geq 12M^2$.

For the Innermost marginally Stable Circular Orbit we need

$$\frac{d^2V^{\text{eff}}}{dr^2} = 0 \tag{5.28}$$

so the square root of the extrema (5.27) has to vanish

$$l^2 = 12M^2 \tag{5.29}$$

$$\Rightarrow r_{\text{ISCO}} = 6M \tag{5.30}$$

We can now, following section 5.2, calculate the maximal redshift for signals from this orbit

$$z_{\text{ISCO}} = \left(1-\frac{2M}{6M}\right)^{-1/2} - 1 \approx 0.2 \tag{5.31}$$

The same can be done for lightlike trajectories to get the Lightlike Unstable Circular Orbit which is always at

$$r_{\text{LUCO}} = 3M \tag{5.32}$$

5.5. Perihelion shift

For calculating the perihelion shift in general relativity we assume that the body is at a stable circular orbit (meaning that we are in a region near r_+) and perturb it slightly. If we would perturb it too much the form of an ellipsis would get lost.

The “radius frequency” of the motion is given by

$${\omega_r}^2 = \left.\frac{d^2V^{\text{eff}}}{dr^2}\right|_{r=r_+} = \frac{1}{{r_+}^4}\left(3l^2 - 2Mr_+ - 4Mr_-\right) \tag{5.33}$$

(in the last term we have used the equation in the parentheses of (5.27) to eliminate a $1/r_+$ term).

With (5.19) we get the angular frequency ω_φ

$${\omega_\varphi}^2 = \dot{\varphi}^2 = \frac{l^2}{r_+^4} \tag{5.34}$$

and derive (by inserting (5.26) and (5.34) into (5.33))

$$\omega_r = \omega_\varphi\left(1-\frac{12M^2}{l^2}\right)^{1/4} \tag{5.35}$$

$$\approx \omega_\varphi\left(1-\frac{3M^2}{l^2}\right) \qquad \text{for } M \ll l\,. \tag{5.36}$$

The precession rate $\Delta\varphi$ is the difference between ω_r and ω_φ

$$\Delta\varphi = T\Delta\omega = 2\pi\left(\omega_\varphi - \omega_r\right) \approx \frac{6\pi M^2}{l^2} \tag{5.37}$$

If the precession rate is zero the orbit is closed perfectly. This is the case for the Newtonian theory where we have a effective potential

$$V^{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{l^2 M}{r^3} \tag{5.38}$$

and

$$\omega_r = \omega_\varphi = \frac{M^2}{l^3} \tag{5.39}$$

The nonzero $\Delta\varphi$ in general relativity leads to a perihelion shift (see figure 5.4).

Combining now the Newtonian formula for bound motion

$$\frac{l^2}{M} \approx A(1-e^2) \tag{5.40}$$

with (5.37) leads to

$$\boxed{\Delta\varphi = \frac{6\pi M}{A(1-e^2)} = \frac{6\pi GM}{c^2 A(1-e^2)}} \tag{5.41}$$

where e is the eccentricity and A is the aphelion of the ellipsis.

This remarkable result can be used to calculate the general relativistic contribution to the perihelion shift of the Mercury. We have to insert

$$M_\odot \approx 2\cdot 10^{30}\text{kg} \approx 10^{38} \tag{5.42}$$

$$A \approx 6\cdot 10^{7}\text{km} \approx 4\cdot 10^{45} \tag{5.43}$$

$$e \approx 0.2 \tag{5.44}$$

into (5.41) to get

$$\Delta\varphi \approx 2\cdot 10^{-5} \approx 0,1''/\text{revolution} \tag{5.45}$$

Since there are around 415 revolutions/century we are now able to compare our calculated to the observed result

$$\Delta\varphi \approx 42''/\text{century} \tag{5.46}$$

$$\Delta\varphi_{\text{obs}} = (43.11 \pm 0.5)''/\text{century} \tag{5.47}$$

When Einstein released his work this result was one of the great achievements of general relativity.

5.6. Gravitational Light Bending

We are now going to derive another remarkable prediction of general relativity, the gravitational light-bending. We are searching for a formula for the deflection angle $\Delta\varphi$ of a light-ray (which moves on a null geodesic) in the gravitational field of a point source (like the Sun). So we use the Schwarzschild metric (5.1) and the results we derived for null geodesics in the Schwarzschild background in section 5.3.

First we establish an integral formula for the azimuthal angle φ as a function of the radial coordinate r. We take

$$\dot\varphi = \frac{l}{r^2} \tag{5.48}$$

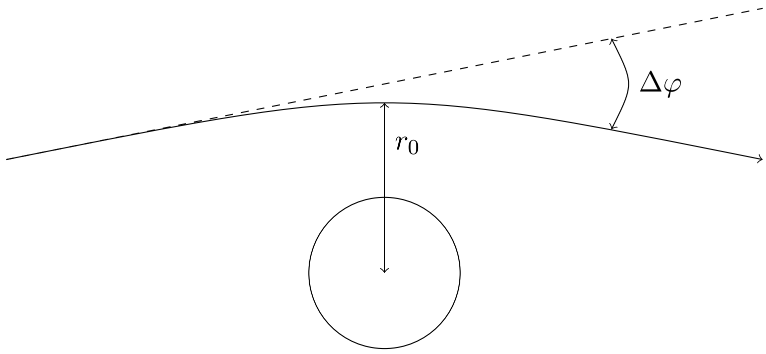

Figure 5.5: Gravitational light bending; $\Delta\varphi$ is related to φ_∞ by $\Delta\varphi = \varphi_\infty - \pi$

and divide it by

$$\dot{r} = \sqrt{2E - \frac{l^2}{r^2} + \frac{2l^2 M}{r^3}} \tag{5.49}$$

to get

$$\frac{d\varphi}{dr} = \frac{1}{r^2\sqrt{\frac{2E}{l^2} - \frac{1}{r^2} + \frac{2M}{r^3}}} \tag{5.50}$$

To get the total change in of the azimuthal angle φ_∞ we have to integrate (5.50) between $-\infty$ and $+\infty$ (see figure 5.5). That is the same as integrating twice from the turning point of the light ray r_0 to infinity

$$\varphi_\infty = 2\int_{r_0}^{\infty} \frac{1}{r^2\sqrt{\frac{2E}{l^2} - \frac{1}{r^2} + \frac{2M}{r^3}}} dr \tag{5.51}$$

The integration is more convenient if we make the variable change $u = 1/r$

$$\varphi_\infty = 2\int_0^{1/r_0} \frac{1}{\sqrt{\frac{2E}{l^2} - u^2 + 2Mu^3}} du \tag{5.52}$$

To eliminate E and l we use the fact that

$$\left.\frac{dr}{d\varphi}\right|_{r_0} = 0 \tag{5.53}$$

which leads to

$$\frac{2E}{l^2} = \frac{1}{{r_0}^2} - \frac{2M}{{r_0}^3} \tag{5.54}$$

For the case of flat spacetime we predict a total change of the azimuthal angle φ_∞ of π which leads to a straight line. So we set $M = 0$ in (5.52) and (5.54) to derive

$$\varphi_\infty = 2\int_0^{1/r_0} \frac{1}{\sqrt{\frac{2E}{l^2} - u^2}} du \tag{5.55}$$

$$= 2 \arctan\left(\frac{u}{\sqrt{\frac{2E}{l^2} - u^2}}\right)\Bigg|_0^{1/r_0} \tag{5.56}$$

$$= \pi \tag{5.57}$$

For $M \neq 0$ the trajectory of the light-ray is no straight line anymore. The deflection is interpreted as the gravitational attraction of the Schwarzschild geometry. To calculate the deflection angle $\Delta\varphi$ to first order in M we first calculate the change of the angular coordinate to first order. We substitute (5.54) into (5.52)

$$\varphi_\infty = 2\int_0^{1/r_0} \frac{du}{({r_0}^{-2} - 2M{r_0}^{-3} - u^2 + 2Mu^3)^{1/2}} \tag{5.58}$$

For the total change of the azimuthal angle φ_∞ in first order of M we have to differentiate φ_∞ by M and evaluate the result at $M = 0$

$$\begin{aligned} \left.\frac{\partial(\varphi_\infty)}{\partial M}\right|_{M=0} &= 2\int_0^{1/r_0} \left.\frac{({r_0}^{-3} - u^3)du}{({r_0}^{-2} - 2M{r_0}^{-3} - u^2 + 2Mu^3)^{3/2}}\right|_{M=0} &(5.59)\\ &= 2\int_0^{1/r_0} \frac{({r_0}^{-3} - u^3)}{({r_0}^{-2} - u^2)^{3/2}} du &(5.60)\\ &= -2\sqrt{r_0^{-2} - u^2}\, \left.\frac{2 + r_0 u}{1 + r_0 u}\right|_0^{1/r_0} &(5.61)\\ &= 4r_0^{-1} &(5.62) \end{aligned}$$

So the deflection angle $\Delta\varphi$ in first order of M is

$$\Delta\varphi = \varphi_\infty - \pi \approx M \left.\frac{\partial(\varphi_\infty)}{\partial M}\right|_{M=0} = \boxed{\frac{4M}{r_0}}. \tag{5.63}$$

Inserting

$$\begin{aligned} M_\odot &\approx 2 \cdot 10^{30}\text{kg} \approx 10^{38} &(5.64)\\ r_\odot &\approx 7 \cdot 10^8\text{m} \approx 7 \cdot 10^{43} &(5.65) \end{aligned}$$

predicts a deflection of

$$\Delta\varphi \approx 1,75'' \tag{5.66}$$

for light-rays which graze the sun. Eddington proved this gravitational light bending of starlight at a solar eclipse in 1919 (up to a measurement accuracy of 10%). Nowadays the the effect can be measured to an accuracy much better than 1% (see figure 5.6).

For more information about the confrontation between General Relativity and experiment see [1].

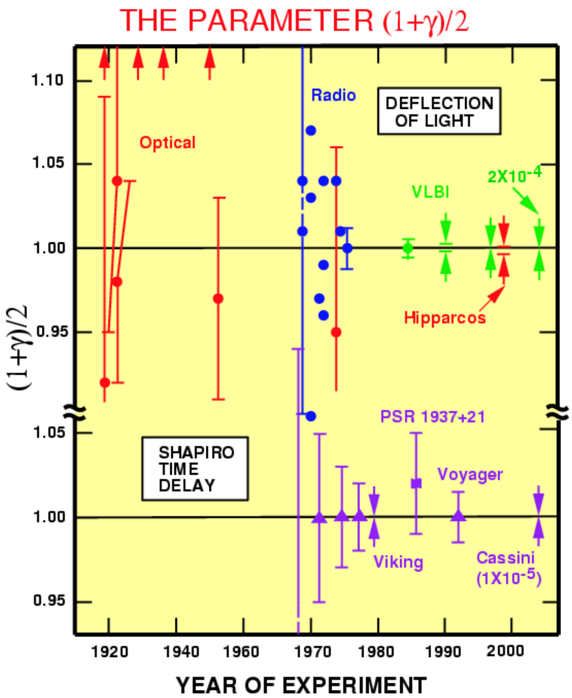

Figure 5.6: Tests for the gravitational light deflection. General relativity predicts $\gamma = 1$. The not very precise optical experiments (Optical) were the first conformation of general relativity (the top arrows denote anomalously large values from early eclipse expeditions). Later radio-interferometery (Radio) and very-long-baseline radio interferometry (VLBI), produced measurements with greatly improved determinations of the deflection of light. (Source: Will - The Confrontation between General Relativity and Experiment)

6. Curvature and Basics of Differential Geometry

General Relativity is a theory of spacetime curvature. Therefore we need to define and introduce some of the basic concepts of differential geometry and develop the mathematical tools to thoroughly describe the phenomenons appearing whilst studying black holes.

In Chapter 4 we have introduced the concept of a geodesic as shortest lines between two points in a given spacetime. Additionally, we briefly discussed parallel transport and "autoparallels" (i.e. straightest lines in a spacetime). In Euclidean spacetime, both notions are identical. In general this is not the case, and we shall see why. It is one of the aims of this chapter to find circumstances and conditions for arbitrary spacetimes, so that this equivalence remains true.

6.1. Manifolds and Tangent Spaces

When visiting an one-year course on topology, you will probably go through sets, enlarge this to topologies and finally arrive at something called a **manifold**. We do not have time enough to do so, therefore we will just focus on a special type of manifold - manifolds with a metric - and define them quite sloppily as "something that locally looks like $\mathbb{R}^n$". This could be a strip of paper, a Moebius strip, our Universe, ...

Next to consider is the concept of **tangent space**. There are various kinds of manifolds. The ones we are exclusively concerned with have a metric and we can can attach to every point x of our manifold a tangent space, a real vector space which intuitively contains the possible "directions" in which one can tangentially pass through x. For example, if the given manifold is a 2-sphere, one can picture the tangent space at a point as the plane which touches the sphere at that point and is perpendicular to the sphere's radius through the point.

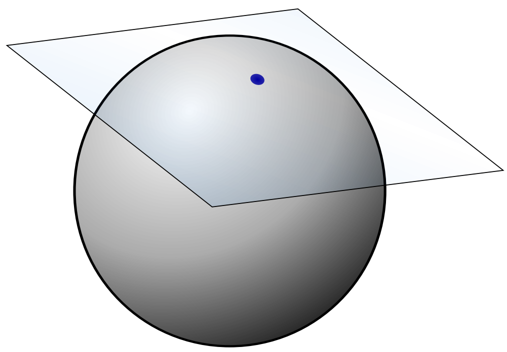

Figure 6.1: Tangent Space of a 2-sphere

By definition, the directional derivatives ∂_i in a certain point form a base of the tangent space attached to this point. This may be illustrated in the following picture:

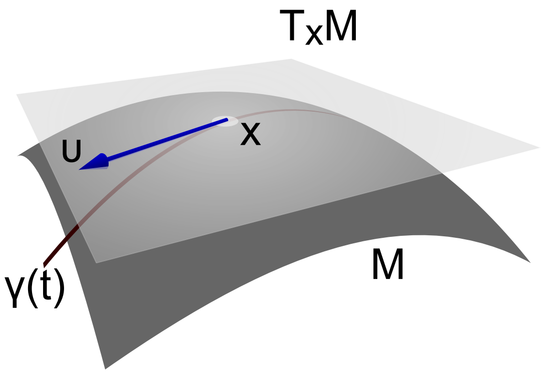

Figure 6.2: Tangent vector of a given path on a 2-sphere

This leads us to the next definition: What is a **vector**? Hopefully, the definition of a vector v as linear combination of the base-vectors ($v := v^\mu \partial_\mu$) is not new to the reader. More formal, we define a vector as follows:

Assuming we have a manifold M and there exist a smooth maps $F : M \to \mathbb{R}$ with $F \in C^\infty$ then a tangent vector at point P maps an element of F to $\mathbb{R}$.

$$v(f) = v^\mu \partial_\mu f \qquad f \in F \tag{6.1}$$

A vector must fulfill two criteria (with $f, g \in F$ and $a, b \in \mathbb{R}$):

1. **linearity:**
$$v(af + bg) = av(f) + bv(g) \tag{6.2}$$

2. **Leibnitz rule:**
$$v(f \cdot g) = f \cdot v(g) + g \cdot v(f) \tag{6.3}$$

   Linearity combined with the Leibnitz rule implies that a vector acting on any constant h vanishes.
$$\Rightarrow h \cdot v(h) = v(h \cdot h) = 2hv(h) = 0 \tag{6.4}$$

There are two important facts about tangent spaces. First, the tangent space in point P (from here on called V_P) fulfills all criteria of a vector space (please, take a look at your linear algebra lecture notes for them) and, second, $dim(V_P) = dim(M)$.

The next important concept is the **dual vector space**. To a given vector space V_P, the dual space V_P^* consists of all linear maps $V_P \to \mathbb{R}$. About the dimension of V_P^* we may say:

$$dim(V_P^*) = dim(V_P) \tag{6.5}$$

With $e_1, \cdots, e_D$ being a basis of V_P ($\frac{\partial}{\partial x^n}$ in a coordinate basis) and $e^1, \cdots, e^D$ of V_P^* (dx^n in a coordinate basis), we get:

$$e^\mu e_\nu = \delta^\mu_\nu \qquad \text{or} \qquad dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu \tag{6.6}$$

This is a generalization of the fact that differentiation (represented by elements of V_P^*) is the dual (i.e. inverse) operation of integration—here represented by the elements of V_P.

Since $V_P^{**} = V_P$ vectors can also be seen as linear maps $V_P^* \to \mathbb{R}$.

6.2. Tensors

A multi-linear map T of the kind

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{\text{p copies}} \otimes \underbrace{V \otimes \cdots \otimes V}_{\text{q copies}} \to \mathbb{R} \tag{6.7}$$

is called a "**tensor** of type (p,q)".
Accordingly a

- vector is a (1,0)-tensor,
- dual-vector is a (0,1)-tensor,
- metric is a (0,2) tensor: $g_{\mu\nu} v^\mu w^\nu = \alpha \in \mathbb{R}$

Taking into account the definition of the basis of vector and dual vector space and definition 6.7 we may write an arbitrary (p,q)-tensor in the following form:

$$T = \widetilde{T}^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q} e_{\mu_1} \otimes \cdots \otimes e_{\mu_p} \otimes e^{\nu_1} \otimes \cdots \otimes e^{\nu_q} \tag{6.8}$$

$$= T^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_p} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_q} \tag{6.9}$$

In this notation, a change of basis can be calculated straightforwardly. With

$$x'^\mu = x'^\mu(x^\nu) \tag{6.10}$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = (\partial_\nu)\frac{\partial x^\nu}{\partial x'^\mu} \tag{6.11}$$

$$dx'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu \tag{6.12}$$

and the requirement that T is invariant under such transformation (tensors are multi-linear maps and do not change when altering the basis), we get:

$$T'^{\alpha_1,\ldots,\alpha_p}{}_{\beta_1,\ldots,\beta_q} = T^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q} \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\alpha_p}}{\partial x^{\mu_p}} \cdot \frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\nu_q}}{\partial x'^{\beta_q}} \tag{6.13}$$

Maybe here is a good point to explain the differences between local and global quantities. When speaking locally, you consider a tensor evaluated at a specific point P. We shall also call a tensor field loosely "tensor" — the same applies to vectors/vector fields and scalars/scalar fields.

6.3. Another View at the Metric

Up until now, we have used the metric only to calculate the length of a vector or the inner product between two vectors.
With our newly acquired knowledge concerning dual vector-spaces, we are able to interpret the metric as a map between a given vector space and its dual space.

$$\boxed{g_{\mu\nu}v^{\nu} = v_{\mu}} \quad \boxed{g^{\mu\nu}v_{\nu} = v^{\mu}} \tag{6.14}$$

Even more, this connection provides us with a natural isomorphism of the vector- and the dual-vector space.

By multiplying equation 6.14 by $g^{\mu\alpha}$ from "left", we obtain

$$g^{\mu\alpha}g_{\mu\nu}v^{\nu} = g^{\mu\alpha}v_{\mu} \tag{6.15}$$

$$g_{\mu\nu}g^{\mu\alpha}v^{\nu} = v_{\alpha} \tag{6.16}$$

$$\Rightarrow g_{\mu\nu}g^{\mu\alpha} = \delta^{\alpha}_{\nu} \tag{6.17}$$

This last result is important for reasons of consistency.

Example: We start with a metric

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = 2dudr - rdu^2 = g'_{\mu'\nu'}dx'^{\mu'}dx'^{\nu'} \tag{6.18}$$

where we have $x^{\mu} = (r, u)$. We want to make a coordinate transformation to the coordinates $x'^{\mu'} = (t, R)$ given by

$$u = t + 2\ln R \tag{6.19}$$

$$t = \frac{R^2}{4} \tag{6.20}$$

In general we could just use the the tensor transformation law (6.13) but it is often more convenient to use

$$du = dt + \frac{2}{R}dR \tag{6.21}$$

$$dr = \frac{R}{2}dR \tag{6.22}$$

to get

$$ds^2 = -\frac{R^2}{4}dt^2 + dR^2 \tag{6.23}$$

or in matrix form

$$g_{\mu\nu} = \begin{pmatrix} -r & 1 \\ 1 & 0 \end{pmatrix}_{\mu\nu} \qquad g'_{\mu'\nu'} = \begin{pmatrix} -\frac{R^2}{4} & 0 \\ 0 & 1 \end{pmatrix}_{\mu'\nu'} \tag{6.24}$$

6.4. Covariant Derivatives

To begin with, we have to define the action of a covariant derivative ∇ on an element of our vector space. With $v \in V$

$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \underbrace{\widetilde{\Gamma}^\alpha{}_{\mu\beta} v^\beta}_{\text{linear transformation}} \tag{6.25}$$

The last term in equation (6.25) accounts for linear transformations acting on a vector as it is transported from an element in V_P to an element in $V_{P'}$, where P' is a point sufficiently close to P. In other words: the covariant derivative is the derivative along the coordinates with correction terms which are unspecified at the moment. Now, by looking at the first term of above equation, we require that the covariant derivative of a vector is a tensor — therefore, we can use what we know about tensor transformation to calculate the covariant derivative in a new basis. Just inserting into 6.13 yields:

$$\nabla'_{\mu'} v^{\alpha'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{\partial x^{\alpha'}}{\partial x^\nu} \left(\partial_\mu v^\nu + \widetilde{\Gamma}^\nu{}_{\mu\lambda} v^\lambda\right) \tag{6.26}$$

We expand the left-hand side of above equation and rewrite it as transformation of the "old" (unprimed) coordinates:

$$\begin{aligned} \partial_{\mu'} v^{\alpha'} + \widetilde{\Gamma}^{\alpha'}{}_{\mu'\nu'} v^{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{\partial x^{\alpha'}}{\partial x^\nu} \partial_\mu v^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{\partial^2 x^{\alpha'}}{\partial x^\mu \partial x^\nu} v^\nu + \widetilde{\Gamma}^{\alpha'}{}_{\mu'\nu'} \frac{\partial x^{\nu'}}{\partial x^\nu} v^\nu \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{\partial x^{\alpha'}}{\partial x^\nu} \left(\partial_\mu v^\nu + \widetilde{\Gamma}^\nu{}_{\mu\lambda} v^\lambda\right) \end{aligned} \tag{6.27}$$

This finally leads us to a generic transformation rule of the $\widetilde{\Gamma}$-element. Note here that we have not defined this object yet — but the fact that we called it $\widetilde{\Gamma}$ might be a hint that it is equal to the known Christoffel-symbol under certain conditions...
The transformation rule is:

$$\boxed{\widetilde{\Gamma}^{\nu'}{}_{\mu'\lambda'} = \widetilde{\Gamma}^\nu{}_{\mu\lambda} \cdot \tfrac{\partial x^\mu}{\partial x^{\mu'}} \cdot \tfrac{\partial x^\lambda}{\partial x^{\lambda'}} \cdot \tfrac{\partial x^{\nu'}}{\partial x^\nu} - \tfrac{\partial x^\mu}{\partial x^{\mu'}} \cdot \tfrac{\partial x^\lambda}{\partial x^{\lambda'}} \cdot \tfrac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}} \tag{6.28}$$

As the $\widetilde{\Gamma}$-components do not transform as the components of a tensor the quantity $\widetilde{\Gamma}$ *it is no tensor*. Only the combination of the partial derivative and the $\widetilde{\Gamma}$-elemtent do transform as a tensor. Generally, the expression $\widetilde{\Gamma}$ is called a "connection", as it allows to "connect" the tangent spaces at different points that are infinitesimally separated from each other.

6.4.1. Properties of the Covariant Derivative

In this short subsection we will just list four important properties of the covariant derivative. We do so without proof, but the identities can be checked by inserting definition (6.25). In all definitions $T, \widetilde{T}$ are a (p,q)-tensors, $\alpha, \beta \in \mathbb{R}$, v is a vector and f a scalar function.

1. Linearity:
$$\nabla_\mu \left(\alpha T + \beta \widetilde{T}\right) = \alpha \nabla_\mu T + \beta \nabla_\mu \widetilde{T} \tag{6.29}$$

2. Leibnitz-Rule:
$$\nabla_\mu \left(T\widetilde{T}\right) = \left(\nabla_\mu T\right) \widetilde{T} + \left(\nabla_\mu \widetilde{T}\right) T \tag{6.30}$$

3. Consistency with directional derivation:
$$v(f) = v^\alpha \nabla_\alpha f = v^\alpha \partial_\alpha f \tag{6.31}$$

4. Absence of torsion
$$\nabla_\alpha \nabla_\beta f = \nabla_\beta \nabla_\alpha f \tag{6.32}$$

5. Metric compatibility (we shall define and use this property below in section 6.7)

Note that 4 and 5 are requirements that are not implicit in the definition (6.25). Let us write the equation (6.32) in the following form:

$$\nabla_a \left(\partial_b f\right) = \nabla_b \left(\partial_a f\right) \tag{6.33}$$
$$\partial_a \partial_b f - \widetilde{\Gamma}^c{}_{ab} \partial_c f = \partial_b \partial_a f - \widetilde{\Gamma}^c{}_{ba} \partial_c f \tag{6.34}$$

$$\Rightarrow \widetilde{\Gamma}^c{}_{[a,b]} = 0 \quad \Leftrightarrow \quad \text{no torsion} \tag{6.35}$$

In equation (6.35) we introduced a new short-hand notation for anti-symmetrization:

$$\widetilde{\Gamma}^c{}_{[a,b]} = \frac{1}{2}\left(\widetilde{\Gamma}^c{}_{ab} - \widetilde{\Gamma}^c{}_{ba}\right) \tag{6.36}$$

6.5. Covariant Derivative acting on Dual Vectors

Up until now, we have only considered a covariant derivative acting on vectors. To give an overview on covariant derivatives, we have to fill this void. We start with the following ansatz

$$\nabla_\mu w_\nu = \partial_\mu w_\nu + \widehat{\Gamma}^\alpha{}_{\mu\nu} w_\alpha \tag{6.37}$$

We use now that the covariant derivative of a scalar is equal to its partial derivative (point 3. in section 6.4.1) and the Leibnitz-Rule (point 2. in section 6.4.1) to get

$$\nabla_\mu(\underbrace{v^\alpha w_\alpha}_{=\text{ scalar}}) = \partial_\mu(v^\alpha w_\alpha) = v^\alpha \partial_\mu w_\alpha + w_\alpha \partial_\mu v^\alpha \tag{6.38}$$

$$= \underline{w_\alpha(\partial_\mu v^\alpha} + \widetilde{\Gamma}^\alpha{}_{\mu\nu} v^\nu) + \underline{\underline{v^\alpha(\partial_\mu w_\alpha}} + \widehat{\Gamma}^\beta{}_{\mu\alpha} w_\beta) \tag{6.39}$$

In the second line above we just used the Leibnitz rule for the covariant derivative. Comparing it with the first line we see that the combination of the non-underlined terms has to be zero and we conclude that (with a little index renaming)

$$w_\alpha \widetilde{\Gamma}^\alpha{}_{\mu\nu} v^\nu + v^\alpha \widehat{\Gamma}^\alpha{}_{\mu\nu} w_\alpha = 0 \tag{6.40}$$

$$w_\alpha v^\nu \left(\widetilde{\Gamma}^\alpha{}_{\mu\nu} + \widehat{\Gamma}^\alpha{}_{\mu\nu}\right) = 0 \tag{6.41}$$

$$\Rightarrow \widehat{\Gamma}^\alpha{}_{\mu\nu} = -\widetilde{\Gamma}^\alpha{}_{\mu\nu} \tag{6.42}$$

Using this result our ansatz now reads:

$$\boxed{\nabla_\mu w_\nu = \partial_\mu w_\nu - \widetilde{\Gamma}^\alpha{}_{\mu\nu} w_\alpha} \tag{6.43}$$

As a consequence we are able to calculate the covariant derivative of a (p,q)-tensor to get an (p,q+1)-tensor:

$$\begin{aligned}
\nabla_\mu T^{\mu_1,\dots,\mu_p}{}_{\nu_1,\dots,\nu_q} = {} & \partial_\mu T^{\mu_1,\dots,\mu_p}{}_{\nu_1,\dots,\nu_q} \\
& + \widetilde{\Gamma}^{\mu_1}{}_{\mu\alpha} T^{\alpha\mu_2,\dots,\mu_p}{}_{\nu_1,\dots,\nu_q} \\
& + \widetilde{\Gamma}^{\mu_2}{}_{\mu\alpha} T^{\mu_1\alpha,\dots,\mu_p}{}_{\nu_1,\dots,\nu_q} \\
& + \dots + \\
& + \widetilde{\Gamma}^{\mu_p}{}_{\mu\alpha} T^{\mu_1,\dots,\alpha}{}_{\nu_1,\dots,\nu_q} \\
& - \widetilde{\Gamma}^{\alpha}{}_{\nu_1\mu} T^{\mu_1,\dots,\mu_p}{}_{\alpha\nu_2,\dots,\nu_q} \\
& - \dots - \\
& - \widetilde{\Gamma}^{\alpha}{}_{\nu_q\mu} T^{\mu_1,\dots,\mu_p}{}_{\nu_1,\dots,\alpha}
\end{aligned} \tag{6.44}$$

Despite looking complicated, the rule governing above derivative is quite simple: each upper index (vector index) leads to a connection term with positive sign, while each lower (dual) index leads to a connection term with negative sign.

6.6. Parallel Transport

Finally, we arrived at the point where our knowledge on differential geometry is sufficient to discuss parallel transport. Take two vectors $v, t \in V$. Then we call a vector v parallel transported along t if:

$$t^a \nabla_a v^b = 0 \tag{6.45}$$

In the beginning of this chapter we mentioned the auto-parallels. With definition (6.45)

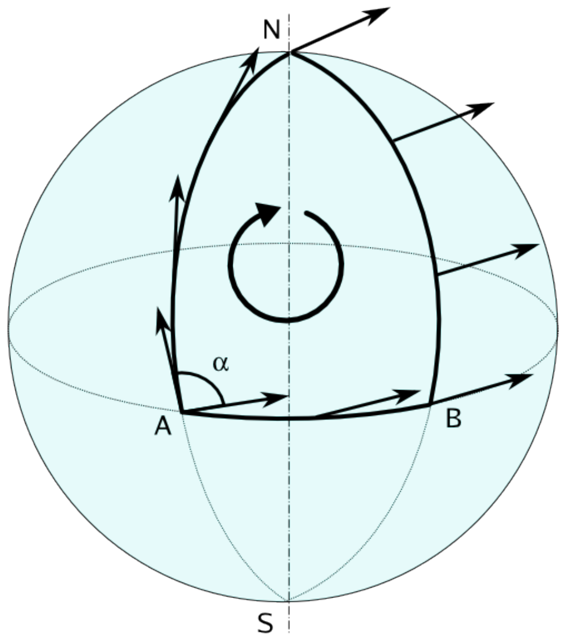

Figure 6.3: Parallel transport on a 2-Sphere (Source: Fred the Oyster, CC BY-SA 4.0, via Wikimedia Commons)

we define an auto-parallel as a curve along a which a vector v is transported parallel to itself.

$$v^a \nabla_a v^b = 0 \tag{6.46}$$

Expanding this expression results in

$$v^a \partial_a v^b + \widetilde{\Gamma}^b{}_{ac} v^a v^c = 0 \tag{6.47}$$

Since $v^b = \dot{x}^b; v^a \partial_a = \partial_\tau$ we arrive at

$$\boxed{\ddot{x}^b + \widetilde{\Gamma}^b{}_{ac} \dot{x}^a \dot{x}^c = 0} \tag{6.48}$$

This is, if $\widetilde{\Gamma} = \Gamma$, exactly the geodesic equation (4.25).

6.7. Fixing $\widetilde{\Gamma}$ Uniquely

To derive the $\widetilde{\Gamma}$ we have to impose another condition on parallel transport, namely that if two vectors $v, w \in V$ are transported parallel along t, the angle between v and w should not change. This means that

$$t^a \nabla_a \left(g_{bc} v^b w^c \right) = 0 \tag{6.49}$$

$\forall t^a, v^b, w^c$. By assuming

$$t^a \nabla_a v^b = t^a \nabla_a w^b = 0 \tag{6.50}$$

we obtain

$$t^a v^b w^c \left(\nabla_a g_{bc} \right) = 0 \tag{6.51}$$

which is only true $\forall t^a, v^b, w^c$ if

$$\boxed{\nabla_a g_{bc} = 0} \tag{6.52}$$

Result (6.52) is called "metricity" or "metric compatibility".

As said before, we can use this result and the torsion free condition to derive a unique $\widetilde{\Gamma}$. We expand (6.52) and rewrite it twice with permuted indicies:

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \underline{\widetilde{\Gamma}^\lambda{}_{\rho\mu} g_{\lambda\nu}} - \underline{\underline{\widetilde{\Gamma}^\lambda{}_{\rho\nu} g_{\mu\lambda}}} = 0 \tag{6.53}$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \widetilde{\Gamma}^\lambda{}_{\mu\nu} g_{\lambda\rho} - \underline{\widetilde{\Gamma}^\lambda{}_{\mu\rho} g_{\nu\lambda}} = 0 \tag{6.54}$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \underline{\underline{\widetilde{\Gamma}^\lambda{}_{\rho\nu} g_{\lambda\mu}}} - \widetilde{\Gamma}^\lambda{}_{\mu\nu} g_{\lambda\rho} = 0 \tag{6.55}$$

No we subtract (6.53) - (6.54) - (6.55) and with identity (6.35) all underlined and double-underlined terms in the above equations cancel, resulting in the determining equation for $\widetilde{\Gamma}$:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\widetilde{\Gamma}^\lambda{}_{\mu\nu} g_{\lambda\rho} = 0 \tag{6.56}$$

After rearranging this equation, we have finally arrived at the key result:

$$\widetilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) = \Gamma^\lambda{}_{\mu\nu} \tag{6.57}$$

In a space-time with no torsion and with metric compatibility, the covariant derivative is determined by the Christoffel-symbols of the second kind. Thus, the connection is determined only by the metric—we call a connection with these properties "Levi-Civita-connection".

6.8. The Riemann-Tensor

This subchapter will be devoted to finding a method of "measuring" curvature in a given geometry. We will do so by calculating the difference vector of a given vector parallel transported among two different paths starting and ending at the same points (see figure 6.8).

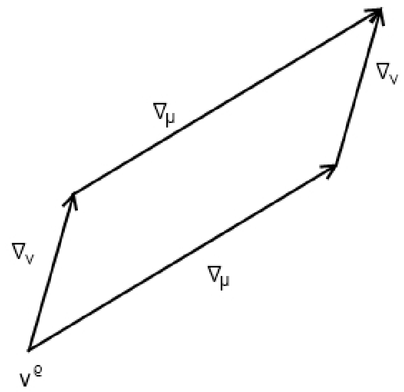

Figure 6.4: Schematic of 2-way parallel transport

In this derivation we do not demand that the connection is torsion free.

$$[\nabla_\mu, \nabla_\nu]\, v^\rho = \nabla_\mu \underbrace{\nabla_\nu v^\rho}_{\text{(1,1)-tensor}} - \nabla_\nu \underbrace{\nabla_\mu v^\rho}_{\text{(1,1)-tensor}} \tag{6.58}$$

$$= \partial_\mu \left(\nabla_\nu v^\rho\right) - \Gamma^\alpha_{\ \mu\nu} \nabla_\alpha v^\rho + \Gamma^\rho_{\ \mu\nu} \nabla_\nu v^\alpha - (\mu \leftrightarrow \nu) \tag{6.59}$$

$$= \partial_\mu \partial_\nu v^\rho + \left(\partial_\mu \Gamma^\rho_{\ \nu\alpha}\right) v^\alpha + \Gamma^\rho_{\ \nu\alpha} \partial_\mu v^\alpha - \Gamma^\alpha_{\ \mu\nu} \nabla_\alpha v^\rho + \Gamma^\rho_{\ \mu\alpha} \partial_\mu v^\alpha + $$
$$+ \Gamma^\rho_{\ \mu\alpha} \Gamma^\alpha_{\ \mu\beta} v^\beta - (\mu \leftrightarrow \nu) \tag{6.60}$$

$$= \left(\partial_\mu \Gamma^\rho_{\ \nu\alpha} - \partial_\nu \Gamma^\rho_{\ \mu\alpha} + \Gamma^\rho_{\ \mu\alpha} + \Gamma^\rho_{\ \mu\lambda} \Gamma^\lambda_{\ \nu\alpha} - \Gamma^\rho_{\ \nu\lambda} \Gamma^\lambda_{\ \mu\alpha}\right) v^\alpha - 2\Gamma^\alpha_{\ [\mu,\nu]} \nabla_\alpha v^\rho \tag{6.61}$$

Identifying the above equation with the following one leads us to the definition of the Riemann and torsion tensors respectively:

$$[\nabla_\mu, \nabla_\nu]\, v^\rho = R^\rho_{\ \alpha\mu\nu} v^\alpha - T^\alpha_{\mu\nu} \nabla_\alpha v^\rho \tag{6.62}$$

With R being the Riemann-Tensor:

$$\boxed{R^\rho_{\ \alpha\mu\nu} = \partial_\mu \Gamma^\rho_{\ \nu\alpha} - \partial_\nu \Gamma^\rho_{\ \mu\alpha} + \Gamma^\rho_{\ \mu\lambda} \Gamma^\lambda_{\ \nu\alpha} - \Gamma^\rho_{\ \nu\lambda} \Gamma^\lambda_{\ \mu\alpha}} \tag{6.63}$$

And T the torsion tensor:

$$T^\alpha_{\mu\nu} = 2\Gamma^\alpha_{\ [\mu\nu]} \tag{6.64}$$

Nota bene: If the Γ-symbols in the torsion tensor are equal to the Christoffel-symbols of the second kind, the torsion tensor vanishes. In these lectures we shall always assume vanishing torsion and metric compatibility.

6.8.1. Properties of the Riemann-Tensor

First,the Riemann-tensor in n dimensions has got $\frac{n^2(n^2-1)}{12}$ algebraically independent components. Evaluating this in various dimensions results in:

- n=1: 0 components

- n=2: 1 component
- n=3: 6 components
- n=4: 20 components
- n=11: 1210 components

The Riemann-tensor has the following symmetries:

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \tag{6.65}$$
$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \tag{6.66}$$
$$R_{\alpha\beta\mu\nu} = +R_{\mu\nu\alpha\beta} \tag{6.67}$$
$$R_{\alpha[\beta\mu\nu]} = 0 \quad (\rightarrow R_{[\alpha\beta\mu\nu]} = 0) \tag{6.68}$$

There are also some noteworthy contractions of the Riemann-tensor - especially the Ricci-tensor and the Ricci-scalar are very important due to their role in Einstein's general theory of relativity.

We obtain the (symmetric) Ricci-tensor by contracting the Riemann-tensor over the "upper" and the third "lower" index:

$$R_{\mu\nu} := R^{\rho}{}_{\mu\rho\nu} \tag{6.69}$$

The Ricci-scalar is equal to the trace of the Ricci-tensor:

$$R := R^{\mu}{}_{\mu} \tag{6.70}$$

When reading literature on differential geometry, you may stumble across the Weyl-tensor (in $n \geq 3$ spacetime dimensions):

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho} + \frac{2}{(n-2)(n-1)}Rg_{\rho[\mu}g_{\nu]\sigma}\right) \tag{6.71}$$

This tensor has sysmmetries equal to those of the Riemann-tensor, but is additionally traceless with respect to all possible index contractions: $C^{\rho}{}_{\mu\rho\nu} = 0$.

6.9. Jacobi / Bianchi Identity

Like any other derivative, the covariant derivative satisfies the Jacobi identity:

$$[[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho] = 0 \tag{6.72}$$

Which in the context of General Relativity is also known as Bianchi identity. This, when applied to the Riemann-curvature-tensor, yields:

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0 \tag{6.73}$$

Or, in our short-hand notation

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \tag{6.74}$$

One quite interesting result, noteworthy here, is obtained by multiplying (6.73) twice with the metric:

$$g^{\nu\sigma}g^{\mu\lambda} \cdot \text{Bianchi identity} = \nabla^\mu R_{\mu\rho} - \nabla_\rho R + \nabla^\nu R_{\nu\rho} = 0 \tag{6.75}$$

This leaves us with:

$$\nabla^\mu \Big(\underbrace{R_{\mu\rho} - \frac{1}{2}g_{\mu\rho}R}_{:=\text{ Einstein tensor}} \Big) = 0 \tag{6.76}$$

With the Einstein tensor:

$$G_{\mu\rho} = R_{\mu\rho} - \frac{1}{2}g_{\mu\rho}R \tag{6.77}$$

In a subsequent chapter - dealing with Einstein's field equations - the importance of this tensor will become obvious.

6.10. Lie Derivatives

The next important concept to introduce during this differential geometry overview is that of Lie derivatives - named after Sophus Lie, the Norwegian mathematician who achieved great breakthroughs in the field of symmetry transformations. Speaking generally, Lie derivatives allow us to evaluate the change of a given vector (or vector field) along the time evolution of another known vector (or vector field).
We start by defining a scalar field $\Phi(x) = \overline{\Phi}(\overline{x})$. Now we introduce a so-called "diffeomorphism", which is an invertible map between two manifolds so that both the function and it's inverse are smooth. In this case you can visualize it as "moving points on a sphere"

$$\overline{x}^\mu = x^\mu - \xi^\mu + O\left(\xi^2\right) \tag{6.78}$$

Applying this to the scalar field Φ results in

$$\Phi(x) = \Phi(\overline{x} + \xi) = \overline{\Phi}(\overline{x}) = \Phi(\overline{x}) + \xi^\mu \partial_\mu \Phi(\overline{x}) + O\left(\xi^2\right) \tag{6.79}$$

In this equation the last term is obtained by Taylor-expansion at $x = \overline{x}$. Rewriting this last statement gives us the definition of the Lie-derivative of $\Phi(x)$ in respect to ξ:

$$\boxed{\mathcal{L}_\xi \Phi(x) := \overline{\Phi}(x) - \Phi(x) = \xi^\mu \partial_\mu \Phi(x)} \tag{6.80}$$

Similarly, we define the Lie-derivative of vectors as the Lie-bracket between these two vectors:

$$\mathcal{L}_\xi v^\mu := [\xi, v]^\mu = \xi^\nu \partial_\nu v^\mu - v^\nu \partial_\nu \xi^\mu \tag{6.81}$$

For dual vectors we use again the Leibniz rule

$$\mathcal{L}_\xi (v^\mu w_\mu) = w_\mu \mathcal{L}_\xi v^\mu + v^\mu \mathcal{L}_\xi w_\mu = \xi^\alpha \partial_\alpha (v^\mu w_\mu) = w_\mu \xi^\alpha \partial_\alpha v^\mu + v^\mu \xi^\alpha \partial_\alpha w_\mu \tag{6.82}$$

and thus can read off the action of the Lie-derivative on dual vectors:

$$\mathcal{L}_\xi w_\mu = \xi^\alpha \partial_\alpha w_\mu + w_\alpha \partial_\mu \xi^\alpha \tag{6.83}$$

Again, like with co- and contravariant derivatives, we can use these results to calculate the Lie-derivative of a (q, p)-tensor:

$$\boxed{\mathcal{L}_\xi T_{\mu_1 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q} := \xi^\mu \partial_\mu T_{\mu_1 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q} + T_{\mu \mu_2 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q} \partial_{\mu_1} \xi^\mu + \cdots - T_{\mu_1 \ldots \mu_p}{}^{\mu \nu_2 \ldots \nu_q} \partial_\mu \xi^{\nu_1} - \ldots} \tag{6.84}$$

This expression is equally valid for any symmetric (torsion free) covariant derivative, i.e., one could substitute everywhere $\partial \to \nabla$ in equation (6.84).

6.11. Killing Vectors

We calculate now the Lie derivative of a metric $g_{\mu\nu}$ along a vector ξ.

$$\begin{aligned}\mathcal{L}_\xi\left(g_{\mu\nu}\right) &= \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha && (6.85)\\ &= \underbrace{\xi^\alpha \nabla_\alpha g_{\mu\nu}}_{=0} + g_{\mu\alpha}\nabla_\nu \xi^\alpha + g_{\nu\alpha}\nabla_\mu \xi^\alpha && (6.86)\end{aligned}$$

This simplifies to

$$\mathcal{L}_\xi\left(g_{\mu\nu}\right) = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu \tag{6.87}$$

A vector that makes the Lie derivate of the metric vanish is called a **Killing vector**; i.e. a vector ξ is a Killing vector if

$$\mathcal{L}_\xi\left(g_{\mu\nu}\right) = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0 \tag{6.88}$$

Equation (6.88) is called the Killing-equation.

Killing vectors generate isometries, meaning that the metric is invariant under the flow generated by a Killing vector. Thus, every Killing vector generates a certain symmetry and by Noether's theorem we expect them to produce conserved quantities. We shall see later how this works in detail when establishing results for the Komar mass and angular momentum in section 8.4.

The existence of Killing vectors can considerably simplify the geometry and often allows to find exact solutions (even with "paper-and-pencil"), which is another pragmatic reason why Killing vectors are useful.

Consider as an example the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{6.89}$$

Here, four Killing vectors exist, namely

$$\begin{aligned}\xi_0 &= \partial_t && (6.90)\\ \xi_1 &= \partial_\phi && (6.91)\\ \xi_2 &= -\cos\phi\partial_\theta + \sin\phi\cot\theta\partial_\phi && (6.92)\\ \xi_3 &= \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi && (6.93)\end{aligned}$$

ξ_0 creates time translations, so the fact that ξ_0 is a Killing vector means that the Schwarzschild metric (6.89) is static. $\xi_{1\ldots3}$ generate rotations on a 2-sphere—so, the Schwarzschild metric is spherically symmetric due to the fact that these are Killing vectors. The existence of four Killing vectors is one way to understand why Schwarzschild was able to find this solution merely a few weeks after Einstein wrote down the field equations of general relativity.

6.12. Tensor Densities

An object ist called a tensor density of weight ω if it transforms like

$$T_{\tilde{\alpha}\ldots\tilde{\beta}} = T_{\alpha\ldots\beta}\frac{\partial x^\alpha}{\partial x^{\tilde{\alpha}}}\cdots\frac{\partial x^\beta}{\partial x^{\tilde{\beta}}}\left(\det\left(\frac{\partial x^{\tilde{\mu}}}{\partial x^\mu}\right)\right)^\omega \tag{6.94}$$

Hence a tensor density transforms like a tensor under coordinate transformation except that it is additionally weighted by a power of the Jacobian.

6.12.1. The Levi-Civita Symbol as a Tensor Density

The Levi-Civita symbol is defined as

$$\tilde{\epsilon}_{\mu_1 \dots \mu_D} = \begin{cases} +1 & \text{for even permutation of } \mu_1 \dots \mu_D \\ -1 & \text{for odd permutation of } \mu_1 \dots \mu_D \\ 0 & \text{if 2 or more indices are equal} \end{cases} \tag{6.95}$$

The determinant of a matrix M can also be written with the Levi-Civita-Symbol

$$\tilde{\epsilon}_{\tilde{\mu}_1 \dots \tilde{\mu}_D} \det(M) = \tilde{\epsilon}_{\mu_1 \dots \mu_D} M^{\mu_1}{}_{\tilde{\mu}_1} \dots M^{\mu_D}{}_{\tilde{\mu}_D} \tag{6.96}$$

Hence the Levi-Civita-Symbol transforms under a general coordinate transformation $M^{\mu}{}_{\tilde{\mu}} = \frac{\partial x^{\mu}}{\partial x^{\tilde{\mu}}}$ as

$$\boxed{\tilde{\epsilon}_{\tilde{\mu}_1 \dots \tilde{\mu}_D} = \det\left(\frac{\partial x^{\tilde{\mu}}}{\partial x^{\mu}}\right) \tilde{\epsilon}_{\mu_1 \dots \mu_D} \frac{\partial x^{\mu_1}}{\partial x^{\tilde{\mu}_1}} \dots \frac{\partial x^{\mu_D}}{\partial x^{\tilde{\mu}_D}}} \tag{6.97}$$

This implies that the Levi-Civita-Symbol is a tensor density of weight 1.

In order to get an expression that will allow us to convert tensor densities of weight ω into tensors, we recall the transformation of an arbitrary metric.

$$g_{\tilde{\mu}\tilde{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\tilde{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\tilde{\nu}}} g_{\mu\nu} \tag{6.98}$$

Taking the determinant of (6.98) yields

$$g := \det\left(g_{\mu\nu}\right) \tag{6.99}$$

$$\tilde{g} = g \left(\det\left(\frac{\partial x^{\tilde{\mu}}}{\partial x^{\mu}}\right)\right)^{-2} \tag{6.100}$$

The transformation law (6.100) implies that the determinant of a metric is a tensor density of weight -2. Therefore we can use the determinant of the metric to promote tensor densities of weight ω to tensors.

$$(\text{tensor density of weight } \omega)\, |g|^{\frac{\omega}{2}} = \text{tensor} \tag{6.101}$$

Remembering that $d^D x$ is a tensor density of weight 1 the result of (6.101) can now be used to create an invariant volume element of the form

$$d^D x\, |g|^{\frac{1}{2}} = \text{invariant volume element} \tag{6.102}$$

$d^D x$ can also be written as

$$dx^0 \wedge dx^1 \wedge \dots \wedge dx^{D-1} = \frac{1}{D!} \tilde{\epsilon}_{\mu_1 \dots \mu_D} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D} \tag{6.103}$$

Combining the invariant volume element (6.102) and equation (6.103) yields

$$\sqrt{|g|} dx^0 \wedge dx^1 \wedge \dots \wedge dx^{D-1} = \frac{1}{D!} \epsilon_{\mu_1 \dots \mu_D} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D} \tag{6.104}$$

$$\rightarrow \underbrace{\epsilon_{\mu_1 \dots \mu_D}}_{\epsilon\text{-tensor}} = \tilde{\epsilon}_{\mu_1 \dots \mu_D} \sqrt{|g|} \tag{6.105}$$

This equation for the epsilon tensor (6.105) also shows that the Levi-Civita-Symbol in flat space(time) equals the epsilon-tensor since in that case $\sqrt{|g|} = 1$.

As a useful application let us consider now the covariant action of a free scalar field in D dimensions and derive the Klein–Gordon equation on some arbitrary curved spacetime:

$$\begin{aligned} S &\propto \int \mathrm{d}^{\mathrm{D}}\mathrm{x}\sqrt{|\mathrm{g}|}\mathrm{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &\rightarrow \delta S \propto \int \mathrm{d}^{\mathrm{D}}\mathrm{x}\delta\phi\partial_\mu\left(\sqrt{|\mathrm{g}|}\mathrm{g}^{\mu\nu}\partial_\nu\phi\right), \end{aligned}$$

using

$$\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = \nabla_\mu\left(\sqrt{|g|}g^{\mu\nu}\nabla_\nu\phi\right) = \sqrt{|g|}g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \Box\phi,$$

this can also be written as

$$\delta S \propto \int \mathrm{d}^{\mathrm{D}}\mathrm{x}\delta\phi\Box\phi.$$

The result above can also be used to derive the Laplace equation in arbitrary coordinate systems.

7. Hilbert Action and Einstein Field Equations

In this chapter we derive the Hilbert action. Its variation yields the Einstein equations through the principle of least action. First of all we want to obtain some kind of action that depends functionally on the metric.

$$S[g] = \int (volume)(scalar) \tag{7.1}$$

7.1. The Action Integral in QED

We have found an invariant expression for the volume in section 6.12. Only the scalar part of the action (7.1) has to be determined. In order to achieve this let us first take a little excursus to quantum electrodynamics. The action in QED should depend on A_μ, the covariant four potential of the electromagnetic field. Using the Minkowski metric the volume element of the action integral becomes d^4x since $\sqrt{|\det(\eta_{\mu\nu})|} = 1$. The only thing missing is a scalar element in the action integral which depends on A_μ. We can also write the scalar part of the action integral in terms of a derivative expansion

$$S[A_\mu] = \int d^4x \left[\alpha_1 A_\mu A^\mu + \alpha_2 \partial_\mu A^\mu + \alpha_3 (\partial_\mu A^\nu)(\partial_\nu A^\mu) \ldots\right] \tag{7.2}$$

The action integral should be invariant under gauge transformations $A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \chi$ with χ being an arbitrary scalar field. Therefore the action should only depend on the gauge invariant Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We hence obtain an expression for the action integral of the form

$$\begin{aligned} S[A_\mu] = \int d^4x \Big[&\alpha F_{\mu\nu}F^{\mu\nu} + \beta F_{\mu\nu}F_{\alpha\beta}\epsilon^{\mu\nu\alpha\beta} + \\ &\gamma(\Box F_{\mu\nu})F^{\mu\nu} + \delta(F_{\mu\nu}F^{\mu\nu})^2 + \ldots\Big] \end{aligned} \tag{7.3}$$

This scalar part of the action integral could have an arbitrary number of terms as long as they are gauge invariant. The dimensions of the coefficients α, β, γ, δ can be obtained by a dimensional analysis. Comparison with the dimension of the volume element which has the dimension of a $(\text{length})^4$ and the Faraday tensor which by definition has a dimension of a $(\text{length})^{-2}$ yields for α and β no dimension, for γ[4] $(\text{length})^2$, for δ $(\text{length})^4$ and for the higher order term scalars a dimension$>(\text{length})^4$.
If a field theory is valid for arbitrary energies/lengths then none of these derivative expansion terms could be neglected in the scalar part of the action integral. However, most field theories are not valid for arbitrary energies/lengths. QED for example is only valid up to a certain energy limit which is called UV cut. Since high energies correspond to small lengths, terms with coefficients that contain higher dimensions of length are suppressed by the UV cut-off scale. Therefore all higher order terms are suppressed by this UV cutoff and only α and β contribute significantly to the action. This leads to an expression of the electromagnetic action integral like

$$S[A_\mu] = \int d^4x \left[\alpha F_{\mu\nu}F^{\mu\nu} + \underbrace{\beta F_{\mu\nu}F_{\alpha\beta}\epsilon^{\mu\nu\alpha\beta}}_{\text{boundary term}}\right] \tag{7.4}$$

[4] $\dim \Box = -2$

7.2. Hilbert Action

The derivative expansion used to find an expression for the action in QED can also be used for the scalar part of the Hilbert action integral

$$S\left[g_{\mu\nu}\right] = -\frac{1}{2\kappa}\int d^4x\,\sqrt{|g|}\,\Big[-2\Lambda + R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + \ldots\Big] \tag{7.5}$$

Λ being the cosmological constant, R the Ricci scalar, $R_{\mu\nu}$ the Ricci Tensor and $C_{\mu\nu\alpha\beta}$ the Weyl tensor. The factor $\kappa = 8\pi$ is called gravitational coupling constant and refers to the strength of the gravitational interaction[5]. Now let us derive which parts of this action can be omitted and which not.

If the Lagrangian for the spacetime metric is to be a scalar, it cannot solely depend on the first derivatives of the metric since $\nabla_\alpha g_{\mu\nu} = 0$. Since the scalar part of the action should contain as few derivations as possible we have to find a scalar consisting of second derivatives. This leads to the Ricci scalar R as it is the only scalar linear in the second derivatives of the metric. By analogy to the QED case above we can neglect the higher order terms proportional to α, β, γ, $\ldots$. Therefore we obtain the Einstein-Hilbert action

$$\boxed{S\left[g_{\mu\nu}\right] = -\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,\left[R - 2\Lambda\right]} \tag{7.6}$$

7.3. Einstein Field Equations

The Einstein field equations can now be obtained by varying the Einstein-Hilbert action (7.6) with respect to the metric ($R = g^{\mu\nu}R_{\mu\nu}$).

$$\delta S \propto \int d^Dx\,\left[\left(\delta\sqrt{-g}\right)(R - 2\Lambda) + \sqrt{-g}\left(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)\right] = 0 \tag{7.7}$$

The variation of the first term is pretty straight forward $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g$. First let us write g as

$$g = \det\left(M^{-1}g_{\mu\nu}M\right) = \prod_{i=1}^{D}\lambda_i = \exp\left(\sum_{i=1}^{D}\ln\left(\lambda_i\right)\right) = e^{\mathrm{Tr}\,(\ln(g_{\mu\nu}))}, \tag{7.8}$$

with M being an invertible matrix that brings $g_{\mu\nu}$ into upper triangular form and λ_i the eigenvalues of $g_{\mu\nu}$. This implies

$$\delta g = g\,g^{\mu\nu}\delta g_{\mu\nu} = -g\,g_{\mu\nu}\delta g^{\mu\nu}, \tag{7.9}$$

where we have made use of the relation

$$\delta\left(g_{\mu\alpha}g^{\alpha\nu}\right) = 0. \tag{7.10}$$

The variation of the determinant of the metric (7.9) follows directly from the following relation

$$g_{\mu\nu}g^{\nu\alpha} = \delta_\mu{}^\alpha \rightarrow \delta\left(g_{\mu\nu}g^{\nu\alpha}\right) = \delta g_{\mu\nu}g^{\nu\alpha} + g_{\mu\nu}\delta g^{\nu\alpha} = 0 \tag{7.11}$$

After this short calculation the variation of the Einstein-Hilbert action is given by

$$\delta S = \int d^Dx\,\sqrt{-g}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda\right)\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right] = 0 \tag{7.12}$$

[5]In SI units this factor equals $\frac{8\pi G_N}{c^4}$

The remaining term $g^{\mu\nu}\delta R_{\mu\nu}$ is a boundary term and can hence be dropped.

$$\begin{aligned}\delta R_{\mu\nu} &= \Big[\partial_\lambda \delta\Gamma^\lambda{}_{\mu\nu} - \partial_\mu \delta\Gamma^\lambda{}_{\lambda\nu} + \delta\Gamma^\alpha{}_{\mu\nu}\Gamma^\lambda{}_{\lambda\alpha} + \Gamma^\alpha{}_{\mu\nu}\delta\Gamma^\lambda{}_{\lambda\alpha} \\ &\quad - \delta\Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\alpha} - \Gamma^\alpha{}_{\mu\lambda}\delta\Gamma^\lambda{}_{\nu\alpha}\Big]\end{aligned} \tag{7.13}$$

This yields

$$\delta R_{\mu\nu} = \underbrace{\nabla_\lambda \delta\Gamma^\lambda{}_{\mu\nu} - \nabla_\mu \delta\Gamma^\lambda{}_{\lambda\nu}}_{\text{boundary term}} \tag{7.14}$$

Therefore we obtain the vacuum Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0} \tag{7.15}$$

For many applications Λ is neglected ($\Lambda \approx 0$). Hence the vacuum Einstein equations can often be simplified to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G_{\mu\nu} = 0 \tag{7.16}$$

and by taking the trace of the vacuum Einstein equaions (7.16)

$$\mathrm{Tr}\left(G_{\mu\nu}\right) = R\left(1 - \frac{D}{2}\right) = 0 \tag{7.17}$$

it follows that the Ricci scalar has to be 0 in higher dimensions i.e $D \neq 2$. $G_{\mu\nu}$ is called Einstein tensor. This reduces the Einstein vacuum equations to

$$\boxed{R_{\mu\nu} = 0} \tag{7.18}$$

Even though equation (7.18) looks simple it nevertheless represents 10 non linear coupled PDEs which are very hard to solve[6].
If we now allow matter to appear in addition to the Einstein-Hilbert action we obtain the inhomogenous Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu}} \tag{7.19}$$

The tensor $T_{\mu\nu}$ is called energy-momentum tensor. Taking the covariant divergence of the left hand side of the inhomogeneous Einstein equations yields zero, since the covariant divergence of the Einstein tensor vanishes — $\nabla_\mu G^{\mu\nu} = \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0$ — and due to metric compatibility $\nabla_\mu\left(\Lambda g^{\mu\nu}\right) = 0$. Therefore, also the covariant divergence of the energy-momentum tensor vanishes.

$$\boxed{\nabla_\mu T^{\mu\nu} = 0} \tag{7.20}$$

Equation (7.20) is called energy-momentum conservation. In chapter 4 we quoted John Wheeler and found out, that the geodesic equation is equivalent to the statement that "space tells matter how to move". We also stated, that "matter tells spacetime how to curve" and promised to derive the equation motivating that statement in this chapter. If we take a closer look at the inhomogeneous Einstein equations (7.19) we see that indeed matter, or to be more accurate the Energy-Momentum tensor tells spacetime how to curve. Hence the geodesic equation (4.25) and the inhomogeneous Einstein equations (7.19) motivated the quote of John Wheeler.

[6] in the case $D = 4$

8. Spherically Symmetric Black Holes and the Birkhoff Theorem

In this section we are going to prove that the only spherically symmetric vacuum solution in general relativity is the Schwarzschild solution which is static. Then we discuss some concepts concerning Killing vector fields namely Killing horizons, the Killing vector lemma and surface gravity. After motivating and introducing global coordinates for the Schwarzschild metric we prove the zeroth law of black hole mechanics. Finally we show how a mass can be defined in general relativity.

8.1. Birkhoff's Theorem

We are going to sketch the proof of Birkhoff's theorem and derive the Schwarzschild solution

> **Birkhoff's theorem:** All spherically symmetric vacuum solutions of Einsteins field equations are static.

First we have to declare what the terms mean:

- **Vacuum solution:** Remembering the Einstein equations and looking back to (7.18) vacuum solution means that the Ricci-tensor vanishes $R_{\mu\nu} = 0$.

- **Stationary spacetime:** In a stationary spacetime it is possible to find a timelike Killing vector field[7] k^μ. This means that if you look at the spacetime after some time is elapsed it should look the same. More practical insert $t \mapsto t + a$ (a is a constant) into our metric $g_{\mu\nu}$ and check if it changes.

  As an example imagine a fluid which is rotating around some axis. For the fluid to be stationary it is allowed to flow as long as the velocity distribution of the fluid does not change in time.

- **Static spacetime:** A spacetime is static if in addition to stationary the Killing vector field is orthogonal to a spacelike hypersurface[8]. That means if coordinates are used in which the killing vector $k^\mu = \partial_t^\mu$ the metric is not allowed to have any cross terms of the form $dtdx^i$. Looking for example at the Schwarzschild solution (6.89) we have the Killing vector field $k^\mu = \partial_t^\mu$; there are no cross terms $dt\, dx^i$. Thus the Schwarzschild spacetime is static.

  As another example we use again the example of our rotating fluid. In a static spacetime the fluid is not allowed to move contrary to the stationary case. It is interesting to note that the failure of a spacetime to be static is due to not being time reflective. While in the case of staticity the time reflection $t \mapsto -t$ makes no difference in stationary spacetimes the cross terms get influenced. The rotating fluid is stationary and not static because the time reflection inverts the rotation direction.

- **Spherical symmetric spacetime:** With an spherical symmetric spacetime[9] we mean a spacetime which has the same symmetries as a 2-sphere i.e. a $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ term.

[7] This is equivalent to the existence of a one-parameter group of isometries whose orbits are timelike curves.

[8] An necessary and sufficient condition for a timelike Killing vector field to be hypersurface orthogonal is $k_{[\mu}\nabla_\nu k_{\pi]} = 0$ (this can be derived using the Frobenius Theorem).

[9] A more precise definition is: A spherical symmetric spacetime has an isometry group which contains a subgroup isomorphic to the group SO(3). The orbits of the SO(3) have to be 2-spheres.

Using spherical symmetry one can show[10] that the spacetime metric has the form

$$ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + X(x^\gamma)d\Omega(\theta,\varphi)^2 \tag{8.1}$$

where in this case the greek indices only have the values $0, 1$. So $g_{\alpha\beta}(x^\gamma)$ is a two dimensional metric and $X(x^\gamma)$ is a scalar field. We are now able to fix locally our coordinates x^γ in such a way that

$$X(x^\gamma) = r^2 \qquad g_{tr}(x^\gamma) = 0 \tag{8.2}$$

where we use $t = x^0$ and $r = x^1$. So it has the form

$$ds^2 = g_{tt}(t,r)dt^2 + g_{rr}(t,r)dr^2 + r^2 d\Omega^2 \tag{8.3}$$

Since our spacetime is Lorentzian we require $g_{tt}(t,r)$ to be negative and $g_{rr}(t,r)$ to be positive which can be achieved by writing

$$ds^2 = -e^{2a(t,r)}dt^2 + e^{2b(t,r)}dr^2 + r^2 d\Omega^2 \tag{8.4}$$

Since our metric should be a solution to the vacuum Einstein equations ($R_{\mu\nu} = 0$) we calculate now the Ricci tensor. The nonzero components are given by (the dot $\dot{}$ means $\partial/\partial t$ while the prime $'$ means $\partial/\partial r$)

$$R_{tt} = e^{2(a-b)}\left(\frac{2}{r}a' + a'^2 - a'b' + a''\right) + \dot{a}\dot{b} - \dot{b}^2 - \ddot{b} = 0 \tag{8.5}$$

$$R_{tr} = \frac{2\dot{b}}{r} = 0 \tag{8.6}$$

$$R_{rr} = e^{2(b-a)}\left(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}\right) - a'' - a'^2 + a'b' + \frac{2}{r}b' = 0 \tag{8.7}$$

$$R_{\theta\theta} = e^{-2b}\left(-1 - ra' + rb'\right) + 1 = 0 \tag{8.8}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\sin^2\theta = 0 \tag{8.9}$$

Looking at equation (8.6) we see that $R_{tr} = 0$ only if

$$\dot{b}(t,r) = 0 \to b(t,r) = b(r) \tag{8.10}$$

which is the first half of Birkhoff's theorem.

Using (8.8) we calculate

$$\dot{R}_{\theta\theta} = -re^{-2b(r)}\dot{a}'(t,r) = 0 \to \dot{a}'(t,r) = 0 \to a(t,r) = a(r) + f(t) \tag{8.11}$$

Inserting this into the metric (8.4) gives an $-e^{2(a(r)+f(t))}dt^2$ term. We can parameterize the time such that $dt \to e^{-f(t)}dt$. Therefore we have $a(t,r) = a(r)$ and we get the metric

$$ds^2 = -e^{2a(r)}dt^2 + e^{2b(r)}dr^2 + r^2 d\Omega^2 \tag{8.12}$$

which is stationary since there is no time dependence in the $g_{\mu\nu}$ terms. There is a timelike Killing vectorfield $k^\mu = \partial_t^\mu$ where t parametrizes the metric and there are no cross terms with dt which implies staticity and proves Birkhoff's theorem.

Furthermore we are now showing that this solution is the Schwarzschild metric. We combine (8.5) and (8.7) suitably to get

$$e^{2(b-a)}R_{tt} + R_{rr} = \frac{2}{r}(a' + b') = 0 \tag{8.13}$$

[10] See for example [2, Section 5.2] or [3, Appendix B].

which is solved by

$$a(r) = -b(r) + \widetilde{c} \tag{8.14}$$

The constant $\widetilde{c}$ can be set to zero since we can again reparametrize $dt \to e^{-\widetilde{c}}dt$.

One of the remaining nonzero Ricci tensor components is

$$R_{\theta\theta} = 1 - e^{2a}(1 + 2ra') = 1 - (re^{2ar})' = 0 \tag{8.15}$$

which leads to

$$e^{2a} = 1 + \frac{c}{r} \tag{8.16}$$

If we compare the integration constant c with the weak field regime in the Newtonian limit (4.26) we get $c = -2M$. So our solution is the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{8.17}$$

So every spherical symmetric vacuum spacetime is described by the Schwarzschild solution. A consequence is that there are no gravitational waves if a spherical symmetric star collapses.

8.2. Killing vectors

Null hypersurfaces

A general co-dimension 1 hypersurface is defined by some condition on the coordinates, $f(x^\mu) = 0$ where f is some smooth function of the coordinates x^μ (a simple example would be a linear function of the coordinates in 3-dimensional Euclidean space, which defines a 2-dimensional plane). The normal vector to this hypersurface is given by $k = \tilde{f}(x^\alpha)g^{\mu\nu}\partial_\nu f \partial_\mu$, where $\tilde{f} \neq 0$ is some arbitrary normalization function. If the normal vector becomes null, $k^2 = 0$, the hypersurface is called a null hypersurface.

Example: consider constant r hypersurfaces in the Schwarzschild spacetime, $f = r - 2M$. Then the normal vector has a norm given by $k^2 = \tilde{f}^2 g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) = \tilde{f}^2 g^{rr} = (1 - 2M/r)\tilde{f}^2$. The hypersurface becomes a null hypersurface at $r = 2M$.

Killing horizon

In a stationary spacetime there exists a timelike Killing vector k^μ. A null hypersurface N is a *Killing horizon* of a Killing vector field k if, on N, k is normal to N. In the case of Schwarzschild geometry with the killing vector $k^\mu = (\partial_t)^\mu$ we get the Killing horizon $r = 2M$

$$k^\mu k_\mu = -\left(1 - \frac{2M}{r}\right) = 0 \Longrightarrow r = 2M \tag{8.18}$$

Killing vector lemma

$$\boxed{\text{Killing vector lemma:} \quad \nabla_\mu \nabla_\nu k^\rho = R^\rho_{\nu\mu\alpha} k^\alpha} \tag{8.19}$$

Before we discuss the consequences of the Killing vector lemma we prove it. Remember Killing's equation (6.88)

$$\nabla_{(\mu} k_{\nu)} = 0 \tag{8.20}$$

We start with the definition of the Riemann tensor

$$\left[\nabla_\mu, \nabla_\nu\right] k^\rho = R^\rho_{\alpha\mu\nu} k^\alpha \tag{8.21}$$

which we expand. With the help of Killing's equation we get

$$\nabla_\mu \nabla_\nu k^\rho + \nabla_\nu \nabla^\rho k_\mu = R^\rho_{\alpha\mu\nu} k^\alpha \tag{8.22}$$

Now we lower all indices and write down the same equation with permutations of $(\mu\nu\rho)$ to $(\nu\rho\mu)$ and $(\rho\mu\nu)$. If we add the $(\mu\nu\rho)$ to the $(\nu\rho\mu)$ term and subtract the $(\rho\mu\nu)$ term we obtain

$$2\nabla_\nu \nabla_\rho k_\mu = (R_{\rho\alpha\mu\nu} + R_{\mu\alpha\nu\rho} - R_{\nu\alpha\rho\mu}) k^\alpha \tag{8.23}$$

Using the antisymmetry condition $R_{\rho[\alpha\mu\nu]} = 0$ (6.68) leads us to

$$\nabla_\nu \nabla_\rho k_\mu = R_{\alpha\nu\rho\mu} k^\alpha \tag{8.24}$$

Permuting the indices of the Riemann tensor gives

$$\nabla_\nu \nabla_\rho k_\mu = R_{\mu\rho\nu\alpha} k^\alpha \tag{8.25}$$

Renaming and raising the indices completes the proof.

As you can see via the Killing vector lemma (8.19) the Killing vector field is completely determined by k^μ and $\nabla_\nu k^\mu$. So for an n dimensional manifold there are maximally as many independent Killing vector fields as we are able to construct initial data for k^μ and $\nabla_\nu k^\mu$. A maximally symmetric space has

$$\underbrace{n}_{k^\mu} + \underbrace{\frac{n(n-1)}{2}}_{\nabla_\nu k^\mu \text{ is antisymmetric}} = \frac{n(n+1)}{2} \tag{8.26}$$

linear independent Killing fields. Consider as an example Euclidean space $\mathbb{R}^n$ where we have n translation and $\frac{n(n-1)}{2}$ rotation symmetries.

Surface gravity

On a stationary black hole there exists a Killing field k^μ which is normal to the horizon. Since $k^\nu k_\nu = 0$ on the horizon $\nabla_\mu (k^\nu k_\nu)$ is also normal to the horizon. So on the horizon there exists a function κ defined by

$$\frac{1}{2} \nabla_\mu (k^\nu k_\nu) = -\kappa k_\mu \tag{8.27}$$

κ is the *surface gravity* of the black hole and is constant on orbits of k^μ.

To get a explicit formula for κ we use the condition for hypersurface orthogonality of section 8.1 which leads us to

$$k_{[\mu} \nabla_\nu k_{\rho]} = 0 \tag{8.28}$$

Expanding (8.28) and using Killing's equation (8.20) we find

$$k_\rho \nabla_\mu k_\nu = -2 k_{[\mu} \nabla_{\nu]} k_\rho \tag{8.29}$$

We multiply it now by the antisymmetric $(\nabla^\mu k^\nu)$ which makes the brackets on the r.h.s. needless

$$k_\rho (\nabla^\mu k^\nu)(\nabla_\mu k_\nu) = -2 (\nabla^\mu k^\nu) k_\mu \nabla_\nu k_\rho \tag{8.30}$$

$$= 2 k_\mu (\nabla^\nu k^\mu) \nabla_\nu k_\rho \tag{8.31}$$

$$= \nabla^\nu (k_\mu k^\mu) \nabla_\nu k_\rho \tag{8.32}$$

Now we use the definition of the surface gravity (8.27) and again Killings's equation to get

$$k_\rho(\nabla^\mu k^\nu)(\nabla_\mu k_\nu) = -2\kappa k^\nu \nabla_\nu k_\rho \tag{8.33}$$
$$= 2\kappa k^\nu \nabla_\rho k_\nu \tag{8.34}$$
$$= \kappa \nabla_\rho (k^\nu k_\nu) \tag{8.35}$$
$$= -2\kappa^2 k_\rho \tag{8.36}$$

So the explicit formula for the surface gravity κ evaluated at the horizon $\mathcal{H}$ is

$$\boxed{\kappa^2 = -\frac{1}{2}(\nabla^\mu k^\nu)(\nabla_\mu k_\nu)\big|_{\mathcal{H}}} \tag{8.37}$$

Evaluating (8.37) for the Schwarzschild metric gives

$$\kappa = \frac{1}{4M} \tag{8.38}$$

Before we come back to Killing vector fields we need to introduce some new concepts.

8.2.1. Spacetime Singularity

We have already mentioned in the beginning of section 5.1 that there are different causes for singularities. In general it is not easy to determine which type of singularity a region or point is but we give here some guidelines:

1. *Real singularity:* The spacetime is in fact singular. One way to prove that a point is a real singularity is by calculating a curvature scalar like $R_{\alpha\beta\chi\delta}R^{\alpha\beta\chi\delta}$ and show that it blows up at the suspected region. You have to show that the point can be reached by an geodesics with some finite affine parameter to show that the singularity is not "at infinity" like e.g. $r \to \infty$ in Schwarzschild. Be aware that it is a sufficient but not a necessary condition. So even if you are not able to find a curvature scalar that blows up it could be that there is a real singularity. Conversely if you are able to find a curvature scalar which blows up at a finite affine parameter you have found a real singularity.

2. *Coordinate singularity:* The spacetime is nonsingular but the coordinates fail to cover the region properly. In this case we try to find a coordinate transformation of the metric where the metric in the new coordinates is not singular anymore. It is possible to extend the original metric if the transformed metric includes the original as a proper set.

In Schwarzschild coordinates the calculation of $R_{\alpha\beta\chi\delta}R^{\alpha\beta\chi\delta}$ shows that there is a real singularity at $r = 0$ which can be reached by a geodesic with a finite affine parameter. The singularity at $r = 2M$ will be discussed in the next two sections.

8.2.2. Near horizon region of Schwarzschild geometry

We are now taking a closer look at the near horizon region $r = 2M$ of the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\Omega^2 \tag{8.39}$$

We now identify

$$r - 2M = \frac{x^2}{8M} \tag{8.40}$$

Transforming this equation and substituting the surface gravity for the Schwarzschild metric (8.38) gives

$$1-\frac{2M}{r}=\frac{(\kappa x)^2}{1+(\kappa x)^2} \tag{8.41}$$

Near the horizon $r=2M$ or accordingly at $x=0$ we get

$$1-\frac{2M}{r}\approx(\kappa x)^2 \tag{8.42}$$

If we differentiate (8.40)

$$dr^2=(\kappa x)^2dx^2 \tag{8.43}$$

and insert this and (8.42) into the Schwarzschild solution we get for the near horizon region $r\approx 2M$ the Rindler spacetime

$$\boxed{ds^2\approx-(\kappa x)^2dt^2+dx^2+\frac{1}{4\kappa^2}d\Omega^2} \tag{8.44}$$

We now analyze the 2-dim Rindler spacetime

$$ds^2=-(\kappa x)^2dt^2+dx^2 \qquad x>0 \qquad -\infty<t<\infty \tag{8.45}$$

This metric seems to be singular at $x=0$. Since no curvature invariant shows a bad behavior at $x\to 0$ we guess that it is maybe a coordinate singularity. So we search for a proper coordinate transformation to eliminate the singularity. We first introduce null coordinates. They are constant along incoming/outgoing null geodesics. The condition for null geodesics parametrized by an affine parameter is

$$g_{\mu\nu}k^\mu k^\nu=-(\kappa x)^2\dot{t}^2+\dot{x}^2=0 \tag{8.46}$$

Transforming (8.46) to

$$\left(\frac{dt}{dx}\right)^2=\frac{1}{(\kappa x)^2} \tag{8.47}$$

and solving this differential equation for t gives

$$\kappa t=\pm\ln x+\text{constant} \tag{8.48}$$

So we define the outgoing null coordinate u and the ingoing null coordinate v by

$$u=\kappa t-\ln x \tag{8.49}$$

$$v=\kappa t+\ln x \tag{8.50}$$

The Rindler metric in our (u,v) coordinates is

$$ds^2=-e^{v-u}du\,dv \tag{8.51}$$

Since our singularity is not removed yet we make another transformation

$$U=-e^{-u} \tag{8.52}$$

$$V=e^{v} \tag{8.53}$$

to get

$$ds^2=-dU\,dV \tag{8.54}$$

The original Rindler coordinates with $x>0$ cover only the $U<0$, $V>0$ region. Since there is no singularity at $U=V=0$ anymore we extend our spacetime to $-\infty<U,V<\infty$.

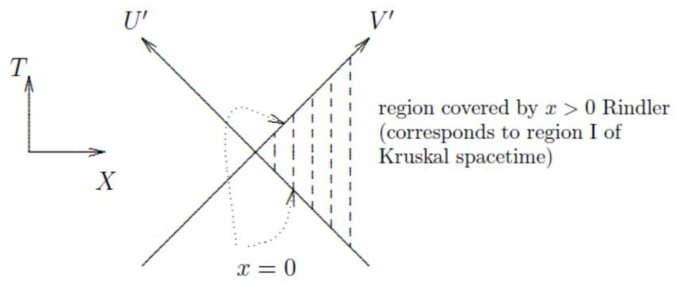

Figure 8.1: Rindler spacetime (Source: Townsend-Black holes: Lecture notes)

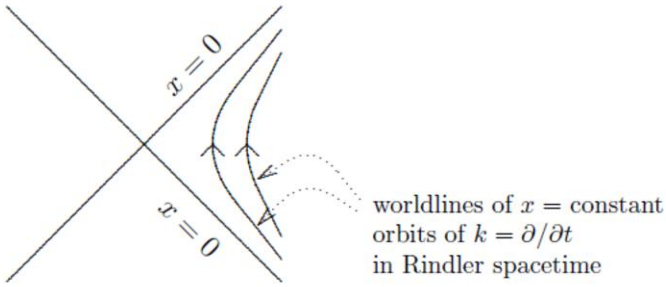

Figure 8.2: Worldlines of particles moving at constant x. It is a hyperbolic motion with constant proper acceleration $a = \frac{1}{x}$. (Source: Townsend-Black holes: Lecture notes)

So in our approximation there are no difficulties at $r = 2M$. We now make the final transformation

$$T = \frac{U+V}{2} \tag{8.55}$$

$$X = \frac{V-U}{2} \tag{8.56}$$

to convert the metric in the well known form

$$ds^2 = -dT^2 + dX^2 \tag{8.57}$$

This shows that the Rindler spacetime is the 2-dimensinal Minkowski spacetime in unusual coordinates.

The original Rindler coordinates with $x > 0$ cover only a certain region (see figure 8.1) of the Minkowski spacetime (see figure 8.1). The time translation symmetry of the Rindler metric corresponds to the boost symmetry of Minkowski space (see figure 8.2).

The name "surface gravity" can be explained by the following proposition: *The surface gravity κ is the acceleration of a static particle near the horizon as measured at spatial infinity.*

So κ is the acceleration a particle needs to stay at a static orbit compared to an observer at infinity. For more information please take a look at [4, Section 2.3.7]

8.2.3. Global coordinates of the Schwarzschild geometry

In this section we will proceed very similar to the previous section. We start again with the Schwarzschild metric (8.39) and search for radial null geodesics. They have to satisfy

$$\left(\frac{dt}{dr}\right)^2 = \frac{1}{\left(1 - \frac{2M}{r}\right)^2} \tag{8.58}$$

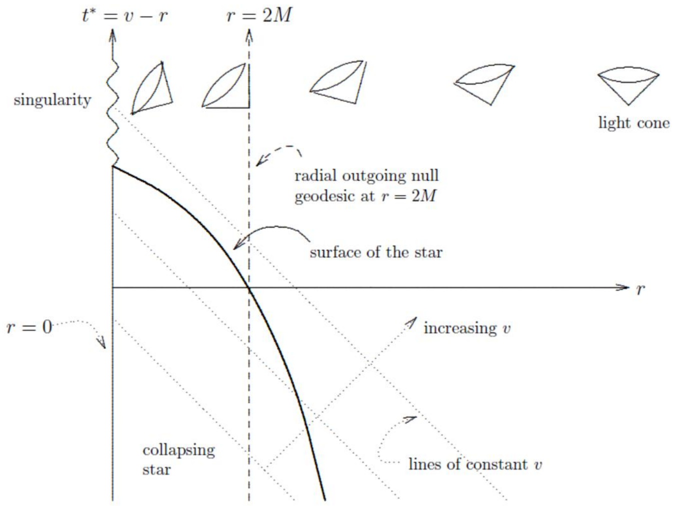

Figure 8.3: Collapsing star in ingoing Eddington-Finkelstein coordinates (Source: Townsend-Black holes: Lecture notes)

which can be solved by

$$t = \pm r^* + \text{constant} \tag{8.59}$$

where r^* denotes the *Regge-Wheeler radial coordinate*

$$r^* = r + 2M \ln \left| \frac{r - 2M}{2M} \right| \tag{8.60}$$

So we define the ingoing radial null coordinate v by

$$v = t + r^* \qquad -\infty < v < \infty \tag{8.61}$$

which leads to the Schwarzschild metric in *ingoing Eddington-Finkelstein coordinates* (v, r, θ, φ)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\ dr\ dv + r^2 d\Omega^2 \tag{8.62}$$

Here we can see already that the metric can be extended to $r > 0$. To see a collapsing star ant the light-cone structure in ingoing Eddington-Finkelstein coordinates look at figure 8.3. We define the outgoing radial null coordinate u by

$$u = t - r^* \qquad -\infty < v < \infty \tag{8.63}$$

and transform the Schwarzschild metric in *outgoing Eddington-Finkelstein* coordinates (u, r, θ, φ)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2\ dr\ du + r^2 d\Omega^2 \tag{8.64}$$

We combine ingoing and outgoing Eddington-Finkelstein coordinates to get

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\ dv + r^2 d\Omega^2 \tag{8.65}$$

We finally introduce

$$U = -e^{-u/4M} \tag{8.66}$$

$$V = e^{-v/4M} \tag{8.67}$$

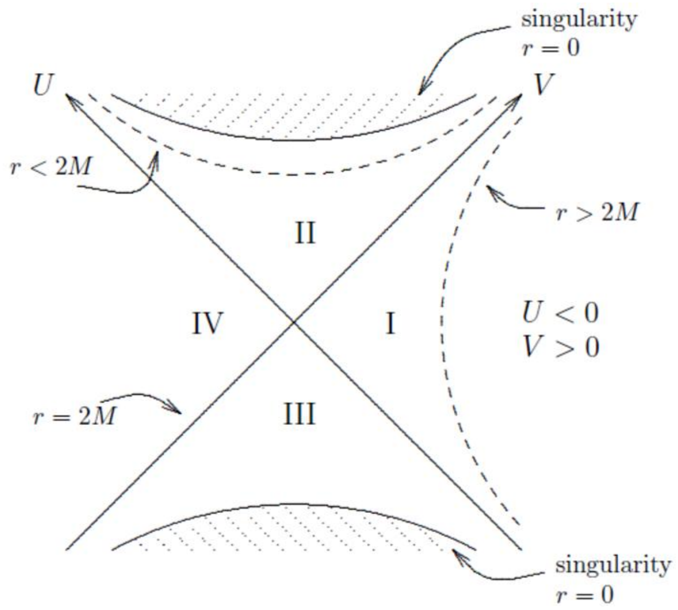

Figure 8.4: Kruskal diagram of Schwarzschild space. Note that each point corresponds to a 2-sphere. (Source: Townsend-Black holes: Lecture notes)

to get the Schwarzschild metric in *Kruskal-Szekeres coordinates*

$$\boxed{ds^2 = -\frac{32M^3}{r}e^{-r/2M}dU\;dV + r^2 d\Omega^2} \tag{8.68}$$

where r is given implicitly by

$$UV = -\frac{r-2M}{2M}\;e^{r/2M} \tag{8.69}$$

There is no singularity at $r = 2M$ (i.e. $U = 0$, $V = 0$) so we are allowed to extend the Schwarzschild solution to all values of U and V which are compatible with $r > 0$.

The Kruskal extension of Schwarzschild spacetime (see figure 8.4) has the following structure:

- Region I: It is our initial "outside region" $r > 2M$ of the Schwarzschild solution.
- Region II: This is the region of the black hole. Once anything crosses the $r = 2M$ border between region I and region II it in finite proper time ends up at $r = 0$.
- Region III: This time-reversed region of region II is called a white hole. Anything staying in this region has to leave region III after finite proper time.
- Region IV: This asymptotically flat space has the same properties as region I. There is no possibility of classical communication between region I and region IV.

How serious should we take this diagram? Since the Schwarzschild solution is a vacuum solution it only describes the region outside of a collapsing star. The past of the spacetime of a collapsing star is not fully described by the Schwarzschild metric so region III and IV are replaced by the spacetime of the matter distribution (see figure 8.5). Region I and II however, should be taken seriously.

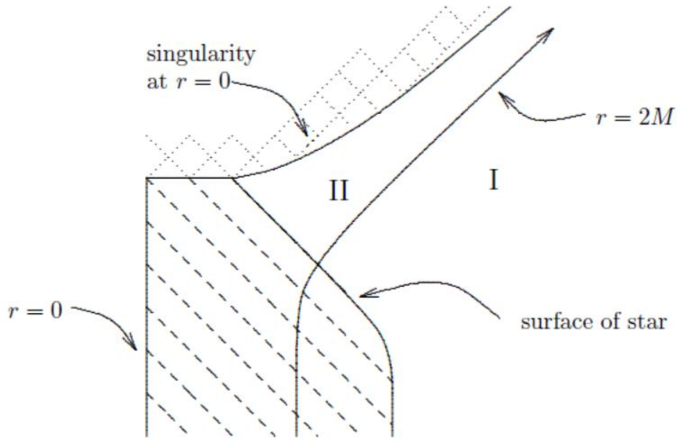

Figure 8.5: Spacetime of a gravitational collapse of a spherical symmetric body. The dashed region is not described by the Schwarzschild solution ($T_{\mu\nu} \neq 0$). (Source: Townsend-Black holes: Lecture notes)

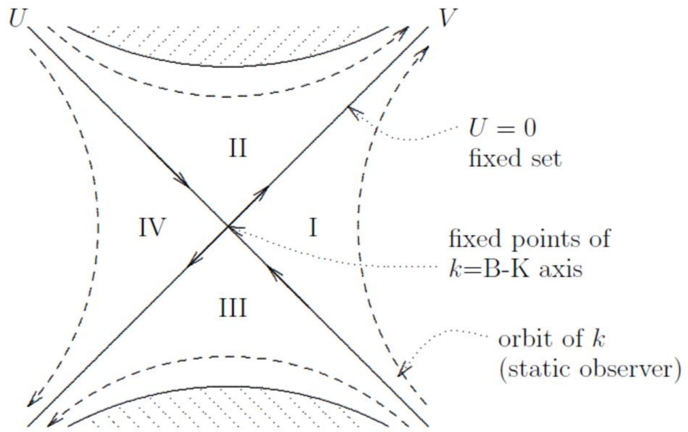

Figure 8.6: Orbits of the Killing vector field in Kruskal coordinates. (Source: Townsend-Black holes: Lecture notes)

8.3. Zeroth law of black hole mechanics

We are now going back to our discussion of Killing vectors. The time translation isometry in Kruskal coordinates is generated by the Killing vector field

$$k^\mu = \frac{1}{4M}(V\partial_V - U\partial_U)^\mu \tag{8.70}$$

which is equal to $k = \partial_t$ in region I. It follows that $\{U = 0\}$ and $\{V = 0\}$ are fixed sets (see figure 8.6). The point $\{U = V = 0\}$ is a fixed point (2-sphere) of k^μ called the bifurcation point (bifurcation 2-sphere). Note that the Killing field vanishes at this locus. In other words, to have a bifurcate Killing horizon we need $k^\mu|_S = 0$, where S is an $(n-2)$-dimensional spacelike hypersurface where the null surfaces generating the Killing horizon intersect.

Now we want to prove that the surface gravity κ is constant on the killing horizon. So we use (8.37) to get

$$k^\mu \nabla_\mu \kappa^2|_{\mathcal{H}} = -\frac{1}{2}k^\mu \nabla_\mu [(\nabla^\alpha k^\nu)(\nabla_\alpha k_\nu)]|_{\mathcal{H}} \tag{8.71}$$

$$= -k^\mu (\nabla^\alpha k^\nu)\nabla_\mu \nabla_\alpha k_\nu|_{\mathcal{H}} \tag{8.72}$$

Now use the Killing vector lemma (8.19) and the fact that $R_{\nu\alpha\mu\rho}$ is antisymmetric (6.65) in $(\mu\rho)$ while the Killing vector fields are symmetric in this indices

$$k^\mu \nabla_\mu \kappa^2\big|_{\mathcal{H}} = -(\nabla^\alpha k^\nu) R_{\nu\alpha\mu\rho} k^\mu k^\rho\big|_{\mathcal{H}} = 0 \tag{8.73}$$

So we have the proof that κ is constant along orbits of k^μ.

Now we want to prove that κ is also constant on the whole bifurcation 2-sphere S. We take a vector field t^μ tangent to S to prove

$$t^\mu \nabla_\mu \kappa^2\big|_S = -t^\mu (\nabla^\alpha k^\nu) \nabla_\mu \nabla_\alpha k_\nu\big|_S = -(\nabla^\alpha k^\nu) R_{\nu\alpha\mu\rho} t^\mu k^\rho\big|_S = 0 \tag{8.74}$$

In the last equality we used that we have a bifurcate Killing horizon, i.e., $k^\rho|_S = 0$. So κ is constant along the bifurcation sphere and we get:

Zeroth law of black hole mechanics: κ is constant on a (bifurcate) Killing horizon $\mathcal{H}$.

Note that this statement resembles the zeroth law of thermodynamics if we identify surface gravity κ with temperature (up to some multiplicative constant). Thus, we have a first weak hint that black holes might be thermal states; of course, we need much more information to really conclude this (see the lectures Black Holes II), but the conclusion turns out to be correct.

8.4. Komar mass

The Komar mass is a concept of mass in general relativity. Note that is is restricted to stationary spacetimes and it is not the only mass definition in general relativity.

In this section we are using Stokes theorem so remember

$$\int_M d\omega = \int_{\partial M} \omega \tag{8.75}$$

$$\int_M d^D x \, \sqrt{|g|} \, \nabla_\mu V^\mu = \int_{\partial M} d^{D-1} x \, \sqrt{|\gamma|} \, n_\mu V^\mu \tag{8.76}$$

where γ is the induced metric at the boundary, ∂M is the boundary of M and n_μ is the unit normal vector to ∂M.

As motivation we discuss the definition of charge Q in electrodynamics. One of Maxwell's equations is

$$\nabla_\mu F^{\nu\mu} = j^\nu \tag{8.77}$$

where $F^{\nu\mu}$ is the antisymmetric electromagnetic field tensor and j^ν is the current density. It is important to note that j^ν is a conserved quantity

$$\nabla_\nu \nabla_\mu F^{\nu\mu} = \frac{1}{2}[\nabla_\nu, \nabla_\mu] F^{\nu\mu} = R_{\nu\mu} F^{\nu\mu} = \nabla_\nu j^\nu = 0 \tag{8.78}$$

Our definition of the 'Komar charge' is

$$Q_K := \frac{1}{4\pi} \int_M d^3 x \, \sqrt{|\gamma|} \, n_\mu j^\mu \tag{8.79}$$

$$= -\frac{1}{4\pi} \int_M d^3 x \, \sqrt{|\gamma|} \, n_\nu \nabla_\mu F^{\mu\nu} \tag{8.80}$$

and leads with the help of Stokes theorem (8.76) to the so called Komar integral

$$Q_K = -\frac{1}{4\pi} \int_{\partial M} d^2 x \, \sqrt{|\hat{\gamma}|} \, n_\nu \sigma_\mu F^{\mu\nu} \tag{8.81}$$

where $\hat{\gamma}$ is the induced metric at the boundary ∂M and σ_n is the outward pointing unit normal. Considering

$$A_0 = \frac{Q_C}{r} \qquad F_{0r} = -\partial_r A_0 = \frac{Q_C}{r^2} \tag{8.82}$$

shows us that our definition reduces to

$$Q_K = -\frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} d\phi\ \sin^2\theta\, r^2 \left(-\frac{Q_C}{r^2}\right) = Q_C \tag{8.83}$$

where Q_C is the Coulomb charge, thus the Komar charge is equivalent to the Coulomb charge.

In general relativity the situation is more difficult. We need again a conserved mass-energy current density

$$\nabla_\nu j^\nu = 0 \tag{8.84}$$

If we would define it equivalently to special relativity

$$j^\nu = T^{\nu\mu} v_\mu \tag{8.85}$$

where v^μ is a velocity our conservation condition would require

$$\nabla_\nu j^\nu = \nabla_\nu (T^{\nu\mu} v_\mu) = (\underbrace{\nabla_\nu T^{\nu\mu}}_{=0}) v_\mu + \underbrace{T^{\nu\mu}(\nabla_\nu v_\mu)}_{=0 \text{ if } \nabla_{(\nu} v_{\mu)}=0} = 0 \tag{8.86}$$

That means that we need a Killing vector field for a conserved quantity.

So we define the conserved current

$$j^\mu := R^{\mu\nu} k_\nu \tag{8.87}$$

We suppose now that there is a timelike killing vector k_ν. We define the Komar mass by

$$M_K := \frac{1}{4\pi}\int_M d^3x\ \sqrt{|\gamma|}\ n_\mu j^\mu \tag{8.88}$$

$$= \frac{1}{4\pi}\int_M d^3x\ \sqrt{|\gamma|}\ n_\mu R^{\mu\nu} k_\nu \tag{8.89}$$

Inserting the contracted Killing vector lemma (8.19)

$$\nabla_\nu \nabla_\mu k^\nu = R_{\mu\nu} k^\nu \tag{8.90}$$

and using Stokes theorem leads to

$$M_K = \frac{1}{4\pi}\int_M d^3x\ \sqrt{|\gamma|}\ n_\mu \nabla_\nu (\nabla^\mu k^\nu) \tag{8.91}$$

$$= \frac{1}{4\pi}\int_{\partial M} d^2x\ \sqrt{|\hat\gamma|}\ n_\mu \sigma_\nu \nabla^\mu k^\nu \tag{8.92}$$

With this procedure we are able to construct conserved quantities for different Killing vector fields.

We calculate the Komar mass for the Schwarzschild black hole. For unit vectors we have the requirements

$$n_\mu n^\mu = -1 \qquad \sigma_\mu \sigma^\mu = 1 \tag{8.93}$$

which lead to

$$n_t = \left(1 - \frac{2M_S}{r}\right)^{\frac{1}{2}} \qquad \sigma_r = \left(1 - \frac{2M_S}{r}\right)^{-\frac{1}{2}} \tag{8.94}$$

$$n_\mu \sigma_\nu \nabla^\mu k^\nu = \nabla^t k^r = g^{tt}(\underbrace{\partial_t k^r}_{=0} + \underbrace{\Gamma^r_{tt} k^t}_{=-\frac{1}{2} g^{rr}\partial_r g_{tt}}) \tag{8.95}$$

$$= -\frac{1}{2}\partial_r g_{tt} = \frac{M_S}{r^2} \tag{8.96}$$

Substitute this into our Komar integral (8.92) to get

$$M_K = \frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} d\phi\ \sin\theta\, r^2 \frac{M_S}{r^2} = M_S \tag{8.97}$$

Hence our Komar mass definition corresponds to the mass we used in the Schwarzschild metric.

9. Rotating Black Holes: The Kerr Solution

In this chapter we will discuss rotating black holes. They are described by the Kerr metric.

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\varphi + \frac{\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right)}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \tag{9.1}$$

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2 \qquad a = \frac{L}{M} = \text{const.} \tag{9.2}$$

L denotes the angular momentum and M the mass of the body. Equation (9.1) is called the Kerr solution of the Einstein field equations and describes the geometry of spacetime around a rotating massive body. It is also quite easy to see that if $a \to 0$ i.e. $L \to 0$ in equation (9.1) we recover the Schwarzschild solution which describes non-rotating black holes. If on the other hand $a \neq 0$ then we have nontrivial rotation and obtain a spacetime which is stationary but not static[11].

This spacetime has two killing vectors i.e. ∂_t and ∂_ϕ. Additionally one can also find an object called killing tensor $k_{\mu\nu}$which also leads to constants of motion like a killing vector, but does not have the same geometrical meaning as a killing vector. This killing tensor satisfies the killing equation $\nabla_{(\mu}k_{\nu\lambda)} = 0$. We also find two classes of singularities, one for $\Delta = 0$ and one for $\Sigma = 0$.

The case $\Delta = 0$ is a coordinate singularity and is the defining equation for the killing horizons of the killing vectors $\xi_\pm = \partial_t + \Omega_\pm\partial_\phi$ with $\Omega_\pm = \frac{a}{{r_\pm}^2+a^2}$ and $r_\pm = M \pm \sqrt{M^2 - a^2}$. To see that $\Delta = 0$ leads to Killing horizons consider the normal vector $n \propto g^{rr}\partial_r$ whose norm vanishes when $g^{rr} = 0$, which implies indeed $\Delta = 0$.

The killing horizon for r_+ is also the event horizon of the rotating black hole. While the Kerr metric for $r > r_+$ is able to describe the space outside the outer horizon r_+ we know very little about the region inside the inner horizon $r < r_-$ and a solution for the Einstein field equations for this region has still not been discovered yet.

The other singularity appears for $\Sigma = 0$ which implies $r = 0$ and $\theta = \frac{\pi}{2}$ and is called ring singularity because the gravitational singularity for $r = 0$ is shaped like a ring. The curvature invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ diverges as we approach $\Sigma = 0$.

If we compare mass and Kerr parameter of the rotating black hole then we can distinguish 3 cases

- $M^2 > a^2$ yields 2 killing horizons at $r = r_\pm$ and therefore a Kerr black hole.
- $M^2 = a^2$ describes an extremal Kerr black hole where the two killing horizons coincide and form one event horizon at $r = M$.
- $M^2 < a^2$ would describe a naked singularity that would not be hidden behind an event horizon i.e. $r_\pm$ would be imaginary.

Since the case $M^2 < a^2$ would describe a naked singularity Roger Penrose conceived in the year 1969 the concept of cosmic censorship conjecture which basically states that no naked singularities exist in the universe. In 1991 John Preskill and Kip Thorne bet against Stephen Hawking that the hypothesis was false. Interestingly this was the bet Hawking lost most clearly even though he was also very close to the up until now accepted solution to the problem. In 1997 Hawking conceded the bet because numerical relativists found initial data leading to a naked singularity which falsified the strict cosmic censorship

[11] Not static means $k_{[\mu}\nabla_\nu k_{\lambda]} \neq 0$ or, equivalently, that the line-element fails to be invariant under time-inversion $t \to -t$ for $a \neq 0$.

conjecture introduced by Penrose. After this violation of the cosmic censorship conjecture was discovered, a not less restrictive definition of a cosmic censorship was introduced which has not been falsified up until now.

Though one may assume that a ratio $\frac{a}{M} \approx 1$ would be rare for real black holes, since it is an extremal case and that $\frac{a}{M} \ll 1$ would be the expected common ratio for most observable black holes, that is interestingly not the case. For most of the observed black holes so far a ratio of $\frac{a}{M} \approx 0.1 - 0.98$ has been observed. In particular the ratio for the black hole GRS1915+105 has been observed to be $1 > \frac{a}{M} > 0.98$ which is very close to the extremal black hole limit.

9.1. Uniqueness Theorem

The uniqueness theorem by Carter and Robinson states that if an asymptotically flat spacetime solving the vacuum Einstein field equations which is stationary, axial symmetric, with an event horizon and no singularities outside the horizon, then it has to be the Kerr solution.

This uniqueness theorem makes the Kerr solution one of the most important spacetimes for black hole physics since nearly every observed black hole up until now is a Kerr black hole and not a Schwarzschild black hole.

9.2. No-hair Theorem

The no-hair theorem postulates that all black hole solutions of the Einstein-Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only three externally observable classical parameters:

- mass
- eletric charge
- angular momentum

All other information about the matter which formed a black hole or is falling into it is lost disappears behind the black hole horizon and is therefore permanently inaccessible to external observers.

9.3. The Ergosphere

According to the Kerr solution rotating black holes do not only have more than one Killing horizon, they also allow an area with very interesting properties to exist called ergosphere.

In order to see where this area is located let us consider the time translation killing vector $k^a = (\partial_t)^a$. With this relation the squared norm is given as

$$k_a k^a = k^b g_{ba} k^a = k^t g_{tt} k^t = 0 \tag{9.3}$$

$$\rightarrow g_{tt} = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} = 0 \tag{9.4}$$

$$\rightarrow r = M + \sqrt{M^2 - a^2 \cos^2\theta} \tag{9.5}$$

The calculated radius (9.5) defines the outer radius of the ergosphere and a sketch of this ergosphere along the killing horizons is given in figure 9.1. The ergosphere is limited by the radius (9.5) and the event horizon.

Note: a particle can enter this ergosphere — unlike the black hole region behind the event horizon - and leave the region again if it is thrown in the ergosphere region. In order to remain stationary, however, an observer would have to go faster than light.

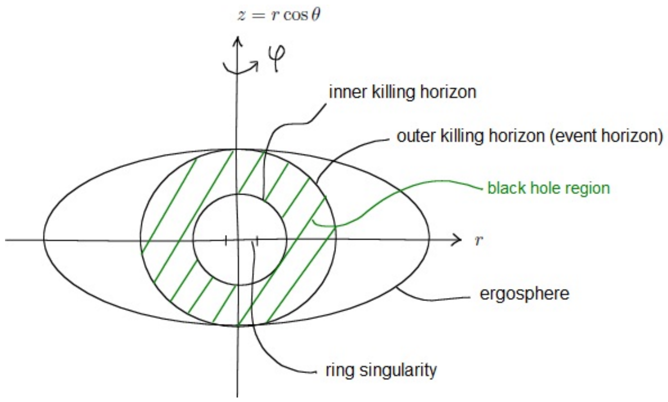

Figure 9.1: Kerr black hole horizons and ergosphere

9.4. The Penrose Process

The existence of the ergoregion allows an interesting process where energy can be extracted from a rotating black hole, which is called the Penrose process.

Assume an unstable particle with Energy $E_1 > 0$ comes from infinity and enters the ergosphere region. Inside this region the particle decays into two new particles, one with energy $-E_2 < 0$ and the other one with energy $E_3 = E_1 + E_2 > E_1$. Now one may ask why a particle with negative energy[12] could exist inside this ergoregion. This is possible because the time translation killing vector $k^a = (\partial_t)^a$ is spacelike in the ergosphere which implies that the energy $E = -p^a k_a$ of a particle with four momentum p^a can be of either sign. If the particle with $-E_2 < 0$ now passes the event horizon and the particle with energy E_3 escapes the ergosphere region, the escaped particle would now have more energy than before the decay inside the ergoregion and the black hole must therefore have lost some of its energy[13]. Since the surface area of the black hole cannot decrease most of the energy lost is rotational energy, hence the black hole loses angular momentum during the process.

Due to this Penrose process one can also find the area theorem mentioned before (see ex. 10.2) for black holes which states that the surface area of a black hole cannot decrease. Using this area theorem it is also possible to calculate the amount of radiated energy during the merger of two axial black holes (see ex. 10.3).

$$\boxed{E_{out} = E_{in} - E_{cap} > E_{in} \quad \text{if} \quad E_{cap} < 0} \tag{9.6}$$

9.5. Frame-dragging/Thirring-Lense Effect [gravimagnetism]

Frame-dragging means that a rotating spacetime can cause inertial observers to rotate even if they have no angular momentum. Let us consider such an observer. As we shall see explicitly in section 10.1 the angular momentum parameter is given by $\ell = g_{t\varphi}\mathrm{d}\,t/\mathrm{d}\,\tau + \mathrm{g}_{\varphi\varphi}\mathrm{d}\,\varphi/\mathrm{d}\,\tau$ so that vanishing ℓ implies $(\mathrm{d}\,\varphi/\mathrm{d}\,\tau)/(\mathrm{d}\,t/\mathrm{d}\,\tau) = -\mathrm{g}_{t\varphi}/\mathrm{g}_{\varphi\varphi}$. The angular coordinate velocity for trajectories with vanishing angular momentum then reads

$$\omega(r,\theta) = \frac{\mathrm{d}\,\varphi}{\mathrm{d}\,t} = \frac{\mathrm{d}\,\varphi}{\mathrm{d}\,\tau}\left(\frac{\mathrm{d}\,t}{\mathrm{d}\,\tau}\right)^{-1} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}\,. \tag{9.7}$$

That fact that $\omega \neq 0$ for $\ell = 0$ is called "frame dragging".

[12] The energy that would be measured by an observer at infinity

[13] For more details on this phenomenon called super radiance see e.g. section 4.4.2 in `gr-qc/9707012`

Relatedly, also the Thirring-Lense effect is a phenomenon predicted by general relativity for non-static (stationary) spacetimes and is caused by rotating bodies which drag space-time around themselves. This effect causes objects to be dragged out of their original position relatively to the rotating body. For example an object which moves contrariwise

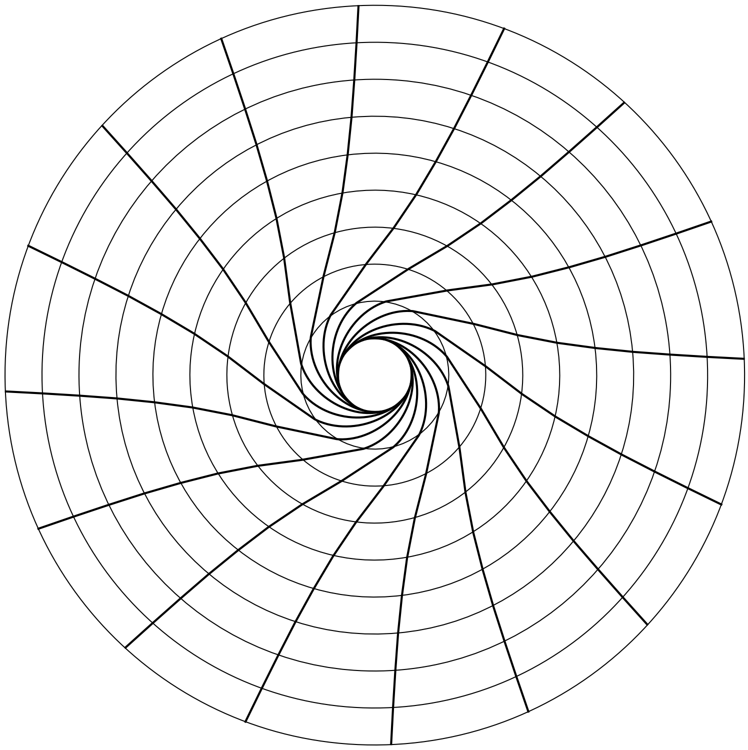

Figure 9.2: Frame dragging

on a circular orbit in respect to the rotation of a rotating black hole, would be dragged along by the black hole spacetime and would be forced to follow the black holes rotation. It is a bit like trying to swim upwards against a current but you are not strong enough to swim upwards so the current drags you along.
A satellite orbiting earth for example would not hit the exact point where it started after one orbit[14] and a gyroscope would undergo a precession depending on the location of the gyroscope.
One experiment to measure the displacement of satellite orbits[15] was done by NASA and the Italian space agency ASI. The orbits and the displacement of the two satellites LAGEOS and LAGEOS 2 have been recorded for years and then evaluated, which proved to be very difficult because of the deviation of Earth's gravitational field from spherical symmetry and many other perturbations. In 2004 evaluation of 11 years of position data led to $99\% \pm 5\%$ of the predicted value of orbital displacement. However these results have not been confirmed by another independent research group evaluating the given data. Another experiment to measure gravitational effects including frame dragging is the Gravity probe B experiment. Gravity probe B is a satellite placed in a polar orbit around Earth containing four gyroscopes with quartz rotors — the roundest objects ever made —. The

[14]The predicted effect is small for a body like the earth - about 0,1 mm for an orbit like that of Gravity Probe B for example.

[15]The original task of this experiment involving the two satellites mentioned was a precise determination of the Earth's gravitational field.

goal is to measure the precession whose frequency can be determined via

$$\omega^{\mu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\xi_{\nu}\nabla_{\rho}\xi_{\sigma}, \tag{9.8}$$

where ξ_{μ} is a timelike killing vector[16], of these four gyroscopes as accurate as possible which proved to be very difficult. The experiment concluded with an article published in the journal *Physical Rewiev Letters* in 2011 which confirmed the precission effect predicted by general relativity caused by frame dragging within an discrepancy of 5%.

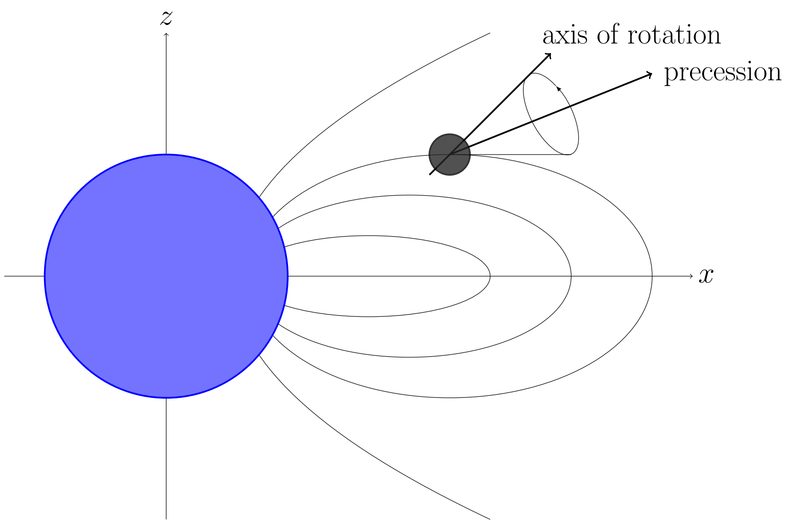

Figure 9.3: Thirring-Lense precession

[16] For more details see `IJMPD 14, Special issue on the Thirring-Lense effect.`

10. Geodesics for Kerr Black Holes

To derive the geodesics for Kerr black holes we proceed similar to section 5.3. We are going to point out the differences of the Kerr geodesics to the Schwarzschild geodesics. Finally we calculate the innermost stable circular orbit.

10.1. Geodesic Equation of the Kerr Black Hole

We assume that $\theta = \pi/2$. So the Kerr solution (9.1) reduces to

$$\Sigma = r^2 \qquad \Delta = r^2 - 2Mr + a^2 \qquad a = \frac{L}{M} = \text{const.} \tag{10.1}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4aM}{r} dt d\varphi + \frac{r^2}{\Delta} dr^2 + \left(r^4 + a^2 + \frac{2Ma^2}{r}\right) d\varphi^2 \tag{10.2}$$

We use again the geodesic action (5.14)

$$\begin{aligned} S &= \frac{1}{2} \int d\tau \left[-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right] && (10.3) \\ &= \frac{1}{2} \int d\tau \left[\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \frac{4aM}{r} \dot{t}\dot{\varphi} - \frac{r^2}{\Delta}\dot{r}^2 - \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) \dot{\varphi}^2\right] && (10.4) \end{aligned}$$

Varying the action in order to derive the Euler-Lagrange equations for t and φ we find two constants of motion

$$\left(1 - \frac{2M}{r}\right) \dot{t} + \frac{2aM}{r}\dot{\varphi} = F = \text{const} \tag{10.5}$$

$$-\frac{2aM}{r}\dot{t} + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) \dot{\varphi} = l = \text{const} \tag{10.6}$$

In addition we can also write down a corresponding Hamiltonian

$$F\dot{t} - l\dot{\varphi} - \frac{r^2}{\Delta}\dot{r}^2 = k = \begin{cases} 1, & \text{spacelike Geodesic} \\ 0, & \text{lightlike Geodesic} \\ -1, & \text{timelike Geodesic.} \end{cases} \tag{10.7}$$

Solving this two equations for $\dot{t}$ and $\dot{\varphi}$ and rearranging the equations to get an equation of the form

$$\frac{\dot{r}^2}{2} + V^{\text{eff}} = E = \frac{F^2 - 1}{2} \tag{10.8}$$

leads to the effective potential V^{eff} $(k = -1)$

$$\mathrm{V}^{\text{eff}} = -\frac{M}{r} + \frac{l^2 - 2a^2E}{2r^2} - \frac{M\left(l - a\sqrt{2E+1}\right)^2}{r^3} \tag{10.9}$$

In comparison to the effective potential of the Schwarzschild case (5.23) we see:

- The energy is also in the potential. It is not possible to solve the equation for $\dot{r}$ and E simultaneous.
- If $l^2 < 2a^2E$ the centrifugal term gets attractive.
- Both effects are most pronounced for extremal Kerr $M^2 = a^2$.

For a graphical comparison see figure 10.1.

Above we have solved the geodesic equation in the plane $\theta = \pi/2$. Thanks to the Killing-tensor of the Kerr solution it turns out that the geodesic equations are integrable even for arbitrary values of θ, so that all orbits around the Kerr black hole can be described in closed form, depending on three constants of motion (in addition to the norm $k = -1, 0$).

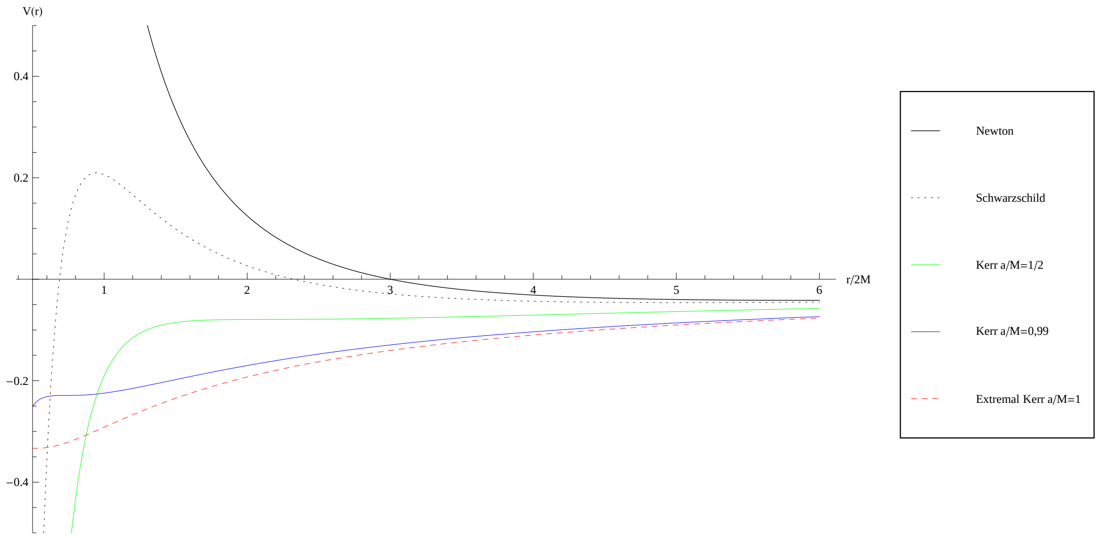

Figure 10.1: Effective potentials for Newton, Schwarzschild and Kerr

10.2. ISCO of the Kerr Black Hole

To calculate the Innermost (marginally) Stable Circular Orbit of the Kerr solution we use three algebraic equations in the three unknown r, E and l

$$V^{\text{eff}} = E \tag{10.10}$$

$$\frac{dV^{\text{eff}}}{dr} = 0 \tag{10.11}$$

$$\frac{d^2V^{\text{eff}}}{dr^2} = 0 \tag{10.12}$$

Since there should be no radial motion $\dot{r} = 0$ we get (10.10). Combined with (10.11) it ensures circularity of the orbit. To get the innermost marginally stable orbit at $r = r_*$ we also need the marginality condition (10.12). We can solve these three algebraic equations for r_*, ℓ and E. Suitably combining them yields a cubic equation for the value of the radius of the ISCO:

$$r_*^2 (r_* - 6M)^2 - 2a^2 r_* (3r_* + 14M) + 9a^4 = 0 \tag{10.13}$$

Solving this equation leads to

$$\frac{r_*}{M} = 3 + \sqrt{x^2 + 3a^2/M^2} \mp \sqrt{(3-x)\,A} \tag{10.14}$$

$$A = 3 + x + 2\sqrt{x^2 + 3a^2/M^2} \tag{10.15}$$

$$x = 1 + \left(1 - a^2/M^2\right)^{1/3} \left[(1 + a/M)^{1/3} + (1 - a/M)^{1/3}\right], \tag{10.16}$$

where the upper sign denotes co-rotation and the lower sign represents counter-rotation. If we set $a = 0$ we see that $x = 3$ and (10.14) reduces to

$$r_* = 6M \tag{10.17}$$

which is the innermost stable circular orbit r_{ISCO} of the Schwarzschild solution.

For the extremal Kerr solution ($a/M = 1 :\to x = 1,\ A = 8$) the equation leads to

$$r_*/M = 5 \mp 4 \tag{10.18}$$

$$\to r_{*\text{CO}} = M \tag{10.19}$$

$$\to r_{*\text{COUNT}} = 9M \tag{10.20}$$

In the case of a counter-rotating test-particle the ISCO of the Kerr Solution is $r_{*\mathrm{COUNT}}$. For co-rotating particles the ISCO is $r_{*\mathrm{CO}}$. Since the innermost stable circular orbit $r_{*\mathrm{CO}}$ equals to the event horizon of the extremal Kerr Solution there is no gap between the accretion disc and the horizon. This means that accretion disks in extremal Kerr black hole backgrounds are able to probe physics close to a black hole horizon. By contrast, in the Schwarzschild black hole there is a gap of minimum $4M$ between the horizon and some circular orbit.

11. Accretion Discs and Black Hole Observations

Up until now we explained and elaborated only theoretical concepts of black hole sciences. This chapter will introduce a method to obtain knowledge about a specific black hole by observing not the hole itself - which is, matter-of-factedly impossible - but a phenomenon associated with black holes in binary systems: the accretion disc.

In the (quite rare) case that a black hole has formed out of the partner in a binary star system or a that black hole was "attracted" by a single, massive star, it could be the case that the black hole starts "sucking" out mass (i.e. matter) from its star-partner. As this matter streams nearer to the black hole it is heated up (due to inter-particle friction) and ultimately starts spiraling into the black hole - and this "spiral" region is what we call accretion disc. Below is depicted an artists' impression of such process:

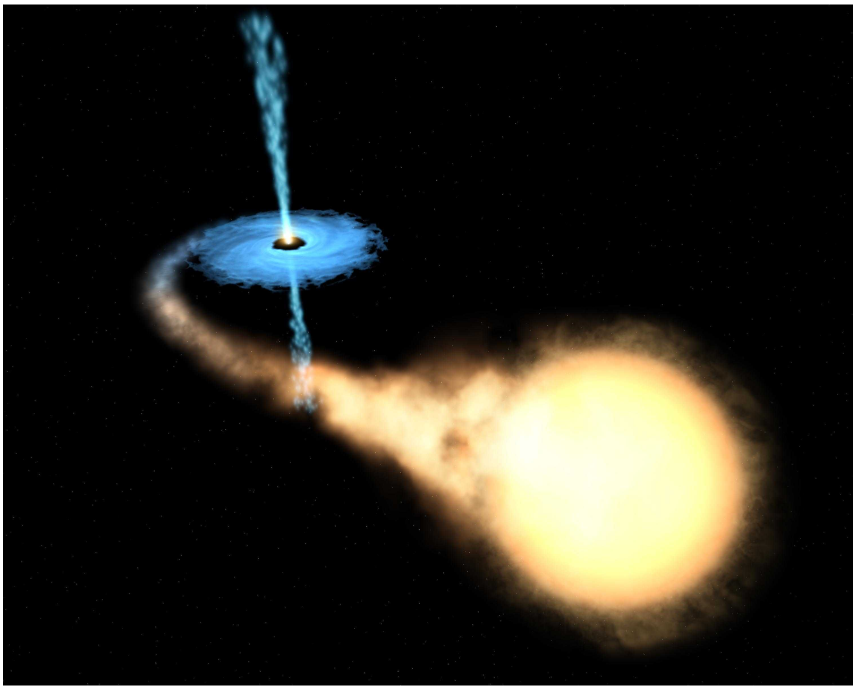

Figure 11.1: Artistic depiction of an accretion disc (Illustration Credit: ESA, NASA, and Felix Mirabel)

So, why are these accretion discs so important to us? As mentioned above, the matter accredited starts to heat up during the process - thus sending out electromagnetic waves up to the X-ray spectrum. And these X-rays (more precisely, their spectrum) contains information about the spin and mass of the black hole[17] and - even more important - they allow us to distinguish between black holes, neutron stars, protostars and white dwarfs - all of which can form accretion discs in binary systems!

This is the point where theory kicks in again - to extract information out of the X-ray spectrum we need to understand and describe the processes in the accretion discs leading to the X-ray spectra. From here on we try to establish a (simplified) model of accretion discs.

[17] One model used to describe accretion discs coined by Kip Thorne yields an angular momentum to mass ratio as $\frac{|a|}{M} \leq 0.998$.

11.1. Simple Theoretical Model: General Relativistic Perfect Fluid

We start with an assumption that greatly simplifies our calculation, though it does not capture all essential aspects of accretion disc physics (in particular, we neglect here viscosity and magnetic fields). Our assumption is that we only consider perfect fluids (i.e. with no viscosity). This is owed to the fact the general relativistic fluid dynamics is quite a demanding topic and it would be far beyond the scope of this course to thoroughly investigate it.
Furthermore, we define:

- fluid velocity: u^μ and its norm: $u^\mu u_\mu = -1$
- a density function: ρ
- and the pressure distribution P.

In General Relativity the stress-energy-tensor for perfect fluids reads:

$$\boxed{T^{\mu\nu} = (\rho + P)\, u^\mu u^\nu + P g^{\mu\nu}} \tag{11.1}$$

This, in flat space with $u^\mu u^\nu$ only having a (0,0)-component, would be $\widehat{T}^{\mu\nu} = \mathrm{diag}\,(\rho, P, P, P)$
For consistency we have to assume conservation of the energy momentum tensor.

$$\nabla_\mu T^{\mu\nu} = 0 \tag{11.2}$$

Equation (11.2) imposes a quite strong restriction on the stress-energy-tensor with an astonishing consequence:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{11.3}$$

$$u_\nu \nabla_\mu \left((\rho + P)\, u^\mu u^\nu + P g^{\mu\nu} \right) = 0 \tag{11.4}$$

$$\underbrace{u_\nu u^\mu u^\nu}_{=-u^\mu} \nabla_\mu (\rho + P) + (\rho + P) \left[\underbrace{u_\nu \left(\nabla_\mu u^\nu\right) u^\mu}_{=-(\nabla_\mu u^\mu)} + \underbrace{u_\nu \left(\nabla_\mu u^\nu\right) u^\mu}_{=\frac{1}{2}\nabla_\mu (u_\nu u^\nu)=0} \right] + u_\mu g^{\mu\nu} \nabla_\mu P = 0 \tag{11.5}$$

$$-\left(\nabla_\mu u^\mu\right)(\rho + P) - u^\mu \nabla_\mu (\rho + P) + u^\mu \nabla_\mu P = 0 \tag{11.6}$$

$$\boxed{\nabla_\mu \left(u^\mu \rho\right) + P \left(\nabla_\mu u^\mu\right) = 0} \tag{11.7}$$

Equation (11.7) is nothing less then the general relativistic continuity equation for ideal fluids! In classical mechanics, we need 2 equations to describe ideal fluids - first, the continuity equation (which we've already found) and second, a force equation (Euler equation), which we have yet to find.
We do so in contracting eq. (11.2) with the projection operator[18] $\delta^\sigma_\nu + u^\sigma u_\nu = \Pi^\sigma{}_\nu$:

$$\left(\delta^\sigma_\nu + u^\sigma u_\nu\right) \nabla_\mu \left[(\rho + P)\, u^\mu u^\nu \right] + \left(\delta^\sigma_\nu + u^\sigma u_\nu\right) \nabla^\nu P = 0 \tag{11.8}$$

$$(\rho + P) \Big[\underbrace{\nabla_\mu \left(u^\mu u^\sigma\right)}_{=u^\mu \nabla_\mu u^\sigma + u^\sigma \nabla_\mu u^\mu} + u^\sigma \underbrace{u_\nu \left(\nabla_\mu u^\mu\right) u^\nu}_{=-\nabla_\mu u^\mu} \Big] + \underbrace{u^\mu u^\nu \left(\delta^\sigma_\nu + u^\sigma u_\nu\right)}_{=u^\mu u^\sigma + u^\mu u^\sigma (u^\mu u_\nu)=0} \nabla_\mu (\rho + P) + $$
$$+ \left(\delta^\sigma_\nu + u^\sigma u_\nu\right) \nabla^\nu P = 0 \tag{11.9}$$

$$\boxed{(\rho + P)\, u^\mu \nabla_\mu u^\sigma + \left(\delta^\sigma_\nu + u^\sigma u_\nu\right) \nabla^\nu P = 0} \tag{11.10}$$

These are the force equations for — again — general relativistic, ideal fluids. The only thing left to do now is to prove that — in non-relativistic limits — equations (11.7) and (11.10) are identical to the known continuitiy and Euler equations (11.11) & (11.12):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{11.11}$$

[18]This projection operator satisfies $\Pi^\mu{}_\nu u^\nu = 0$ and $\Pi^\mu{}_\nu \Pi^\nu{}_\alpha = \Pi^\mu{}_\alpha$.

$$\frac{\partial\left(\rho\vec{v}\right)}{\partial t}+\vec{\nabla}\cdot\left[\vec{v}\otimes\rho\vec{v}\right]+\vec{\nabla}P=0 \tag{11.12}$$

As done before, we introduce the classic limit by setting

$$u^{\mu}=\left(1+O\left(v^{2}\right),v^{i}\right)\qquad\qquad P\ll\rho \tag{11.13}$$

With this eq. (11.7) reads (N.B.: no curvature):

$$\nabla_{t}\Big(\rho\underbrace{u^{t}}_{=1}\Big)+\nabla_{i}\left(\rho v^{i}\right)+\underbrace{P\left(\nabla_{t}u^{t}+\nabla_{i}v^{i}\right)}_{\approx 0+\mathcal{O}(P\cdot v)}=0 \tag{11.14}$$

$$\frac{\partial\rho}{\partial t}+\partial_{i}\left(\rho v^{i}\right)\approx 0+\mathcal{O}\left(P\cdot v\right) \tag{11.15}$$

And this is the classic continuity equation. No we apply all limits to eq. (11.10) and result in:

$$\rho\left[\partial_{t}\vec{v}+\left(\vec{v}\vec{\nabla}\right)\vec{v}\right]+\vec{\nabla}P=0 \tag{11.16}$$

Taking the difference of (11.16) and (11.12) shows that they are equivalent if the continuity equation holds. Thus we have proved that — in classic limits — both general relativistic fluid equations are identical to the Euler and continuity equations, respectively.

Actually, there are some types of perfect fluids — with special equations of state — these equations apply to:

- dust: $P=0\Rightarrow\quad T^{\mu\nu}=\rho u^{\mu}u^{\nu}$
- light-like fluid: $\rho=3P\Rightarrow\quad T^{\mu}_{\ \mu}=0$
- polytropic: $P=\alpha\rho^{n}$
- barotropic: $P=P(\rho)$

Summarizing, the main purpose of this chapter was to learn something about the influence of the background geometry on a relativistic fluid and the corresponding equation of state.

In order to treat real accretion discs one would also have to take into account viscosity and electromagnetic interactions. This is hard to do in a full general relativistic framework and therefore is often done in quasi-Newtonian simulations, where the Newton potential is replaced by some effective potential similar to (10.9).

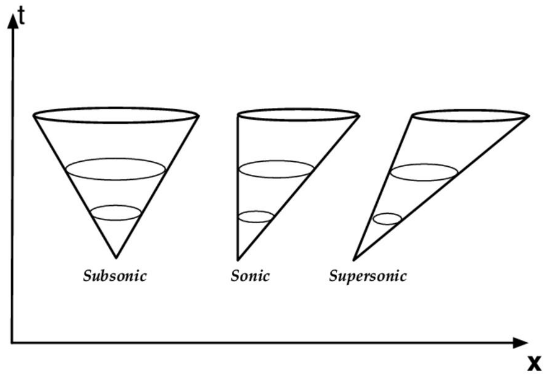

Figure 12.1: Tilted sound cones in moving fluid (Source: Barceló et al. - Analogue Gravity)

12. Black hole analogs in condensed matter physics

As we have already seen analogies provide a rich source of inspiration and understanding. That is especially valuable in fields that are unfamiliar to us like general relativity. Analog gravity[19] is motivated among other things by:

- Classical analogies: Giving easy and pedagogical examples for complex phenomena. For example the fishy Gedankenexperiment 3.1 and the draining bathtub 12.4.
- Semiclassical: The predicted Hawking effect for stellar black holes is in practice unobservable. There are approches to show this effect with analog models.
- Quantum gravity: As toy model for quantum gravity.

As a motivation look at figure 12.1 where you can see sound cones tilted by the flow of fluid in analogy to light cones in a gravitational field.

In this section we are going to proof that sound waves in some special fluid behave similar to fields in general relativity. Furthermore we bring an example which is a good analogy to the Kerr black hole.

12.1. Analogy theorem

We assume the fluid to be

- barotropic
- inviscid (ideal)

and the flow to be

- irrotational.

Then the equation of motion for the velocity potential ϕ that describes the acoustic disturbance is equal to the general relativistic massless Klein-Gordon equation

$$\boxed{\Delta\phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\;g^{\mu\nu}\;\partial_\nu\phi\right) = 0} \tag{12.1}$$

[19]For more information see: Barceló et al. - Analogue Gravity [5].

with the acoustic metric

$$\boxed{g_{\mu\nu}(t,\vec{x}) = \frac{\rho}{c}\begin{bmatrix} -\left(c^2-v^2\right) & -v^i \\ -v^j & \delta_{ij} \end{bmatrix}} \tag{12.2}$$

(where $i,j = 1..3$). The acoustic metric depends on the density ρ, the flow velocity $\vec{v}$ and the local speed of sound in the fluid c.

12.2. Proof

We start with the continuity equation

$$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{12.3}$$

and Euler's equation of inviscid flow

$$\rho\left[\partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right] + \vec{\nabla}P = 0 \tag{12.4}$$

We use the fact that

$$\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = \frac{1}{2}\vec{\nabla}v^2 \tag{12.5}$$

and get with some manipulations on Euler's equation

$$\partial_t\vec{v} = \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) - \frac{1}{\rho}\vec{\nabla}P - \frac{1}{2}\vec{\nabla}v^2 \tag{12.6}$$

We assume the flow to be irrotational $\vec{\nabla}\times\vec{v} = 0$. Locally the velocity flow is given by the velocity potential $\vec{v} = -\vec{\nabla}\phi$. Our fluid is also barotropic which means that ρ is a function of P. That makes it possible to define the specific enthalpy $h(P)$

$$h(P) = \int_0^p \frac{dP'}{\rho(P')} \qquad \Longrightarrow \qquad \vec{\nabla}h = \frac{1}{\rho}\vec{\nabla}P \tag{12.7}$$

Using the above informations Euler's equation (12.6) simplifies to

$$-\partial_t\phi + h + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 = 0 \tag{12.8}$$

Now we split our equations of motion in background (ρ_0, P_0, ϕ_0) and linearized fluctuations ($\epsilon\rho_1$, ϵP_1, $\epsilon\phi_1$)

$$\rho = \rho_0 + \epsilon\rho_1 + O(\epsilon^2) \tag{12.9}$$
$$P = P_0 + \epsilon P_1 + O(\epsilon^2) \tag{12.10}$$
$$\phi = \phi_0 + \epsilon\phi_1 + O(\epsilon^2) \tag{12.11}$$

You can imagine the background as the motion of the fluid "in total". The linearized fluctuations are the small relative oscillatory motions known by definition as sound. As an example you can think of some floating liquid (background) where somewhere in the middle of this liquid a small disturbance (sound) occurred.

We use the Taylor expansions (12.9) and (12.11) to linearize the continuity equation (12.3) to zeroth

$$\partial_t\rho_0 + \vec{\nabla}\cdot(\rho_0\vec{v}_0) = 0 \tag{12.12}$$

and first order in ϵ

$$\partial_t\rho_1 + \vec{\nabla}\cdot(\rho_1\vec{v}_0 + \rho_0\vec{v}_1) = 0 \tag{12.13}$$

Using the barotropic condition we derive

$$h(P) = h(P_0 + \epsilon P_1 + O(\epsilon^2)) = h(P_0) + \epsilon h'(P_0)P_1 + O(\epsilon^2) = h_0 + \epsilon \frac{P_1}{\rho_0} + O(\epsilon^2) \quad (12.14)$$

With the above result we linearize the Euler equation (12.8) to zeroth

$$-\partial_t \phi_0 + h_0 + \frac{1}{2}(\vec{\nabla}\phi_0)^2 = 0 \quad (12.15)$$

and first order in ϵ

$$-\partial_t \phi_1 + \frac{P_1}{\rho_0} - \vec{v_0} \cdot \vec{\nabla}\phi_1 = 0 \quad (12.16)$$

We transform the last equation to get

$$P_1 = \rho_0(\partial_t \phi_1 + \vec{v_0} \cdot \vec{\nabla}\phi_1) \quad (12.17)$$

The fact that our fluid is barotropic leads us to

$$\rho_1 = \frac{\partial \rho}{\partial P} P_1 = \frac{\partial \rho}{\partial P} \rho_0(\partial_t \phi_1 + \vec{v_0} \cdot \vec{\nabla}\phi_1) \quad (12.18)$$

In this equation we recognize the term for the speed of sound

$$c^{-2} := \frac{\partial \rho}{\partial P} \quad (12.19)$$

We substitute ρ_1 into our linearized continuity equation (12.13) to get the wave equation

$$\partial_t \left(\frac{\partial \rho}{\partial P} \rho_0(\partial_t \phi_1 + \vec{v_0} \cdot \vec{\nabla}\phi_1) \right) + \vec{\nabla} \cdot \left(\frac{\partial \rho}{\partial P} \rho_0(\partial_t \phi_1 + \vec{v_0} \cdot \vec{\nabla}\phi_1)\vec{v_0} - \rho_0 \vec{\nabla}\phi_1 \right) = 0 \quad (12.20)$$

Substituting the speed of light in this equation and rearranging the terms leads us to

$$-\partial_t \left(\frac{\rho_0}{c^2}(\partial_t \phi_1 + \underbrace{\vec{v_0}}_{v_0^j} \cdot \vec{\nabla}\phi_1) \right) - \vec{\nabla} \cdot \left(\frac{\rho_0}{c^2}[\underbrace{\vec{v_0}}_{v_0^i} \partial_t \phi_1 + \underbrace{\vec{v_0}(\vec{v_0}}_{v_0^i v_0^j} \cdot \vec{\nabla}\phi_1) - c^2 \vec{\nabla}\phi_1] \right) = 0 \quad (12.21)$$

Since we got rid of the ρ_1 and P_1 terms we have a wave equation where we just have to know ρ_0 and v_0 to solve for ϕ_1. Once we have solved this equation we are able to calculate ρ_1 and P_1 by (12.17) and (12.18). Also observe that if there is no background speed $v_0 = 0$ and if the background density is constant the equation (12.21) reduces to the well known form

$$\partial_t{}^2 \phi_1 = c^2 \Delta \phi_1 \quad (12.22)$$

We now define the matrix

$$f^{\mu\nu}(t, \vec{x}) := \frac{\rho_0}{c^2} \begin{bmatrix} -1 & -v_0^j \\ -v_0^i & (c^2 \delta^{ij} - v_0^i v_0^j) \end{bmatrix} \quad (12.23)$$

and observe that it is possible to write our wave equation (12.21) compactly as

$$\partial_\mu (f^{\mu\nu} \partial_\nu \phi_1) = 0 \quad (12.24)$$

That looks already similar to our Klein-Gordon equation (g =det($g_{\mu\nu}$))

$$\Delta \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \phi \right) = 0 \quad (12.25)$$

For them to coincide we have to set

$$\sqrt{-g}\, g^{\mu\nu} := f^{\mu\nu} \tag{12.26}$$

We use the determinant property $\det(aA) = a^n \det(A)$ for $n \times n$ matrices to get

$$\det(f^{\mu\nu}) = (\sqrt{-g})^4 g^{-1} = g \tag{12.27}$$

Expanding the determinant by minors leads to

$$\det(f^{\mu\nu}) = -\frac{\rho_0{}^4}{c^2} \overset{(12.27)}{\Longrightarrow} g = -\frac{\rho_0{}^4}{c^2} \implies \sqrt{-g} = \frac{\rho_0{}^2}{c} \tag{12.28}$$

So we get the inverse acoustic metric

$$g^{\mu\nu}(t,\vec{x}) = \frac{1}{\sqrt{-g}} f^{\mu\nu} = \frac{1}{\rho_0 c} \left[\begin{array}{cc} -1 & -v_0^j \\ -v_0^i & (c^2\delta^{ij} - v_0^i v_0^j) \end{array} \right] \tag{12.29}$$

and by matrix inversion the acoustic metric

$$g_{\mu\nu}(t,\vec{x}) = \frac{\rho_0}{c} \left[\begin{array}{cc} -\left(c^2 - v^2\right) & -v^i \\ -v^j & \delta^{ij} \end{array} \right] \tag{12.30}$$

This completes the proof. $\square$ Q.e.d

12.3. General remaks

- $g_{\mu\nu}$ has the signature $(-,+,+,+)$
- There are two different metrics
  - Physical spacetime metric: Since we neglect general relativity it is the flat metric of Minkowski space. It is used for all fluid particles.
  - Acoustic metric: The sound waves propagate effectively on the acoustic metric.
- Concepts like killing horizon, ergo-region and event horizon generalize to acoustic metrics:
  - Killing horizon: Equivalent to general relativity (8.18) we calculate $k^\mu k_\mu = 0$ with $k^\mu = \partial_t^\mu$ and get $|\vec{v}| = c$ for the killing horizon.
  - Ergo-region: The region where $|\vec{v}| > c$ i.e. the region of supersonic flow is an ergo-region.
  - Event horizon: Is the boundary of the region from where sound with the speed of c is not able to escape (see figure 12.2).

12.4. Example: Vortex geometry

As an example we are now considering the vortex geometry which looks similar to a draining bathtub (see figure 12.3). The background velocity potential has the form

$$\phi_0 = -A\ln(r/a) - B\,\theta \tag{12.31}$$

which leads to a velocity of the fluid flow of

$$v_0^i = (\partial^r \phi, \partial^\theta \phi) = (\partial_r \phi, \partial_\theta \phi / r^2) = -\left(\frac{A}{r}, \frac{B}{r^2}\right) \tag{12.32}$$

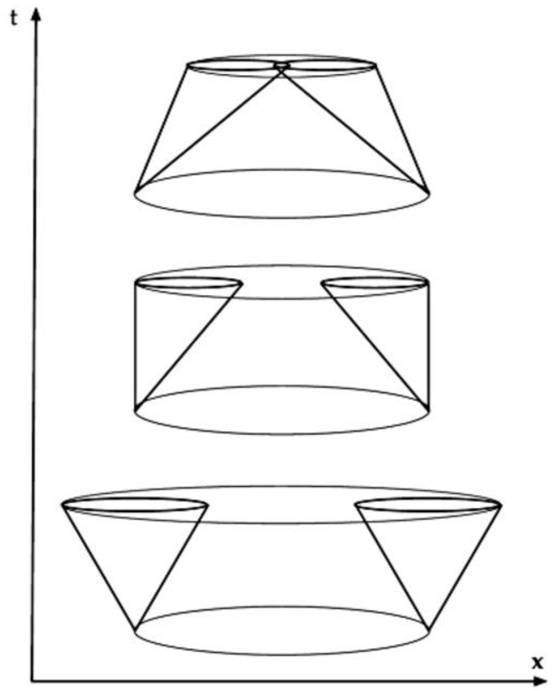

Figure 12.2: Trapped surfaces formed by moving fluid. (Source: Barceló et al. - Analogue Gravity)

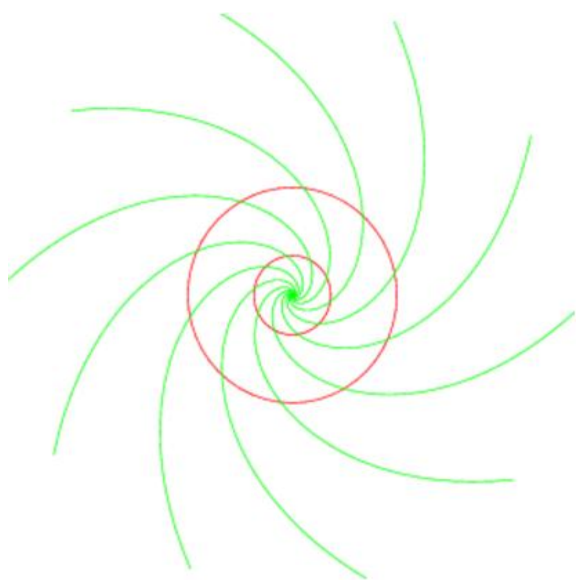

Figure 12.3: Vortex Geometry: The spirals are the streamlines of the fluid flow. The outer circle is the killing horizon of ∂_t^μ while the inner is the event horizon. The region in between is the ergo-region. (Source: Barceló et al. - Analogue Gravity)

and to

$$v_0^2 = \frac{A^2 + B^2}{r^2} \tag{12.33}$$

Calculating the acoustic metric explicitly leads to

$$ds^2 = -\left(c^2 - \frac{A^2 + B^2}{r^2}\right) dt^2 - \frac{2A}{r}\, dr\, dt - 2B\, d\theta\, dt + dr^2 + r^2\, d\theta^2 + dz^2 \tag{12.34}$$

The killing horizon is at

$$r_{\text{ergo}} = \frac{\sqrt{A^2 + B^2}}{c} \tag{12.35}$$

Since for the event horizon only the radial part counts we set $B = 0$ and get the event horizon

$$r_{\text{horizon}} = \frac{|A|}{c} \tag{12.36}$$

Given this similarities with the Kerr black hole makes it an interesting and acceptable analogy.

A. Useful formulas

The following formulas were collected by Robert McNess and are available online at http://jacobi.luc.edu/Useful.html.

Conventions, Definitions, Identities, and Other Useful Formulae

Contents

1. Curvature tensors

Consider a $d+1$ dimensional manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. The covariant derivative on $\mathcal{M}$ that is metric-compatible with $g_{\mu\nu}$ is ∇_μ.

Christoffel Symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) \tag{1}$$

Riemann Tensor

$$R^\lambda{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\kappa_{\mu\nu}\Gamma^\lambda_{\kappa\sigma} - \Gamma^\kappa_{\mu\sigma}\Gamma^\lambda_{\kappa\nu} \tag{2}$$

Ricci Tensor

$$R_{\mu\nu} = \delta^\sigma{}_\lambda R^\lambda{}_{\mu\sigma\nu} \tag{3}$$

Schouten Tensor

$$S_{\mu\nu} = \frac{1}{d-1}\left(R_{\mu\nu} - \frac{1}{2\,d}\, g_{\mu\nu} R\right) \tag{4}$$

$$\nabla^\nu S_{\mu\nu} = \nabla_\mu S^\nu{}_\nu \tag{5}$$

Weyl Tensor

$$C^\lambda{}_{\mu\sigma\nu} = R^\lambda{}_{\mu\sigma\nu} + g^\lambda{}_\nu\, S_{\mu\sigma} - g^\lambda{}_\sigma\, S_{\mu\nu} + g_{\mu\sigma}\, S^\lambda{}_\nu - g_{\mu\nu}\, S^\lambda{}_\sigma \tag{6}$$

Commutators of Covariant Derivatives

$$[\nabla_\mu, \nabla_\nu]\, A_\lambda = R_{\lambda\sigma\mu\nu} A^\sigma \tag{7}$$

$$[\nabla_\mu, \nabla_\nu]\, A^\lambda = R^\lambda{}_{\sigma\mu\nu} A^\sigma \tag{8}$$

Bianchi Identity

$$\nabla_\kappa R_{\lambda\mu\sigma\nu} - \nabla_\lambda R_{\kappa\mu\sigma\nu} + \nabla_\mu R_{\kappa\lambda\sigma\nu} = 0 \tag{9}$$

$$\nabla^\nu R_{\lambda\mu\sigma\nu} = \nabla_\mu R_{\lambda\sigma} - \nabla_\lambda R_{\mu\sigma} \tag{10}$$

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2}\,\nabla_\mu R \tag{11}$$

Bianchi Identity for Weyl

$$\nabla^\nu C_{\lambda\mu\sigma\nu} = (d-2)\, \Big(\nabla_\mu S_{\lambda\sigma} - \nabla_\lambda S_{\mu\sigma}\Big) \tag{12}$$

$$\begin{aligned}\nabla^\lambda \nabla^\sigma C_{\lambda\mu\sigma\nu} = \frac{d-2}{d-1}\, \Bigg[& \nabla^2 R_{\mu\nu} - \frac{1}{2d}\, g_{\mu\nu} \nabla^2 R - \frac{d-1}{2d}\, \nabla_\mu \nabla_\nu R - \left(\frac{d+1}{d-1}\right)\, R_\mu{}^\lambda R_{\nu\lambda} \\ & + C_{\lambda\mu\sigma\nu} R^{\lambda\sigma} + \frac{(d+1)}{d(d-1)}\, R\, R_{\mu\nu} + \frac{1}{d-1}\, g_{\mu\nu} \left(R^{\lambda\sigma} R_{\lambda\sigma} - \frac{1}{d}\, R^2 \right) \Bigg]\end{aligned} \tag{13}$$

2. Conventions for Differential Forms

p-Form Components

$$\mathbf{A}_{(p)} = \frac{1}{p!}\, A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{14}$$

Exterior Derivative

$$\Big(d\mathbf{A}_{(p)}\Big)_{\mu_1 \dots \mu_{p+1}} = (p+1)\, \partial_{[\,\mu_1} A_{\mu_2 \dots \mu_{p+1}\,]} \tag{15}$$

$$B_{[\,\mu_1 \dots \mu_n]} := \frac{1}{n!}\, \Big(B_{\mu_1 \dots \mu_n} + \text{permutations} \Big) \tag{16}$$

Hodge-Star

$$\Big(\star\, \mathbf{A}_{(p)}\Big)_{\mu_1 \dots \mu_{d+1-p}} = \frac{1}{p!}\, \epsilon_{\mu_1 \dots \mu_{d+1-p}}{}^{\nu_1 \dots \nu_p}\, A_{\nu_1 \dots \nu_p} \tag{17}$$

$$\star\star = \big(-1\big)^{p(d+1-p)+1} \tag{18}$$

Wedge Product

$$\Big(\mathbf{A}_{(p)} \wedge \mathbf{B}_{(q)}\Big)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!}\, A_{[\,\mu_1 \dots \mu_p}\, B_{\mu_{p+1} \dots \mu_{p+q}\,]} \tag{19}$$

3. Euler Densities

Let $\mathcal{M}$ be a manifold with dimension $d+1 = 2n$ an even number. Normalized so that $\chi(S^{2n}) = 2$.

Euler Number

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} d^{2n}x\sqrt{g}\,\mathcal{E}_{2n} \tag{20}$$

$$= \int_{\mathcal{M}} \mathbf{e}_{2n} \tag{21}$$

Euler Density

$$\mathcal{E}_{2n} = \frac{1}{(8\pi)^n\,\Gamma(n+1)}\,\epsilon_{\mu_1\cdots\mu_{2n}}\,\epsilon_{\nu_1\cdots\nu_{2n}}\,R^{\mu_1\mu_2\nu_1\nu_2}\ldots R^{\mu_{2n-1}\mu_{2n}\nu_{2n-1}\nu_{2n}} \tag{22}$$

$$\mathbf{e}_{2n} = \frac{1}{(4\pi)^n\,\Gamma(n+1)}\,\epsilon_{a_1\ldots a_{2n}}\,\mathbf{R}^{a_1a_2}\wedge\ldots\wedge\mathbf{R}^{a_{2n-1}a_{2n}} \tag{23}$$

Curvature Two-Form

$$\mathbf{R}^a{}_b = \frac{1}{2}\,R^a{}_{b\,c\,d}\,\mathbf{e}^c\wedge\mathbf{e}^d \tag{24}$$

Examples

$$\mathcal{E}_2 = \frac{1}{8\pi}\,\epsilon_{\mu\nu}\epsilon_{\lambda\rho}\,R^{\mu\nu\lambda\rho} \tag{25}$$

$$= \frac{1}{4\pi}\,R$$

$$\mathcal{E}_4 = \frac{1}{128\pi^2}\,\epsilon_{\mu\nu\lambda\rho}\,\epsilon_{\alpha\beta\gamma\delta}\,R^{\mu\nu\alpha\beta}\,R^{\lambda\rho\gamma\delta} \tag{26}$$

$$= \frac{1}{32\pi^2}\left(R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} - 4\,R^{\mu\nu}R_{\mu\nu} + R^2\right)$$

$$= \frac{1}{32\pi^2}\,C^{\mu\nu\lambda\rho}C_{\mu\nu\lambda\rho} - \frac{1}{8\pi^2}\left(\frac{d-2}{d-1}\right)\left(R^{\mu\nu}R_{\mu\nu} - \frac{d+1}{4\,d}\,R^2\right)$$

4. Hypersurfaces

Let $\Sigma \subset \mathcal{M}$ be a d dimensional hypersurface whose embedding is described locally by an outward-pointing, unit normal vector n^μ. Rather than keeping track of the signs associated with n^μ being either spacelike or timelike, we will just assume that n^μ is spacelike. Indices are lowered and raised using $g_{\mu\nu}$ and $g^{\mu\nu}$, and symmetrization of indices is implied when appropriate.

First Fundamental Form / Induced Metric on Σ

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \tag{27}$$

Projection onto Σ

$$\perp T^{\mu\,\cdots}{}_{\nu\,\cdots} = h^\mu{}_\lambda\ldots h^\sigma{}_\nu\ldots T^{\lambda\,\cdots}{}_{\sigma\,\cdots} \tag{28}$$

Second Fundamental Form / Extrinsic Curvature of Σ

$$K_{\mu\nu} = \perp(\nabla_\mu n_\nu) = h_\mu{}^\lambda h_\nu{}^\sigma\nabla_\lambda n_\sigma = \frac{1}{2}\,\pounds_n h_{\mu\nu} \tag{29}$$

Trace of Extrinsic Curvature

$$K = \nabla_\mu n^\mu \tag{30}$$

'Acceleration' Vector

$$a^\mu = n^\nu \nabla_\nu n^\mu \tag{31}$$

Surface-Forming Normal Vectors

$$n_\mu = \frac{1}{\sqrt{g^{\nu\lambda}\,\partial_\nu\alpha\partial_\lambda\alpha}}\,\partial_\mu\alpha \quad \Rightarrow \quad \perp\nabla_{[\,\mu}n_{\nu]} = 0 \tag{32}$$

Covariant Derivative on Σ compatible with $h_{\mu\nu}$

$$\mathcal{D}_\mu T^{\alpha\,\cdots}{}_{\beta\,\ldots} = \perp \nabla_\mu T^{\alpha\,\cdots}{}_{\beta\,\ldots} \quad \forall \quad T = \perp T \tag{33}$$

Intrinsic Curvature of (Σ, h)

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\, A^\lambda = \mathcal{R}^\lambda{}_{\sigma\mu\nu} A^\sigma \quad \forall \quad A^\lambda = \perp A^\lambda \tag{34}$$

Gauss-Codazzi

$$\perp R_{\lambda\mu\sigma\nu} \;=\; \mathcal{R}_{\lambda\mu\sigma\nu} - K_{\lambda\sigma}K_{\mu\nu} + K_{\mu\sigma}K_{\nu\lambda} \tag{35}$$

$$\perp \left(R_{\lambda\mu\sigma\nu}\, n^\lambda\right) \;=\; \mathcal{D}_\nu K_{\mu\sigma} - \mathcal{D}_\sigma K_{\mu\nu} \tag{36}$$

$$\perp \left(R_{\lambda\mu\sigma\nu}\, n^\lambda n^\sigma\right) \;=\; -\,\mathcal{L}_n\, K_{\mu\nu} + K_\mu{}^\lambda\, K_{\lambda\nu} + \mathcal{D}_\mu a_\nu - a_\mu a_\nu \tag{37}$$

Projections of the Ricci tensor

$$\perp (R_{\mu\nu}) \;=\; \mathcal{R}_{\mu\nu} + \mathcal{D}_\mu a_\nu - a_\mu a_\nu - \mathcal{L}_n\, K_{\mu\nu} - K\, K_{\mu\nu} + 2K_\mu{}^\lambda\, K_{\nu\,\lambda} \tag{38}$$

$$\perp (R_{\mu\nu}\, n^\mu) \;=\; \mathcal{D}^\mu K_{\mu\nu} - \mathcal{D}_\nu K \tag{39}$$

$$R_{\mu\nu} n^\mu n^\nu \;=\; -\,\mathcal{L}_n\, K - K^{\mu\nu}\, K_{\mu\nu} + \mathcal{D}_\mu a^\mu - a_\mu a^\mu \tag{40}$$

Decomposition of the Ricci scalar

$$R \;=\; \mathcal{R} - K^2 - K^{\mu\nu}\, K_{\mu\nu} - 2\,\mathcal{L}_n\, K + 2\,\mathcal{D}_\mu a^\mu - 2\, a_\mu a^\mu \tag{41}$$

Lie Derivatives along n^μ

$$\pounds_n K_{\mu\nu} \;=\; n^\lambda \nabla_\lambda K_{\mu\nu} + K_{\lambda\nu}\nabla_\mu n^\lambda + K_{\mu\lambda}\nabla_\nu n^\lambda \tag{42}$$

$$\perp (\pounds_n\, \mathcal{F}^{\mu\,\cdots}{}_{\nu\,\ldots}) \;=\; \pounds_n\, \mathcal{F}^{\mu\,\cdots}{}_{\nu\,\ldots} \quad \forall \quad \perp\mathcal{F} = \mathcal{F} \tag{43}$$

5. Sign Conventions for the Action

These conventions follow Weinberg, keeping in mind that he defines the Riemann tensor with a minus sign relative to our definition. They are appropriate when using signature $(-,+,\ldots,+)$. The $d+1$-dimensional Newton's constant is $2\kappa^2 = 16\pi G_{d+1}$. The sign on the boundary term follows from our definition of the extrinsic curvature.

Gravitational Action

$$
\begin{aligned}
I_G = &\ \frac{1}{2\,\kappa^2} \int_{\mathcal{M}} d^{\,d+1}x \sqrt{g}\,\Big(R - 2\,\Lambda\Big) + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^{\,d}x \sqrt{\gamma}\,K && (44)\\
= &\ \frac{1}{2\,\kappa^2} \int_{\mathcal{M}} d^{\,d+1}x \sqrt{g}\,\Big(\mathcal{R} + K^2 - K^{\mu\nu}\,K_{\mu\nu} - 2\,\Lambda\Big) && (45)
\end{aligned}
$$

Gauge Field Coupled to Particles

$$
\begin{aligned}
I_M = &\ -\frac{1}{4} \int_{\mathcal{M}} d^{\,d+1}x \sqrt{g}\,F^{\mu\nu} F_{\mu\nu} && (46)\\
&- \sum_n m_n \int dp \left(-g_{\mu\nu}(x_n(p))\,\frac{dx_n^\mu(p)}{dp}\frac{dx_n^\nu(p)}{dp}\right)^{1/2} && (47)\\
&+ \sum_n e_n \int dp\,\frac{dx_n^\mu(p)}{dp}\,A_\mu(x_n(p)) && (48)
\end{aligned}
$$

Gravity Minimally Coupled to a Gauge Field

$$
I = \int_{\mathcal{M}} d^{\,d+1}x \sqrt{g} \left[\frac{1}{2\,\kappa^2}\,\left(R - 2\,\Lambda\right) - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}\right] + \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^{\,d}x \sqrt{\gamma}\,K \tag{49}
$$

6. Hamiltonian Formulation

The canonical variables are the metric $h_{\mu\nu}$ on Σ and its conjugate momenta $\pi^{\mu\nu}$. The momenta are defined with respect to evolution in the *spacelike* direction n^μ, so this is not the usual notion of the Hamiltonian as the generator of time translations.

Bulk Lagrangian Density

$$
\mathscr{L}_{\mathcal{M}} = \frac{1}{2\,\kappa^2}\,\Big(K^2 - K^{\mu\nu}\,K_{\mu\nu} + \mathcal{R} - 2\,\Lambda\Big) \tag{50}
$$

Momentum Conjugate to $h_{\mu\nu}$

$$
\pi^{\mu\nu} = \frac{\partial \mathscr{L}_{\mathcal{M}}}{\partial\left(\pounds_n h_{\mu\nu}\right)} = \frac{1}{2\,\kappa^2}\,\left(h^{\mu\nu}\,K - K^{\mu\nu}\right) \tag{51}
$$

Momentum Constraint

$$
\mathcal{H}_\mu = \frac{1}{\kappa^2} \perp (n^\nu G_{\mu\nu}) = 2\,\mathcal{D}^\nu \pi_{\mu\nu} = 0 \tag{52}
$$

Hamiltonian Constraint

$$
\mathcal{H} = -\frac{1}{\kappa^2}\,n^\mu n^\nu G_{\mu\nu} = 2\,\kappa^2\,\left(\pi^{\mu\nu}\,\pi_{\mu\nu} - \frac{1}{d-1}\,\pi^2\right) + \frac{1}{2\,\kappa^2}\,\Big(\mathcal{R} - 2\,\Lambda\Big) = 0 \tag{53}
$$

7. Conformal Transformations

The dimension of spacetime is $d+1$. Indices are raised and lowered using the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

Metric

$$
\widehat{g}_{\mu\nu} = e^{\,2\,\sigma}\,g_{\mu\nu} \tag{54}
$$

Christoffel

$$\widehat{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \Theta^{\lambda}_{\mu\nu} \tag{55}$$

$$\Theta^{\lambda}_{\mu\nu} = \delta^{\lambda}{}_{\mu}\, \nabla_{\nu}\sigma + \delta^{\lambda}{}_{\nu}\, \nabla_{\mu}\sigma - g_{\mu\nu}\, \nabla^{\lambda}\sigma \tag{56}$$

Riemann Tensor

$$\begin{aligned} \widehat{R}^{\lambda}{}_{\mu\rho\nu} = \;& R^{\lambda}{}_{\mu\rho\nu} + \delta^{\lambda}{}_{\nu}\, \nabla_{\mu}\nabla_{\rho}\sigma - \delta^{\lambda}{}_{\rho}\, \nabla_{\mu}\nabla_{\nu}\sigma + g_{\mu\rho}\, \nabla_{\nu}\nabla^{\lambda}\sigma - g_{\mu\nu}\, \nabla_{\rho}\nabla^{\lambda}\sigma & (57)\\ & + \delta^{\lambda}{}_{\rho}\, \nabla_{\mu}\sigma\, \nabla_{\nu}\sigma - \delta^{\lambda}{}_{\nu}\, \nabla_{\mu}\sigma\, \nabla_{\rho}\sigma + g_{\mu\nu}\, \nabla_{\rho}\sigma\, \nabla^{\lambda}\sigma - g_{\mu\rho}\, \nabla_{\nu}\sigma\, \nabla^{\lambda}\sigma & (58)\\ & + \Big(g_{\mu\rho}\, \delta^{\lambda}{}_{\nu} - g_{\mu\nu}\, \delta^{\lambda}{}_{\rho} \Big)\, \nabla^{\alpha}\sigma\, \nabla_{\alpha}\sigma & (59) \end{aligned}$$

Ricci Tensor

$$\begin{aligned} \widehat{R}_{\mu\nu} = \;& R_{\mu\nu} - g_{\mu\nu}\, \nabla^{2}\sigma - (d-1)\, \nabla_{\mu}\nabla_{\nu}\sigma + (d-1)\, \nabla_{\mu}\sigma\, \nabla_{\nu}\sigma & (60)\\ & - (d-1)\, g_{\mu\nu} \nabla^{\lambda}\sigma\, \nabla_{\lambda}\sigma & (61) \end{aligned}$$

Ricci Scalar

$$\widehat{R} = e^{-2\,\sigma}\, \Big(R - 2\, d\, \nabla^{2}\sigma - d\, (d-1)\, \nabla^{\mu}\sigma\, \nabla_{\mu}\sigma \Big) \tag{62}$$

Schouten Tensor

$$\widehat{S}_{\mu\nu} = \; S_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\sigma + \nabla_{\mu}\sigma\, \nabla_{\nu}\sigma - \frac{1}{2}\, g_{\mu\nu} \nabla^{\lambda}\sigma\, \nabla_{\lambda}\sigma \tag{63}$$

Weyl Tensor

$$\widehat{C}^{\lambda}{}_{\mu\rho\nu} = C^{\lambda}{}_{\mu\rho\nu} \tag{64}$$

Normal Vector

$$\widehat{n}^{\mu} = e^{-\sigma}\, n^{\mu} \qquad \widehat{n}_{\mu} = e^{\,\sigma}\, n_{\mu} \tag{65}$$

Extrinsic Curvature

$$\widehat{K}_{\mu\nu} = e^{\,\sigma}\, \Big(K_{\mu\nu} + h_{\mu\nu}\, n^{\lambda} \nabla_{\lambda}\sigma \Big) \tag{66}$$

$$\widehat{K} = e^{-\sigma}\, \Big(K + d\, n^{\lambda} \nabla_{\lambda}\sigma \Big) \tag{67}$$

8. Small Variations of the Metric

Consider a small perturbation to the metric of the form $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. All indices are raised and lowered using the unperturbed metric $g_{\mu\nu}$ and its inverse. All quantities are expressed in terms of the perturbation to the metric with lower indices, and never in terms of the perturbation to the inverse metric. As in the previous sections, ∇_{μ} is the covariant derivative on $\mathcal{M}$ compatible with $g_{\mu\nu}$ and $\mathcal{D}_{\mu}$ is the covariant derivative on a hypersurface Σ compatible with $h_{\mu\nu}$.

Inverse Metric

$$g^{\mu\nu} \to g^{\mu\nu} - g^{\mu\alpha}\, g^{\nu\beta}\, \delta g_{\alpha\beta} + g^{\mu\alpha}\, g^{\nu\beta}\, g^{\lambda\rho}\, \delta g_{\alpha\lambda}\, \delta g_{\beta\rho} + \ldots \tag{68}$$

Square Root of Determinant

$$\sqrt{g} \rightarrow \sqrt{g}\, \left(1 + \frac{1}{2}\, g^{\mu\nu} \delta g_{\mu\nu} + \dots\right) \tag{69}$$

Variational Operator

$$\delta(g_{\mu\nu}) = \delta g_{\mu\nu} \qquad \delta^{\,2}(g_{\mu\nu}) = \delta(\delta g_{\mu\nu}) = 0 \tag{70}$$

$$\delta(g^{\mu\nu}) = -g^{\mu\alpha}\, g^{\nu\beta}\, \delta g_{\alpha\beta} \qquad \delta^{\,2}(g^{\mu\nu}) = \delta\left(-g^{\mu\lambda}\, g^{\nu\rho}\, \delta g_{\lambda\rho}\right) = 2\, g^{\mu\alpha}\, g^{\nu\beta}\, g^{\lambda\rho}\, \delta g_{\alpha\lambda}\, \delta g_{\beta\rho} \tag{71}$$

$$\mathcal{F}(g + \delta g) = \mathcal{F}(g) + \delta \mathcal{F}(g) + \frac{1}{2}\, \delta^{\,2} \mathcal{F}(g) + \dots + \frac{1}{n!}\, \delta^{\,n} \mathcal{F}(g) + \dots \tag{72}$$

Christoffel (All Orders)

$$\delta\, \Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}\, \left(\nabla_\mu\, \delta g_{\rho\nu} + \nabla_\nu\, \delta g_{\mu\rho} - \nabla_\rho\, \delta g_{\mu\nu}\right) \tag{73}$$

$$\delta^{\,2}\, \Gamma^{\lambda}_{\mu\nu} = -g^{\lambda\alpha}\, g^{\rho\beta}\, \delta g_{\alpha\beta}\, \left(\nabla_\mu\, \delta g_{\rho\nu} + \nabla_\nu\, \delta g_{\mu\rho} - \nabla_\rho\, \delta g_{\mu\nu}\right) \tag{74}$$

$$\delta^{\,n}\, \Gamma^{\lambda}_{\mu\nu} = \frac{n}{2}\, \delta^{\,n-1} \left(g^{\lambda\rho}\right)\, \left(\nabla_\mu\, \delta g_{\rho\nu} + \nabla_\nu\, \delta g_{\mu\rho} - \nabla_\rho\, \delta g_{\mu\nu}\right) \tag{75}$$

Riemann Tensor

$$\delta\, R^{\lambda}{}_{\mu\sigma\nu} = \nabla_\sigma \delta\, \Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\, \Gamma^{\lambda}_{\mu\sigma} \tag{76}$$

Ricci Tensor

$$\begin{aligned} \delta\, R_{\mu\nu} = {} & \nabla_\lambda \delta\, \Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\, \Gamma^{\lambda}_{\mu\lambda} & (77) \\ = {} & \frac{1}{2}\, \left(\nabla^\lambda \nabla_\mu\, \delta g_{\lambda\nu} + \nabla^\lambda \nabla_\nu\, \delta g_{\mu\lambda} - g^{\lambda\rho}\, \nabla_\mu \nabla_\nu\, \delta g_{\lambda\rho} - \nabla^2\, \delta g_{\mu\nu}\right) & (78) \end{aligned}$$

Ricci Scalar

$$\delta\, R = -R^{\mu\nu}\, \delta g_{\mu\nu} + \nabla^\mu \left(\nabla^\nu\, \delta g_{\mu\nu} - g^{\lambda\rho}\, \nabla_\mu\, \delta g_{\lambda\rho}\right) \tag{79}$$

Surface Forming Normal Vector

$$\delta\, n_\mu = \frac{1}{2}\, n_\mu\, n^\nu n^\lambda \delta g_{\nu\lambda} = \frac{1}{2}\, \delta g_{\mu\nu}\, n^\nu + c_\mu \tag{80}$$

$$c_\mu = \frac{1}{2}\, n_\mu\, n^\nu n^\lambda \delta g_{\nu\lambda} - \frac{1}{2}\, \delta g_{\mu\nu}\, n^\nu = -\frac{1}{2}\, h_\mu{}^\lambda\, \delta g_{\lambda\nu} n^\nu \tag{81}$$

Extrinsic Curvatures

$$\begin{aligned} \delta\, K_{\mu\nu} = {} & \frac{1}{2}\, n^\alpha n^\beta \delta g_{\alpha\beta}\, K_{\mu\nu} + \delta g_{\lambda\rho}\, n^\rho \left(n_\mu K^\lambda{}_\nu + n_\nu K_\mu{}^\lambda\right) \\ & - \frac{1}{2}\, h_\mu{}^\lambda\, h_\nu{}^\rho n^\alpha \left(\nabla_\lambda \delta g_{\alpha\rho} + \nabla_\rho \delta g_{\lambda\alpha} - \nabla_\alpha \delta g_{\lambda\rho}\right) \end{aligned} \tag{82}$$

$$\delta\, K = -\frac{1}{2}\, K^{\mu\nu}\, \delta g_{\mu\nu} - \frac{1}{2}\, n^\mu \left(\nabla^\nu \delta g_{\mu\nu} - g^{\nu\lambda}\, \nabla_\mu \delta g_{\nu\lambda}\right) + \mathcal{D}_\mu c^\mu \tag{83}$$

9. The ADM Decomposition

The conventions and notation in this section (and the next) are different than what was used in the preceding sections. We consider a d-dimensional spacetime with metric h_{ab}.

We start by identifying a scalar field t whose isosurfaces Σ_t are normal to the timelike unit vector given by

$$u_a = -\alpha\, \partial_a t \ , \tag{84}$$

where the lapse function α is

$$\alpha := \frac{1}{\sqrt{-h^{ab}\, \partial_a t\, \partial_b t}} \ . \tag{85}$$

An observer whose worldline is tangent to u_a experiences an acceleration given by the vector

$$a_b = u^c \cdot {}^d\nabla_c u_b \ , \tag{86}$$

which is orthogonal to u_a. The (spatial) metric on the $d-1$ dimensional surface Σ_t is given by

$$\sigma_{ab} = h_{ab} + u_a u_b \ . \tag{87}$$

The intrinsic Ricci tensor built from this metric is denoted by $\mathcal{R}_{ab}$, and its Ricci scalar is $\mathcal{R}$. The covariant derivative on Σ_t is defined in terms of the d dimensional covariant derivative as

$$D_a V_b := \sigma_a{}^c \sigma_b{}^e \big({}^d\nabla_c V_e\big) \quad \text{for any} \quad V_b = \sigma_b{}^c V_c \ . \tag{88}$$

The extrinsic curvature of Σ_t embedded in the ambient d dimensional spacetime (the constant r surfaces from the previous section) is

$$\theta_{ab} := -\sigma_a{}^c \sigma_b{}^d \big({}^d\nabla_c u_d\big) = -{}^d\nabla_a u_b - u_a a_b = -\frac{1}{2}\, \pounds_u \sigma_{ab} \ . \tag{89}$$

This definition has an additional minus sign, compared to the extrinsic curvature $K_{\mu\nu}$ for the constant r surfaces of the previous section. This is merely for compatibility with the standard conventions in the literature.

Now we consider a 'time flow' vector field t^a, which satisfies the condition

$$t^a\, \partial_a t = 1 \ . \tag{90}$$

The vector t^a can be decomposed into parts normal and along Σ_t as

$$t^a = \alpha\, u^a + \beta^a \ , \tag{91}$$

where α is the lapse function (85) and $\beta^a := \sigma^a{}_b t^b$ is the shift vector. An important result in the derivations that follow relates the Lie derivative of a scalar or spatial tensor (one that is orthogonal to u^a in all of its indices) along the time flow vector field, to Lie derivatives along u^a and β^a. Let S be a scalar. Then

$$\pounds_t S = \pounds_{\alpha u} S + \pounds_\beta S = \alpha \pounds_u S + \pounds_\beta S \ . \tag{92}$$

Rearranging this expression then gives

$$\pounds_u S = \frac{1}{\alpha}\left(\pounds_t S - \pounds_\beta S\right) \ . \tag{93}$$

Similarly, for a spatial tensor with all lower indices we have

$$\pounds_t W_{a\ldots} = \alpha \pounds_u W_{a\ldots} + \pounds_\beta W_{a\ldots} \ . \tag{94}$$

This is not the case when the tensor has any of its indices raised. In a moment, these identities will allow us to express certain Lie derivatives along u^a in terms of regular time derivatives and Lie derivatives along the shift vector β^a.

Next, we construct the coordinate system that we will use for the decomposition of the equations of motion. The adapted coordinates (t, x^i) are defined by

$$\partial_t x^a := t^a \ . \tag{95}$$

The x^i are d dimensional coordinates along the surface Σ_t. If we define

$$P_i{}^a := \frac{\partial x^a}{\partial x^i} \ , \tag{96}$$

then it follows from the definition of the coordinates that $P_i{}^a \partial_a t = 0$ and we can use $P_i{}^a$ to project tensors onto Σ_t. For example, in the adapted coordinates the spatial metric, extrinsic curvature, and acceleration and shift vectors are

$$\sigma_{ij} = P_i{}^a P_j{}^b \sigma_{ab} \tag{97}$$
$$\theta_{ij} = P_i{}^a P_j{}^b \theta_{ab} \tag{98}$$
$$a_j = P_j{}^b a_b \tag{99}$$
$$\beta_i = P_i{}^a \beta_a = P_i{}^a t_a \ . \tag{100}$$

The line element in the adapted coordinates takes a familiar form:

$$h_{ab} dx^a dx^b = h_{ab} \left(\frac{\partial x^a}{\partial t} dt + \frac{\partial x^a}{\partial x^i} dx^i \right) \left(\frac{\partial x^b}{\partial t} dt + \frac{\partial x^b}{\partial x^j} dx^j \right) \tag{101}$$
$$= h_{ab} \big(t^a dt + P_i{}^a dx^i \big) \big(t^b dt + P_j{}^b dx^j \big) \tag{102}$$
$$= t^a t_a dt^2 + 2 t_a dt P_i{}^a dx^i + h_{ab} P_i{}^a P_j{}^b dx^i dx^j \tag{103}$$
$$= \big(-\alpha^2 + \beta^i \beta_i \big) dt^2 + 2\beta_i dt dx^i + \sigma_{ij} dx^i dx^j \tag{104}$$
$$\Rightarrow h_{ab} dx^a dx^b = -\alpha^2 dt^2 + \sigma_{ij} \big(dx^i + \beta^i dt \big) \big(dx^j + \beta^j dt \big) \ . \tag{105}$$

Thus, in the adapted coordinate system we can express the components of the (d dimensional) metric h_{ab} and its inverse h^{ab} as

$$h_{ab} = \left(\begin{array}{c|c} -\alpha^2 + \beta^i \beta_i & \sigma_{ij} \beta^j \\ \hline \sigma_{ij} \beta^j & \sigma_{ij} \end{array} \right) \tag{106}$$
$$h^{ab} = \left(\begin{array}{c|c} -\frac{1}{\alpha^2} & \frac{1}{\alpha^2} \beta^i \\ \hline \frac{1}{\alpha^2} \beta^i & \sigma^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j \end{array} \right) \tag{107}$$
$$\det(h_{ab}) = -\alpha^2 \det(\sigma_{ij}) \tag{108}$$

Obtaining the components of the inverse is a short algebraic calculation. Note that the spatial indices '$i, j, \ldots$' in the adapted coordinates are lowered and raised using the spatial metric σ_{ij} and its inverse σ^{ij}.

In adapted coordinates there are several results concerning the projections of Lie derivatives of scalars and tensors which will be important in what follows. The first, which is trivial, is that the Lie derivative of a scalar S along the time-flow vector t^a is just the regular time-derivative

$$\pounds_t S = t^a \partial_a S = \frac{\partial x^a}{\partial t} \frac{\partial S}{\partial x^a} = \partial_t S \ . \tag{109}$$

Next, we consider the projector $P_i{}^a$ applied to the Lie derivative along t^a of a general vector W_a, which gives

$$P_i{}^a \pounds_t W_a = \partial_t W_a \quad \forall \quad W_a \ . \tag{110}$$

The important point is that this applies not just to spatial vectors but to *any* vector W_a, as a consequence of the result

$$P_i^{\ a} \pounds_t u_a = 0 \ . \tag{111}$$

Finally, we can show that the Lie derivative along t^a of any contravariant spatial vector satisfies

$$P^i{}_a \pounds_t V^a = \partial_t V^i \quad \forall \quad V^i = P^i{}_a V^a \ . \tag{112}$$

This follows from a lengthier calculation than what is required for the first two results.

Given these results, we can express various geometric quantities and their projections normal to and along Σ_t in terms of quantities intrinsic to Σ_t and simple time derivatives. First, the extrinsic curvature is

$$\theta_{ij} = -\frac{1}{2} P_i^{\ a} P_j^{\ b} \pounds_u \sigma_{ab} \tag{113}$$

$$= -\frac{1}{2} P_i^{\ a} P_j^{\ b} \Big(\frac{1}{\alpha} \left(\pounds_t \sigma_{ab} - \pounds_\beta \sigma_{ab} \right) \Big) \tag{114}$$

$$\Rightarrow \theta_{ij} = -\frac{1}{2\alpha} \Big(\partial_t \sigma_{ab} - \big(D_a \beta_b + D_b \beta_a \big) \Big) \ . \tag{115}$$

Since θ_{ab} is a spatial tensor, projections of its Lie derivative along u^a can be expressed in a similar manner

$$P_i^{\ a} P_j^{\ b} \pounds_u \theta_{ab} = \frac{1}{\alpha} \left(\partial_t \theta_{ab} - \pounds_\beta \theta_{ab} \right) \ . \tag{116}$$

Now we present the Gauss-Codazzi and related equations in adapted coordinates:

$$P_i^{\ a} P_j^{\ b} \big({}^d R_{ab} \big) = \mathcal{R}_{ij} + \theta \theta_{ij} - 2\theta_i^{\ k} \theta_{jk} - \frac{1}{\alpha} \left(\partial_t \theta_{ij} - \pounds_\beta \theta_{ij} \right) - \frac{1}{\alpha} D_i D_j \alpha \tag{117}$$

$$P_i^{\ a} \big({}^d R_{ab} u^b \big) = D_i \theta - D^j \theta_{ij} \tag{118}$$

$${}^d R_{ab} u^a u^b = \frac{1}{\alpha} \left(\partial_t \theta - \beta^i \partial_i \theta \right) - \theta^{ij} \theta_{ij} + \frac{1}{\alpha} D_i D^i \alpha \tag{119}$$

$${}^d R = \mathcal{R} + \theta^2 + \theta^{ij} \theta_{ij} - \frac{2}{\alpha} \left(\partial_t \theta - \beta^i \partial_i \theta \right) - \frac{2}{\alpha} D_i D^i \alpha \ . \tag{120}$$

10. Converting to ADM Variables

The metric is often presented in the form

$$h_{ab} dx^a dx^b = h_{tt} dt^2 + 2 h_{ti} dt dx^i + h_{ij} dx^i dx^j \ . \tag{121}$$

We would like to relate these components to the ADM variables: the lapse function α, the shift vector β_i, and the spatial metric σ_{ij}. This is a fairly straightforward exercise in linear algebra. Comparing with (105), we first note that

$$\sigma_{ij} = h_{ij} \ . \tag{122}$$

The inverse spatial metric, σ^{ij}, is literally the inverse of h_{ij}, which is *not* the same thing as h^{ij}

$$\sigma^{ij} = (\sigma_{ij})^{-1} = (h_{ij})^{-1} \neq h^{ij} \ . \tag{123}$$

For the shift vector we have

$$h_{ti} = \sigma_{ij} \beta^j \quad \rightarrow \quad \sigma^{ik} h_{tk} = \sigma^{ik} \sigma_{kl} \beta^l = \beta^i \tag{124}$$

$$\Rightarrow \beta^i = \sigma^{ij} h_{tj} \ . \tag{125}$$

Finally, for the lapse we obtain

$$\alpha^2 = \sigma^{ij} h_{ti} h_{tj} - h_{tt} \ . \tag{126}$$

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